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BIFURCATION ANALYSIS OF A SINGLE SPECIES REACTION-DIFFUSION MODEL WITH NONLOCAL DELAY

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Abstract. A reaction-diffusion model with spatiotemporal delay modeling the dynamical behavior of a single species is investigated. The parameter regions for the local stability, global stability and instability of the unique positive constant steady state solution are derived. The conditions of the occurrence of Turing (diffusion-driven) instability are obtained. The existence of time-periodic solutions, the existence and nonexistence of nonconstant positive steady state solutions are proved by bifurcation method and energy method. Numerical simulations are presented to verify and illustrate the theoretical results.

1. Introduction

In 1952, Alan Turing published a seminal paper "The chemical basis of morphogenesis"[38]. His intriguing ideas influenced the thinking of theoretical biologists and scientists from many fields, successfully developed on the theoretical backgrounds. The Turing mechanism has been used to describe the structure changes of interacting species or reactants of ecology, chemical reaction and gene formation in nature (see for example, [1, 4–6, 8–15, 17–21, 23, 25–29, 31–36, 40, 42–47, 49, 51–57]).

In this paper, we investigate spatial, temporal and spatiotemporal patterns of following single species reaction-diffusion model with nonlocal delay:

$$
(1)\quad \begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u = ru\Big(1 + au - bu^2 \\ \qquad\qquad -c\displaystyle\int_{-\infty}^{t}\int_{\Omega}\frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}\frac{1}{\tau}e^{-\frac{t}{\tau}}u(y,s)dyds\Big), & x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases}
$$

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where Ω is a bounded and open domain in $\mathbf{R}^N$ with $N \geq 1$ is an integer, ν is the unit outward normal on $\partial\Omega$, and the homogeneous Neumann boundary condition indicates that the predator-prey system is self-contained with zero population flux across the boundary, a, b, c, d, r are positive parameters, $u_0(x) \in C^1(\overline{\Omega})$ is a nonnegative nontrivial function and satisfies $\partial u_0/\partial\nu = 0$ on $\partial\Omega$. Model (1) can be regarded as the dynamical behavior of a single species with effects of diffusion, aggregation, reproduction and competition for apace and resources. The biology interpretations of the terms in model (1) are as follows.

- The term au is a measure of the advantage to individuals in aggregating or grouping;
- The term $-bu^2$ denotes competition for space (rather than resources), which impedes population growth and stops the population density from ever exceeding a certain value;
- The integral term of model is called spatiotemporal delay or nonlocal effects in space and time, which reflects competition between the individuals for food sources.

Model (1) with $\Omega = \mathbf{R}^2$ was firstly proposed by Britton [2, 3], and they demonstrated that there were three kinds of bifurcation solutions, i.e., steady spatially periodic structure solutions, periodic standing wave solutions, and periodic traveling wave solutions. These bifurcation solutions would lead to the formation of spatiotemporal patterns, including uniform temporal oscillations, stationary spatially periodic patterns, standing waves and wave trains. Recently, in [48], by using the multiple scale method, the authors considered the conditions of both spot and stripe patterns.

Based on the above works, in this paper, we further study the spatial, temporal and spatiotemporal patterns of the solutions to problem (1). For the simplicities of our research, we let

$$v(x,t) = \int_\Omega \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}} \frac{1}{\tau} e^{-\frac{t}{\tau}} u(y,s)dyds, \tag{2}$$

and $\lambda = 1/\tau$, then (1) can be transformed as the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u = ru(1 + au - bu^2 - cv), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} - \Delta v = \lambda(u - v), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3}$$

The state problem corresponding to (3) is the following semilinear elliptic equations:

$$\begin{cases} -d\Delta u = ru(1+au-bu^2-cv), & x \in \Omega, \\ -\Delta v = \lambda(u-v), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

Throughout this paper, we denote

$$\alpha = \frac{a-c+\sqrt{(a-c)^2+4b}}{2b}, \tag{5}$$

then (3) possesses a unique constant equilibrium $(u, v) = (\alpha, \alpha)$.

