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Comparison of EWMA and CUSUM Charts with Variable Sampling Intervals for Monitoring Variance-Covariance Matrix

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Abstract

To monitor all elements simultaneously of variance-covariance matrix Σ of several correlated quality characteristics under multivariate normal process $N_p(\underline{\mu}, \Sigma)$, multivariate exponentially weighted moving average (EWMA) chart and cumulative sum (CUSUM) chart are considered and compared. Numerical performances of the considered variable sampling interval (VSI) charts are evaluated using average run length (ARL), average time to signal (ATS), average number of switches (ANSW) to signal, and the probability of switch Pr(switch) between two sampling interval d_1 and d_2 where $d_1 < d_2$. For small or moderate changes of Σ, the performances of multivariate EWMA chart is approximately equivalent to that of multivariate CUSUM chart

Keywords : LRT Statistic, ARL, ATS, ANSW, Probability of Switches

1. Introduction

The Quality of a product is usually determined by jointed levels of multiple correlated quality characteristics not by single characteristic. When the multiple quality characteristics are correlated, a quality engineer can obtain better sensitivity by using multivariate control chart than separate control charts which use each of the characteristic or process parameter. The first work on multivariate control chart to detect changes in the process was introduced by Hotelling[1]. Alt[2] and Jackson[3] conducted many studies on multivariate quality control procedures. Jackson[4], Ghare and Togersen[5] and Alt[2] considered multivariate Shewhart control charts based on Hotelling's T^2 statistic.

The EWMA control chart was first introduced by Roberts[6]. Crowder and Hamilton[7] proposed an EWMA chart to monitor a process standard deviation, and they also showed that the proposed EWMA chart is superior to the R-chart or S^2-chart in terms of its ability to quickly detect small increasing in the standard deviation of a normal process $N(\mu, \sigma^2)$.

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Woodall and Ncube[8] considered a single multivariate CUSUM procedure for controlling the $\underline{\mu}$ of multivarite normal process. They described how a p-variate normal process $N_p(\mu, \sigma^2)$ can be monitored by using p two-sided univariate CUSUM charts. Through many researchers' numerical outcomes for evaluating the performances of control schemes, it is known that the performance of the EWMA control scheme is approximately equivalent to that of CUSUM control scheme and in some ways EWMA scheme is more easier to operate and interpret. Vargas et al.[9] studied and concluded that the efficiencies of CUSUM charts are similar to that EMMA charts in terms of time to signal (TS).

The efficiency of a control chart is determined by the length of time required to signal when a production process has changed. Thus, a good chart detects changes quickly of a process while producing few false alarms. In fixed sampling interval (FSI) chart, the run length (RL) is defined as the number of samples required for a chart to signal and the average run length (ARL) is the expected value of the RL. For VSI chart, the sampling time interval $t_{i+1}-t_i$ depends on the samples $\underline{X}_1$, $\underline{X}_2$, ..., $\underline{X}_i$.

Hence the ability of a control chart can be determined by ARL and the average time to signal (ATS). In VSI chart, frequent switching between different sampling intervals d_1 and $d_2 (d_1 < d_2)$ is one disadvantage. Therefore the average number of switches (ANSW) between different sampling intervals from the start of the production process until the chart signals, and the probabilities of switches are also measured to evaluate the ability of the control chart in VSI procedures. Chang[10] studied two sampling interval and three sampling interval VSI charts for monitoring both means $\mu_i's$ and variances $\sigma_i^2's(i = 1, 2,..., p)$ of multivariate normal process.

Most of studies on multivariate control chart have been focused on monitoring mean vector $\underline{\mu}$ of p-variate normal process $N_p(\underline{\mu}, \Sigma)$. In this study, a single EWMA chart and CUSUM chart are presented to monitor simultaneously both variances $\sigma_i^2(i = 1, 2,..., p)$ and two quality characteristics' covariances $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j(i = 1, 2,..., p, i \neq j)$ in variance-covariance matrix Σ.