The remaining part of this paper is organized as follows. In Section 2, we investigate the local stability, global stability and instability of the positive equilibrium (α, α) and occurrence of Turing instability. In Section 3, we study the existence of Hopf bifurcation. In Section 4, we consider the existence and nonexistence of positive solutions for problem (4) by bifurcation theory and energy method. In Section 5, we present some numerical simulations to verify and illustrate the theoretical results. Throughout this paper, $\mathbf{N}$ is the set of natural numbers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The eigenvalues of the operator $-\Delta$ with homogeneous Neumann boundary condition in Ω are denoted by $0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n \le \cdots$, and the eigenfunction corresponding to μ_n is $\phi_n(x)$.

2. Stability analysis

In this section, we consider the local and global asymptotic stability of unique constant equilibrium (α, α) defined as (5). The local stability of (α, α) with respect to (3) is determined by the following eigenvalue problem which is got by linearizing the (4) about (α, α) and the fact $b\alpha^2 = 1 + (a-c)\alpha$:

$$\begin{cases} d\Delta\phi + r((2c-a)\alpha - 2)\phi - cr\alpha\psi = \mu\phi, & x \in \Omega, \\ \Delta\psi + \lambda\phi - \lambda\psi = \mu\psi, & x \in \Omega, \\ \dfrac{\partial \phi}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{6}$$

Denote

$$L(\lambda) = \begin{pmatrix} d\Delta + r((2c-a)\alpha - 2) & -cr\alpha \\ \lambda & \Delta - \lambda \end{pmatrix}. \tag{7}$$

For each $n \in \mathbf{N}_0$, we define a 2×2 matrix

$$L_n(\lambda) = \begin{pmatrix} -d\mu_n + r((2c-a)\alpha - 2) & -cr\alpha \\ \lambda & -\mu_n - \lambda \end{pmatrix}. \tag{8}$$

The following statements hold true by using Fourier decomposition

(1) If μ is an eigenvalue of (6), then there exists $n \in \mathbf{N}_0$ such that μ is an eigenvalue of $L_n(\lambda)$.

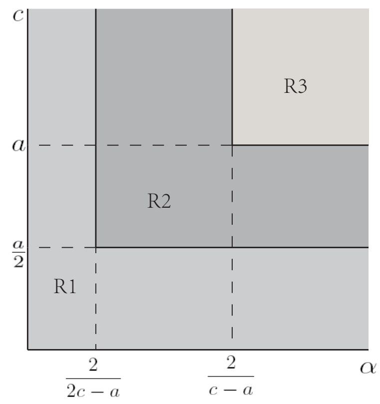

FIGURE 1. Illusion the stable and unstable regions in (α, c)-plane, where R1 is the stable region, R3 is the unstable region, and R2 the coexistence of stable and unstable region.

(2) The equilibrium (α, α) is locally asymptotically stable with respect to (3) if and only if for every $n \in \mathbf{N}_0$, all eigenvalues of $L_n(\lambda)$ have negative real part, and it is unstable if there exists an $n \in \mathbf{N}_0$ such that $L_n(\lambda)$ has at least one eigenvalue with positive real part.

The characteristic equation of $L_n(\lambda)$ is

$$\mu^2 - T_n(\lambda)\mu + D_n(\lambda) = 0, \tag{9}$$

where

$$T_n(\lambda) = -(d+1)\mu_n + r((2c-a)\alpha - 2) - \lambda,$$
$$D_n(\lambda) = d\mu_n^2 + [d\lambda - r((2c-a)\alpha - 2)]\mu_n + \lambda r(2 + (a-c)\alpha).$$

Then (α, α) is locally asymptotically stable if $T_n(\lambda) < 0$ and $D_n(\lambda) > 0$ for all $n \in \mathbf{N}_0$, and it is unstable if there exists $n \in \mathbf{N}_0$ such that $T_n(\lambda) > 0$ or $D_n(\lambda) < 0$.

Obviously, if $(2c-a)\alpha \leq 2$, then $T_n(\lambda) < 0$ and $D_n(\lambda) > 0$ for all $n \in \mathbf{N}_0$. That is (α, α) is locally asymptotically stable if (see R1 of Fig. 1)

(1) $c \leq \frac{a}{2}$; or
(2) $c > \frac{a}{2}$ and $\alpha \leq \frac{2}{2c-a}$.

If $c > a$ and $\alpha > \frac{2}{c-a}$, we get $D_0(\lambda) < 0$, so (α, α) is unstable (see R3 of Fig. 1).

So in the following, we consider the case (see R2 of Fig. 1)

$$\frac{a}{2} < c \leq a \text{ and } \alpha > \frac{2}{2c-a} \tag{10}$$

or the case

$$c > a \text{ and } \frac{2}{2c-a} < \alpha < \frac{2}{c-a}. \tag{11}$$

We define

$$\lambda_0 = r((2c-a)\alpha - 2) = cr\alpha - A > 0, \tag{12}$$

$$T(\lambda, \mu) = -(d+1)\mu + \lambda_0 - \lambda, \tag{13}$$

$$D(\lambda, \mu) = d\mu^2 + (d\lambda - \lambda_0)\mu + \lambda A, \tag{14}$$

and

$$H = \{(\lambda, \mu) \in (0, \infty) \times [0, \infty) : T(\lambda, \mu) = 0\},$$
$$S = \{(\lambda, \mu) \in (0, \infty) \times [0, \infty) : D(\lambda, \mu) = 0\},$$

where

$$A = r(2 + (a-c)\alpha), \tag{15}$$

which is positive if (10) or (11) holds. Then H is the Hopf bifurcation curve and S is the steady state bifurcation curve. Furthermore, the sets H and S are graphs of functions defined as follows

$$\lambda_H(\mu) = -(d+1)\mu + \lambda_0, \tag{16}$$

$$\lambda_S(\mu) = \frac{\lambda_0\mu - d\mu^2}{d\mu + A}. \tag{17}$$

Our next lemma is about the properties of $\lambda_H(\mu)$ and $\lambda_S(\mu)$. To give this lemma, let's firstly introduce some constants:

$$\mu_1^* = \frac{\sqrt{A^2 + A\lambda_0} - A}{d} \in \left(0, \frac{\lambda_0}{d}\right), \tag{18}$$

$$\mu_2^* = \frac{\lambda_0}{d+1}, \tag{19}$$

$$\mu_3^* = \frac{\lambda_0}{d}, \tag{20}$$

$$\mu_H = \frac{-[A(d+1) - d\lambda_0 + \lambda_0] + \sqrt{[A(d+1) - d\lambda_0 + \lambda_0]^2 + 4Ad^2\lambda_0}}{2d^2}, \tag{21}$$

$$D_1^* = \frac{A\lambda_0}{\sqrt{A^2 + A\lambda_0}}, \tag{22}$$

$$D_2^* = \sqrt{1 + \frac{\lambda_0}{A}} - 1, \tag{23}$$

$$D_3^* = \frac{(\sqrt{A+\lambda_0} - \sqrt{A})^2}{\lambda_0} < 1, \tag{24}$$

$$D_4^* = \frac{(\sqrt{A+\lambda_0} + \sqrt{A})^2}{\lambda_0} > D_3^*, \tag{25}$$

$$\lambda_S^* = \lambda_S(\mu_1^*) = \frac{\left(\sqrt{A^2+A\lambda_0}-A\right)(\lambda_0+2A)-A\lambda_0}{d\sqrt{A^2+A\lambda_0}}, \tag{26}$$

$$\lambda_H^* = \lambda_H(\mu_1^*) = \lambda_0 - \frac{d+1}{d}\left(\sqrt{A^2+A\lambda_0}-A\right), \tag{27}$$

$$\mu_L = \frac{(1-d)\lambda_0 - \lambda_0\sqrt{(d-D_3^*)(d-D_4^*)}}{2d}, \tag{28}$$

$$\mu_R = \frac{(1-d)\lambda_0 + \lambda_0\sqrt{(d-D_3^*)(d-D_4^*)}}{2d}. \tag{29}$$