2. Control Statistic for Variance-Covariance Matrix

Assume that a process in interest has p quality characteristics whose distribution is multivariate normal $N_p(\underline{\mu}, \Sigma)$, and $(\underline{\mu}_0, \Sigma_0)$ is the known target process values for $(\underline{\mu}, \Sigma)$. The target $\underline{\mu}_0$ and Σ_0 of p quality characteristics is represented as $\underline{\mu}_0 = (\mu_{10}, \mu_{20},..., \mu_{p0})'$ and $\Sigma_0 = (\rho_{ij0}\sigma_{i0}\sigma_0)_{p\times p}$ where the target covariance of characteristics X_r and X_s is $\sigma_{rs0} = \rho_{rs0}\sigma_{r0}\sigma_{s0}$ for $r,s = 1,2,..., p$.

At each sampling time $i(i = 1, 2,...)$, we take a sequence of random vector $\underline{X}_i' = (\underline{X}_i', \underline{X}_{i2}',..., \underline{X}_{in}')$ where $\underline{X}_{ij}' = (\underline{X}_{ij1}', \underline{X}_{ij2}',..., \underline{X}_{ijp}')$. Thus $\underline{X}_i$ is an $np\times 1$ column vector. The jkth element X_{ijk} of ith sampling time $\underline{X}_i$ is the jth observation for kth quality characteristic at each $i(j = 1, 2,..., n;\ k = 1, 2,..., p)$. In this paper, we also assume that the sequential observation vectors between and within samples are independent and identically distributed.

To control the matrix $\Sigma_{p\times p}$ of multivariate normal process, Alt[2] proposed the control statistic

$$W_i = tr\,(A_i\,\Sigma_0^{-1}) - n\ln|A_i| + n\ln|\Sigma_0| + np\ln n - np \quad (1)$$

where $A_i = \sum_{j=1}^{n}(\underline{X}_{ij}-\overline{\underline{X}}_i)(\underline{X}_{ij}-\overline{\underline{X}}_i)'$. Hence likelihood ratio test (LRT) statistic W_i for testing $H_0 : \Sigma = \Sigma_0$ vs $H_1 : \Sigma \neq \Sigma_0$, where target mean vector $\underline{\mu}_0$ is known, can be used as the control statistic for monitoring Σ.

3. CUSUM Chart with VSI Procedure

Multivariate CUSUM chart for Σ at the ith sample is

$$Y_{W1,i} = \max\{Y_{W1,i-1}, 0\} + (W_i - k_W) \quad (2)$$

where $Y_{W1,0} = \omega_1(\omega_1 \geq 0)$ and reference value $k_W \geq 0$. This CUSUM chart signals whenever $Y_{W1,i} \geq h_{W1}$. For the two sampling interval VSI scheme, suppose that the sampling interval $(d_1<d_2)$;

$$d_1 \text{ is used when } Y_{W1,i} \in (g_{W1}, h_{W1}], \quad (3)$$
$$d_2 \text{ is used when } Y_{W1,i} \in (-k_W, h_{W1}]$$

where $-k_W < g_{W1} \leq h_{W1}$. In this paper, we assume that this chart is started at time 0 and the sampling interval used before the first sample is a fixed constant, say d_0. Since it is difficult to obtain the distribution of (2), the process parameter g_{W1}, h_{W1} and the performances of this multivariate CUSUM chart can be evaluated by simulation with 10,000 iterations when the parameters of the process are on-target or changed.

To evaluate the efficiency of two sampling interval VSI charts, we count ANSW, the average number of switches made from the start of the process until the chart signals between two sampling intervals d_1 and d_2. The ANSW is caculated by $ANSW = (ARL - 1)\cdot$ Pr(switch).

And, the switch probability Pr(switch) is given by

$$\Pr(\text{switch}) = P(d_1)\cdot P(d_2|d_1) + P(d_2)\cdot P(d_1|d_2) \quad (4)$$

where $P(d_i)(i = 1, 2)$ is the probability of sampling interval d_i, and $P(d_i|d_j)$ is the conditional probability of d_i on the current sampling interval $d_j(d_i \neq d_j)$.

4. EWMA Chart with VSI procedure

Multivariate EWMA chart for Σ at the ith sample is

$$Y_{W2,i} = (1-\lambda)Y_{W2,i-1} + \lambda W_i \quad (5)$$

where $Y_{W2,0} = \omega_2(\omega_2 \geq 0)$ and $0 < \lambda \leq 1$. This EWMA chart signals whenever $Y_{W2,i} \geq h_{W2}$. For the two sampling interval VSI EWMA scheme based on LRT statistic W_i, suppose that the two sampling interval $(d_1 < d_2)$;

d_1 is used when $Y_{W2,i} \in (g_{W2}, h_{W2}]$, (6)
d_2 is used when $Y_{W2,i} \in (0, g_{W2}]$.