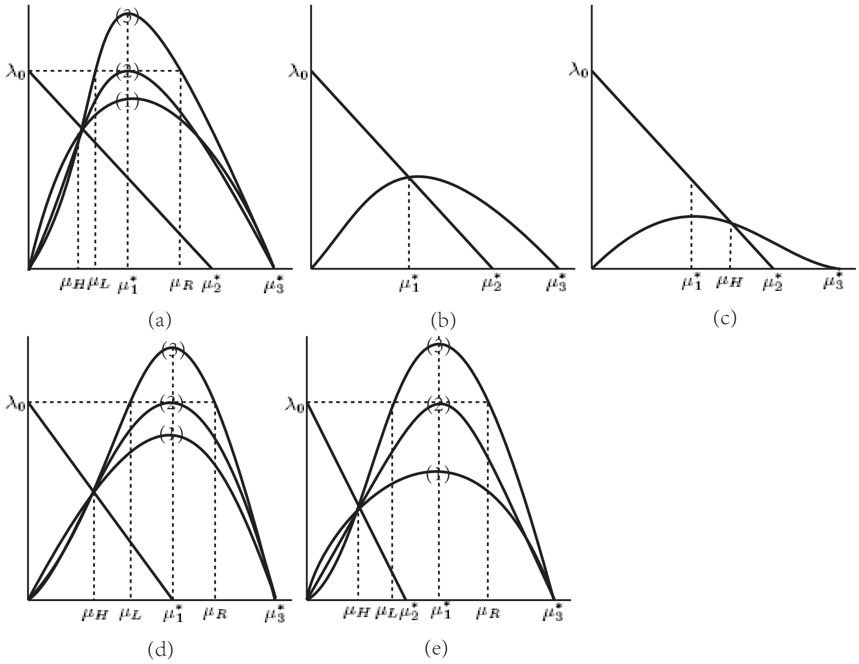

FIGURE 2. Illusion of Lemma 2.1. The curves are the graphs of $\lambda_S(\mu)$ and the lines are graphs of $\lambda_H(\mu)$. (a) is the case of $D_1^* < d < 1$; (b) is the case $1 = d > D_1^*$; (c) is the case $d > \max\{D_1^*, 1\}$; (d) is the case $d_1 = D_1^*$; (e) is the case $d_1 < D_1^*$. For all, the curves (1) represents $\lambda_S^* < \lambda_0$; the curves (2) represent $\lambda_S^* = \lambda_0$; the curves (1) represent $\lambda_S^* > \lambda_0$

Lemma 2.1 (see Fig. 2)**.** *Suppose* (10) *or* (11) *holds. Then the functions* $\lambda_H(\mu)$ *and* $\lambda_S(\mu)$ *admit the following properties*:

(1) *The function* $\lambda_H(\mu)$ *is strictly decreasing for* $\mu \in (0, \infty)$ *and*

$$\lambda_H(0) = \lambda_0,\ \lambda_H(\mu_2^*) = 0,\ \lambda_H(\mu) < 0\ for\ \mu > \mu_2^*,\ \lim_{\mu\to\infty} \lambda_H(\mu) = -\infty.$$

(2) $\mu = \mu_1^*$ *is the unique critical value of* $\lambda_S(\mu)$, *the function* $\lambda_S(\mu)$ *is strictly increasing for* $\mu \in (0, \mu_1^*)$, *and it is strictly decreasing for* $\mu \in (\mu_1^*, \infty)$. *Furthermore,*

$$\lambda_S(0) = \lambda_S(\mu_3^*) = 0, \ \max_{\mu\in[0,\infty)} \lambda_S(\mu) = \lambda_S^*, \ \lim_{\mu\to\infty} \lambda_S(\mu) = -\infty.$$

(3) $\lambda_H(\mu)$ *and* $\lambda_S(\mu)$ *cross at the point* $(\mu_H, \lambda_H(\mu_H))$ *and* $\lambda_H(\mu) > \lambda_S(\mu)$ *for* $0 \le \mu < \mu_H$, $\lambda_H(\mu) < \lambda_S(\mu)$ *for* $\mu_H < \mu \le \mu_3^*$.

(4) $\mu_1^* > \mu_2^*$ *if* $d < D_1^*$, $\mu_1^* = \mu_2^*$ *if* $d = D_1^*$, $\mu_1^* < \mu_2^*$ *if* $d > D_1^*$.

(5) $\mu_H < \mu_1^*$ *if and only if* $\lambda_H^* < \lambda_S^*$ *if and only if* $d < 1$; $\mu_H = \mu_1^*$ *if and only if* $\lambda_H^* = \lambda_S^*$ *if and only if* $d = 1$; $\mu_H > \mu_1^*$ *if and only if* $\lambda_H^* > \lambda_S^*$ *if and only if* $d > 1$.

(6) *If* $d \ge D_2^*$ *or* $D_3^* < d < D_2^*$, *then* $\lambda_S^* < \lambda_0$; *If* $D_3^* = d < D_2^*$, *then* $\lambda_S^* = \lambda_0$; *If* $d < D_2^*$ *and* $d < D_3^*$, *then* $\lambda_S^* > \lambda_0$.

(7) *If* $d < D_2^*$ *and* $d < D_3^*$, *then we have*
   (a) $0 < \mu_L < \mu_1^* < \mu_R$ *and* $\lambda_S(\mu_L) = \lambda(\mu_R) = 0$,
   (b) $\lambda_S(\mu) > \lambda_0$ *for* $\mu \in (\mu_L, \mu_R)$ *and* $0 < \lambda_S(\mu) < \lambda_0$ *for* $\mu \in (0, \mu_L) \cup (\mu_R, \mu_3^*)$.

Based on the above analysis, we can give a stability/instability result regarding to the positive equilibrium (α, α) with respect to (3).

Theorem 2.2. *Let* λ_0, D_2^*, D_3^* *and* λ_S^* *be the constants defined as* (12), (23), (24) *and* (26) *respectively. Then* (α, α) *is locally asymptotically stable with respect to* (3) *if*

(1) $c \le \frac{a}{2}$; *or*
(2) $c > \frac{a}{2}$ *and* $\alpha \le \frac{2}{2c-a}$; *or*
(3) (10) *or* (11) *holds and* $\lambda > \max\{\lambda_0, \overline{\lambda}\}$,

where

$$\overline{\lambda} = \max_{n\in\mathbf{N}} \lambda_S(\mu_n) \le \lambda_S^*. \tag{30}$$

In particular $\lambda > \max\{\lambda_0, \overline{\lambda}\}$ *holds if*

$$\lambda > \max\{\lambda_0, \lambda_S^*\} = \begin{cases} \lambda_S^*, & \text{if } d < \min\{D_2^*, D_3^*\}; \\ \lambda_0, & \text{otherwise.} \end{cases}$$

The positive equilibrium (α, α) *is unstable with respect to* (3) *if*

(1) $c > a$ *and* $\alpha > \frac{2}{c-a}$; *or*
(2) (10) *or* (11) *holds and* $\lambda < \max\{\lambda_0, \overline{\lambda}\}$.

Next we derive some conditions for the Turing instability with respect to the positive equilibrium (α, α), which means (α, α) is locally asymptotically stable with respect to the following local system

$$\begin{cases} \dfrac{du}{dt} = ru(1 + au - bu^2 - cv), & t > 0, \\ \dfrac{dv}{dt} = \lambda(u - v), & t > 0, \end{cases} \tag{31}$$

but it is unstable with respect to (3). It is easy to see (α, α) is locally asymptotically stable with respect to (31) if $\lambda > \lambda_0$. Then in view of Lemma 2.1 and Theorem (2.2), we have

Theorem 2.3. *Assume* (10) *or* (11) *holds. Then turing instability happens if*
(1) $d < \min\{D_2^*, D_3^*\}$ *and*
(2) *there exists* $k \in \mathbf{N}$ *such that* $\mu_k \in (\mu_L, \mu_R)$,
where $D_2^*, D_3^*, \mu_L, \mu_R$ *are given in* (23), (24), (28) *and* (29) *respectively.*

Now we consider the global stability of (α, α) with respect to (3) by using monotone iterative methods. Firstly, let's introduce the following lemma (see [41]):

Lemma 2.4. *Assume* $f(s) \in C^1([0, +\infty))$, $\kappa > 0$, $\rho \geq 0$, $T \in [0, +\infty)$ *are constants,* $w \in C^{2,1}(\Omega \times (T, +\infty)) \cap C^{1,0}(\bar{\Omega} \times [T, +\infty))$ *is positive. If* w *satisfies*

$$\begin{cases} \dfrac{\partial w}{\partial t} - \kappa \Delta w \leq (\geq) w^{1+\rho} f(w)(\gamma - w), & x \in \Omega,\ t > T, \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > T, \end{cases}$$

where $\gamma > 0$ *is a constant, we have*

$$\limsup_{t \to +\infty} \max_{\bar{\Omega}} w(\cdot, t) \leq \gamma \quad (\liminf_{t \to +\infty} \min_{\bar{\Omega}} w(\cdot, t) \geq \gamma).$$

Theorem 2.5. *If* $b > ac + c^2$, *then the positive constant equilibrium* (α, α) *is globally asymptotically stable with respect to* (3).