We also assume that the first sampling interval d_0 is a fixed constant. Since it is difficult to obtain the distribution of (5), the process parameters h_{W2}, g_{W2} and the performances of this chart can be evaluated by simulation under the parameters of the process being on-target or changed. When the smoothing constant $\lambda(0 < \lambda \le 1)$ is 1, the multivariate EWMA chart changes to multivariate Shewhart chart.

5. Numerical Performances and Concluding Remarks

In order to evaluate and compare the performances of the matched FSI and VSI multivariate charts, we let that the sampling interval is a unit time $d = 1$ in FSI chart and that two sampling intervals are $d_1 = 0.1$ and $d_2 = 1.9$ in the two sampling interval VSI chart with $d_0 = 1.0$. In our computation, the ARL and ATS of the proposed charts when the process is in-control were fixed to be 200 and the sample size n for each characteristic was 5 for $p = 4$. For computational convenience, we assume that the known target mean vector is $\underline{\mu}_0 = \underline{0}$ and we also assume that $\sigma_{r0}^2 = 1$, $\rho_{rs0} = 0.30$ for $r,s = 1, 2,..., p$ in target variance-covariance matrix Σ_0.

Since it is not possible to investigate all of the different scale of shifts in which Σ could change, we consider the following typical types of shifts for comparison in the process parameters :

1) component σ_1 : σ_{10} of Σ_0 starts at 1.0 and increases by 0.2 from 1.1 to 2.1.
2) components ρ_{12} and ρ_{21} C_i : ρ_{120} and ρ_{210} of Σ_0 start at 0.3 and increases by 0.1 from 0.4 to 0.9.
3) components (σ_1, ρ_{12}) : (σ_1, ρ_{12}) start at (1.0, 0.3) and increases by (0.2, 0.1) from (1.1, 0.4) to (2.1, 0.9).
4) Σ_0 is changed to $c_i\ \Sigma_0$ where c_i starts at target 1.0 and increases by 0.1 from 1.1 to 1.9.

The process parameters g_{W1}, h_{W1}, g_{W2} and h_{W2} of the chart are obtained to guarantee an in-control ARL and ATS. After the reference value k and smoothing constant λ of the proposed multivariate charts in (2) and (5) have been determined, the parameters g_{W1}, h_{W1}, g_{W2} and h_{W2} were obtained by simulation with 10,000 iterations.

The numerical performances for matched FSI and

Table 1. Performances of EWMA chart based on W_i when σ_1^2 in Σ_0 is changed.

	EWMA ($\lambda = 0.1$)				EWMA ($\lambda = 0.3$)			
σ_1	ARL	ATS	ANSW	Pr(switch)	ARL	ATS	ANSW	Pr(switch)
in-control	200.00	200.00	27.29	0.14	199.97	200.00	50.28	0.25
$\sigma_1 = 1.1$	181.08	176.98	24.49	0.14	185.98	181.92	46.71	0.25
$\sigma_1 = 1.3$	94.76	83.45	11.28	0.12	108.35	90.38	26.20	0.24
$\sigma_1 = 1.5$	44.61	41.62	4.51	0.10	44.50	30.47	9.60	0.22
$\sigma_1 = 1.7$	26.06	27.85	2.78	0.11	19.93	13.22	4.08	0.22
$\sigma_1 = 1.9$	17.68	20.82	2.32	0.14	11.23	8.16	2.63	0.26
$\sigma_1 = 2.1$	13.22	16.37	2.13	0.17	7.42	5.96	2.13	0.33

Table 2. Performances of CUSUM chart based on W_i when σ_1^2 in Σ_0 is changed.

	CUSUM ($k_W = 16.0$)				CUSUM ($k_W = 17.0$)			
σ_1	ARL	ATS	ANSW	Pr(switch)	ARL	ATS	ANSW	Pr(switch)
in-control	200.00	199.99	25.93	0.13	200.00	200.02	42.16	0.21
$\sigma_1 = 1.1$	177.28	171.53	22.91	0.13	180.25	175.76	37.85	0.21
$\sigma_1 = 1.3$	80.01	61.79	10.03	0.13	85.72	69.09	17.10	0.20
$\sigma_1 = 1.5$	33.25	20.46	4.27	0.13	32.81	19.90	6.00	0.19
$\sigma_1 = 1.7$	17.46	9.90	2.69	0.16	15.51	8.04	2.99	0.21
$\sigma_1 = 1.9$	11.00	6.11	2.14	0.21	9.20	4.59	2.07	0.25
$\sigma_1 = 2.1$	7.69	4.25	1.82	0.27	6.30	3.17	1.65	0.31

Table 3. Performances of EWMA chart based on W_i when ρ_{12} in Σ_0 is changed.