Proof. By the first equation of (3), we have

$$u_t - d\Delta u \leq ru(1 + au - bu^2) = bru\left(u - \frac{a - \sqrt{a^2 + 4b}}{2b}\right)\left(\frac{a + \sqrt{a^2 + 4b}}{2b} - u\right).$$

Then it follows from Lemma 2.4 that

$$\limsup_{t \to \infty} \max_{\overline{\Omega}} u(\cdot, t) \leq \frac{a + \sqrt{a^2 + 4b}}{2b} =: \overline{u}_1. \tag{32}$$

So for any $\epsilon > 0$, there exists $T_1^\epsilon \gg 1$ such that

$$u(x, t) \leq \overline{u}_1 + \epsilon,\ x \in \overline{\Omega},\ t \geq T_1^\epsilon.$$

By the second equation of (3) and the above inequality, we get

$$v_t - \Delta v \leq (\overline{u}_1 + \epsilon - v),\ x \in \Omega,\ t \geq T_1^\epsilon.$$

Then it follows Lemma 2.4 and the arbitrariness of ϵ that

$$\limsup_{t \to \infty} \max_{\Omega} v(\cdot, t) \leq \overline{u}_1 =: \overline{v}_1. \tag{33}$$

Since $b > ac + c^2$, there exists a constant $\epsilon_0 > 0$ such that $c(\overline{v}_1 + \epsilon) < 1$ for $\epsilon \in (0, \epsilon_0)$. For such ϵ, there exists $T_2^\epsilon \gg 1$ such that

$$v(x, t) \leq \overline{v}_1 + \epsilon,\ x \in \overline{\Omega},\ t \geq T_2^\epsilon.$$

By the first equation of (3) and the above inequality, for $x \in \Omega$ and $t \geq T_2^\epsilon$, we get

$$\begin{aligned} & u_t - d\Delta u \\ \geq\ & ru(1 - c(\overline{v}_1 + \epsilon) + au - bu^2) \\ =\ & bru\left(u - \frac{a - \sqrt{a^2 + 4b(1 - c(\overline{v}_1 + \epsilon))}}{2b}\right)\left(\frac{a + \sqrt{a^2 + 4b(1 - c(\overline{v}_1 + \epsilon))}}{2b} - u\right). \end{aligned}$$

Then it follows Lemma 2.4 and the arbitrariness of $\epsilon \in (0, \epsilon_0)$ that

$$\liminf_{t\to\infty} \min_{\overline{\Omega}} u(\cdot, t) \geq \frac{a + \sqrt{a^2 + 4b(1 - c\overline{v}_1)}}{2b} =: \underline{u}_1 > 0, \tag{34}$$

and $\overline{u}_1 \geq \underline{u}_1$. So for $\epsilon > 0$ small enough, there exists $T_3^\epsilon \gg 1$ such that

$$u(x, t) \geq \underline{u}_1 - \epsilon > 0,\ x \in \overline{\Omega},\ t \geq T_3^\epsilon.$$

By the second equation of (3) and the above inequality, we get

$$v_t - \Delta v \geq \lambda(\underline{u}_1 - \epsilon - v),\ x \in \overline{\Omega},\ t \geq T_3^\epsilon.$$

Then it follows Lemma 2.4 and the arbitrariness of ϵ that

$$\liminf_{t\to\infty} \min_{\overline{\Omega}} v(\cdot, t) \geq \underline{u}_1 =: \underline{v}_1 > 0, \tag{35}$$

and $\overline{v}_1 \geq \underline{v}_1$. So for any $\epsilon \in \underline{v}_1$, there exists $T_4^\epsilon \gg 1$ such that

$$v(x, t) \geq \underline{v}_1 - \epsilon,\ x \in \overline{\Omega},\ t \geq T_4^\epsilon.$$

By the first equation of (3) and the above inequality, for $x \in \Omega$ and $t \geq T_4^\epsilon$, we get

$$\begin{aligned} & u_t - d\Delta u \\ \leq\ & ru(1 - c(\underline{v}_1 - \epsilon) + au - bu^2) \\ =\ & bru\left(u - \frac{a - \sqrt{a^2 + 4b(1 - c(\underline{v}_1 - \epsilon))}}{2b}\right)\left(\frac{a + \sqrt{a^2 + 4b(1 - c(\underline{v}_1 - \epsilon))}}{2b} - u\right). \end{aligned}$$

Then it follows Lemma 2.4 and the arbitrariness of $\epsilon \in (0, \epsilon_0)$ that

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} u(\cdot, t) \leq \frac{a + \sqrt{a^2 + 4b(1 - c\underline{v}_1)}}{2b} =: \overline{u}_2 > 0, \tag{36}$$

and $\overline{u}_1 \geq \overline{u}_2$.