	EWMA ($\lambda = 0.1$)				EWMA ($\lambda = 0.3$)			
ρ_{12}	ARL	ATS	ANSW	Pr(switch)	ARL	ATS	ANSW	Pr(switch)
in-control	200.00	200.00	27.29	0.14	199.97	200.00	50.28	0.25
$\rho_{12} = 0.4$	186.38	183.03	25.23	0.14	190.02	186.92	47.76	0.25
$\rho_{12} = 0.5$	150.54	139.78	19.62	0.13	164.24	152.81	40.97	0.25
$\rho_{12} = 0.6$	104.01	89.94	12.34	0.12	123.35	102.76	29.73	0.24
$\rho_{12} = 0.7$	64.36	53.85	6.40	0.10	80.14	55.39	17.51	0.22
$\rho_{12} = 0.8$	37.69	34.39	3.26	0.09	42.72	22.85	7.34	0.18
$\rho_{12} = 0.9$	21.25	22.93	2.21	0.11	17.52	8.91	2.53	0.15

Table 4. Performances of CUSUM chart based on W_i when ρ_{12} in Σ_0 is changed.

	CUSUM ($k_W = 16.0$)				CUSUM ($k_W = 17.0$)			
ρ_{12}	ARL	ATS	ANSW	Pr(switch)	ARL	ATS	ANSW	Pr(switch)
in-control	200.00	199.99	25.93	0.13	200.00	200.02	42.16	0.21
$\rho_{12} = 0.4$	182.96	178.49	23.68	0.13	186.78	183.35	39.22	0.21
$\rho_{12} = 0.5$	138.39	124.84	17.62	0.13	148.12	136.71	30.58	0.21
$\rho_{12} = 0.6$	88.07	68.78	10.87	0.13	98.07	80.07	19.38	0.20
$\rho_{12} = 0.7$	49.08	31.80	5.79	0.12	53.87	35.75	9.74	0.18
$\rho_{12} = 0.8$	26.37	14.65	3.26	0.13	26.21	13.25	4.23	0.17
$\rho_{12} = 0.9$	13.33	6.72	2.13	0.17	11.63	4.70	2.01	0.19

Table 5. Performances of EWMA chart based on W_i when (σ_1, ρ_{12}) in Σ_0 are changed.

	EWMA ($\lambda = 0.1$)				EWMA ($\lambda = 0.3$)			
(σ_1, ρ_{12})	ARL	ATS	ANSW	Pr(switch)	ARL	ATS	ANSW	Pr(switch)
in-control	200.00	200.00	27.29	0.14	199.97	200.00	50.28	0.25
(1.1, 0.4)	172.63	166.62	23.17	0.13	179.69	173.71	45.09	0.25
(1.3, 0.5)	83.37	72.91	9.59	0.12	97.01	77.69	23.09	0.24
(1.5, 0.6)	38.29	36.67	3.78	0.10	36.86	23.96	7.59	0.21
(1.7, 0.7)	22.02	24.45	2.50	0.12	15.76	10.36	3.21	0.22
(1.9, 0.8)	14.52	17.63	2.14	0.16	8.56	6.38	2.20	0.29
(2.1, 0.9)	9.97	12.65	2.00	0.22	5.26	4.34	1.85	0.43

Table 6. Performances of CUSUM chart based on W_i when (σ_1, ρ_{12}) in Σ_0 are changed.

	CUSUM ($k_W = 16.0$)				CUSUM ($k_W = 17.0$)			
(σ_1, ρ_{12})	ARL	ATS	ANSW	Pr(switch)	ARL	ATS	ANSW	Pr(switch)
in-control	200.00	199.99	25.93	0.13	200.00	200.02	42.16	0.21
(1.1, 0.4)	166.43	158.27	21.45	0.13	171.29	164.87	35.80	0.21
(1.3, 0.5)	68.36	50.22	8.46	0.13	74.02	56.85	14.45	0.20
(1.5, 0.6)	27.58	16.31	3.64	0.14	26.58	15.11	4.79	0.19
(1.7, 0.7)	14.21	7.86	2.37	0.18	12.31	6.04	2.42	0.21
(1.9, 0.8)	8.58	4.60	1.87	0.25	7.07	3.35	1.71	0.28
(2.1, 0.9)	5.47	2.85	1.53	0.34	4.41	2.06	1.30	0.38

Table 7. Performances of EWMA chart based on W_i when Σ_0 is changed to $c\Sigma_0$.

	EWMA ($\lambda = 0.1$)				EWMA ($\lambda = 0.3$)			
$c\Sigma_0$	ARL	ATS	ANSW	Pr(switch)	ARL	ATS	ANSW	Pr(switch)
in-control	200.00	200.00	27.29	0.14	199.97	200.00	50.28	0.25
$1.1\times\Sigma_0$	142.44	131.95	18.50	0.13	155.63	143.58	38.73	0.25
$1.2\times\Sigma_0$	68.47	58.89	7.28	0.11	80.10	59.39	18.35	0.23
$1.3\times\Sigma_0$	35.06	33.89	3.36	0.10	34.06	20.74	6.55	0.20
$1.4\times\Sigma_0$	21.93	24.15	2.46	0.12	16.40	10.11	3.13	0.20
$1.5\times\Sigma_0$	15.55	18.51	2.18	0.15	9.51	6.70	2.28	0.27
$1.6\times\Sigma_0$	11.80	14.72	2.06	0.19	6.50	5.10	2.01	0.37
$1.7\times\Sigma_0$	9.38	12.00	2.01	0.24	4.87	4.09	1.85	0.48
$1.8\times\Sigma_0$	7.71	9.98	1.96	0.29	3.86	3.35	1.68	0.59
$1.9\times\Sigma_0$	6.47	8.42	1.91	0.35	3.19	2.81	1.50	0.69

Table 8. Performances of CUSUM chart based on W_i when Σ_0 is changed to $c\Sigma_0$.

	CUSUM ($k = 16.0$)				CUSUM ($k = 17.0$)			
$c\Sigma_0$	ARL	ATS	ANSW	Pr(switch)	ARL	ATS	ANSW	Pr(switch)
in-control	200.00	199.99	25.93	0.13	200.00	200.02	42.16	0.21
$1.1\times\Sigma_0$	130.15	115.75	16.63	0.13	139.01	126.85	28.67	0.21
$1.2\times\Sigma_0$	53.58	36.16	6.50	0.12	57.71	40.28	10.83	0.19
$1.3\times\Sigma_0$	24.68	14.10	3.28	0.14	23.54	12.47	4.10	0.18
$1.4\times\Sigma_0$	14.09	7.54	2.32	0.18	12.20	5.67	2.33	0.21
$1.5\times\Sigma_0$	9.26	4.80	1.91	0.23	7.67	3.47	1.74	0.26
$1.6\times\Sigma_0$	6.68	3.39	1.66	0.29	5.42	2.46	1.45	0.33
$1.7\times\Sigma_0$	5.10	2.61	1.49	0.36	4.07	1.92	1.26	0.41
$1.8\times\Sigma_0$	4.07	2.07	1.34	0.44	3.24	1.57	1.11	0.50
$1.9\times\Sigma_0$	3.35	1.74	1.21	0.52	2.68	1.38	0.99	0.59

VSI charts are given in Table 1 through Table 8. For the CUSUM chart, our computational results show that large reference value k is efficient for large shifts and smaller reference value k is efficient for small shifts of the process parameters in terms of ARL, ATS and ANSW. In EWMA chart, we also found that smaller values of smoothing constant λ are more efficient for small changes. Ryu and Wan[11] studied the optimal selection of reference value k in multivariate CUSUM chart for a mean shift of unknown size.

From the simulation results of Table 1 through Table 8, we can see that as the scales of shifts increase the ARL, ATS, and ANSW greatly decrease but the P(swith) dose change a little.

The optimal selection of reference value k in CUSUM or smoothing constant λ in EWMA charts depend on the size of the shift in monitoring parameters, and so we need to try to find the optimal k and λ which fit the scales of shifts of interest.

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