Let

$$\phi(s) = \frac{a + \sqrt{a^2 + 4b(1 - cs)}}{2b}, \tag{37}$$

where $s \in [0, \overline{v}_1]$. Then $1 - cs > 0$, and the constants $\overline{u}_1$, $\overline{v}_1$, $\underline{u}_1$, $\underline{v}_1$ and $\overline{u}_2$ constructed above satisfy

$$\begin{aligned} & \underline{v}_1 = \underline{u}_1 = \phi(\overline{v}_1) \leq \phi(\underline{v}_1) = \overline{u}_2 \leq \overline{u}_1 = \overline{v}_1, \\ & \underline{u}_1 \leq \liminf_{t\to\infty} \min_{\overline{\Omega}} u(\cdot, t) \leq \limsup_{t\to\infty} \max_{\overline{\Omega}} u(\cdot, t) \leq \overline{u}_2, \end{aligned} \tag{38}$$

$$\underline{v}_1 \le \liminf_{t\to\infty} \min_{\overline{\Omega}} v(\cdot,t) \le \limsup_{t\to\infty} \max_{\overline{\Omega}} v(\cdot,t) \le \overline{v}_1.$$

By induction, we can construct four sequence $\{\underline{u}_i\}_{i=1}^{\infty}$, $\{\underline{v}_i\}_{i=1}^{\infty}$, $\{\overline{u}_i\}_{i=1}^{\infty}$, $\{\overline{v}_i\}_{i=1}^{\infty}$ as follows

$$\overline{v}_i = \overline{u}_i,\ \underline{u}_i = \phi(\overline{v}_i), \overline{u}_{i+1} = \phi(\underline{v}_i), \underline{v}_i = \underline{u}_i, \tag{39}$$

such that

$$\underline{u}_i \le \liminf_{t\to\infty} \min_{\overline{\Omega}} u(\cdot,t) \le \limsup_{t\to\infty} \max_{\overline{\Omega}} u(\cdot,t) \le \overline{u}_i,$$
$$\underline{v}_i \le \liminf_{t\to\infty} \min_{\overline{\Omega}} v(\cdot,t) \le \limsup_{t\to\infty} \max_{\overline{\Omega}} v(\cdot,t) \le \overline{v}_i.$$

By (38), (39) and the decreasing property of ϕ, we can prove $\{\underline{u}_i\}_{i=1}^{\infty}$ and $\{\underline{v}_i\}_{i=1}^{\infty}$ are increasing, $\{\overline{u}_i\}_{i=1}^{\infty}$ and $\{\overline{v}_i\}_{i=1}^{\infty}$ are decreasing. So there exist two constants σ and δ such that

$$\lim_{i\to\infty} \underline{u}_i = \lim_{i\to\infty} \underline{v}_i = \sigma,\ \lim_{i\to\infty} \overline{u}_i = \lim_{i\to\infty} \overline{v}_i = \delta,$$
$$\sigma \le \liminf_{t\to\infty} \min_{\overline{\Omega}} u(\cdot,t) \le \limsup_{t\to\infty} \max_{\overline{\Omega}} u(\cdot,t) \le \delta,$$
$$\sigma \le \liminf_{t\to\infty} \min_{\overline{\Omega}} v(\cdot,t) \le \limsup_{t\to\infty} \max_{\overline{\Omega}} v(\cdot,t) \le \delta,$$
$$\sigma = \frac{a+\sqrt{a^2+4b(1-c\delta)}}{2b},\ \delta = \frac{a+\sqrt{a^2+4b(1-c\sigma)}}{2b}.$$

Then (α,α) is globally asymptotically stable if $\sigma=\delta$, since we can get $\sigma=\delta=\alpha$ by the fact

$$\delta = \frac{a+\sqrt{a^2+4b(1-c\delta)}}{2b}.$$

By contradiction, we assume $\sigma\ne\delta$, i.e., $\delta>\sigma$. Then we have

$$\begin{aligned}\delta-\sigma &= \frac{\sqrt{a^2+4b(1-c\sigma)}-\sqrt{a^2+4b(1-c\delta)}}{2b}\\ &= \frac{2c(\delta-\sigma)}{\sqrt{a^2+4b(1-c\sigma)}+\sqrt{a^2+4b(1-c\delta)}}\\ &= \frac{2c(\delta-\sigma)}{2b(\delta+\sigma)-2a},\end{aligned}$$

which implies

$$\delta+\sigma = \frac{a+c}{b}.$$

Then both δ and σ are the solutions of the following equation

$$\frac{a+c}{b} - z = \frac{a+\sqrt{a^2+4b(1-cz)}}{2b},\ z>0,$$

which is equivalent to

$$b^2z^2 - b(a+c)z + ac + c^2 - b,\ z>0. \tag{40}$$

Since $b > ac + c^2$, (40) has a unique solution, which implies $\delta = \sigma$, a contradiction. □

3. Hopf bifurcation

The main purpose of this section is to study the existence of periodic solutions of (3) by using a Hopf bifurcation results developed in [16, 49, 50]. The first result of this section is the following theorem:

Theorem 3.1. *Assume* (10) *or* (11) *holds. Let* Ω *be a smooth domain so that all eigenvalues* μ_i, $i \in \mathbf{N}_0$, *are simple,* $\phi_i(x)$ *is the corresponding eigenfunction. Then there exists a* $n_0 \in \mathbf{N}_0$ *such that* $\mu_{n_0} < \mu_H \le \mu_{n_0+1}$, *and* $\lambda_{i,H}$, *defined as*

$$\lambda_{i,H} = \lambda_H(\mu_i) \tag{41}$$

is a Hopf bifurcation value for $i \in \{0, \dots, n_0\}$, *where* μ_H *and* λ_H *are given in* (21) *and* (16) *respectively. At each* $\lambda_{i,H}$, *the system* (3) *undergoes a Hopf bifurcation, and the bifurcation periodic orbits near* $(\lambda, u, v) = (\lambda_{i,H}, \alpha, \alpha)$ *can be parameterized as* $(\lambda(s), u(s), v(s))$, *so that* $\lambda(s)$ *is the form of* $\lambda(s) = \lambda_{i,H} + o(s)$ *for* $s \in (0, \rho)$ *for some constant* $\rho > 0$, *and*

$$u(s)(x,t) = \alpha + s a_i \cos(\omega(\lambda_{i,H})t)\phi_i(x) + o(s),$$
$$v(s)(x,t) = \alpha + s b_i \cos(\omega(\lambda_{i,H})t)\phi_i(x) + o(s),$$

where $\omega(\lambda_{i,H}) = \sqrt{D_i(\lambda_i(H))}$ *with* $D_i(\lambda)$ *given in* (9) *is the corresponding time frequency,* $\phi_i(x)$ *is the corresponding spatial eigenfunction, and* (a_i, b_i) *is the corresponding eigenvector, i.e.,*

$$(L(\lambda_{i,H}) - i\omega(\lambda_{i,H})I)\begin{pmatrix} a_i\phi_i(x) \\ b_i\phi_i(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $L(\lambda)$ *is given in* (7). *Moreover,*

1. *The bifurcation periodic orbit from* $\lambda_{0,H} = \lambda_0$ *are spatially homogeneous, where* λ_0 *is given in* (12);
2. *The bifurcation periodic orbit from* $\lambda_{i,H}$, $i \in \{1, \dots, n_0\}$, *are spatially nonhomogeneous.*

Proof. We use λ as the main bifurcation parameter. To identify possible Hopf bifurcation value λ_H, we recall the following necessary and sufficient condition from [16, 49, 50].

(HS) There exists $i \in \mathbf{N}_0$ such that

$$T_i(\lambda_H) = 0,\ D_i(\lambda_H) > 0 \text{ and } T_j(\lambda_H) \neq 0,\ D_j(\lambda_H) \neq 0 \text{ for } j \in \mathbf{N}_0 \setminus \{0\}, \tag{42}$$

where $T_i(\lambda)$ and $D_i(\lambda)$ are given in (9), and for the unique pair of complex eigenvalues $A(\lambda) \pm iB(\lambda)$ near the imaginary axis,

$$A'(\lambda_H) \neq 0 \text{ and } B(\lambda_H) > 0. \tag{43}$$

By the definition of $\lambda_{i,H}$ in (41), we have $T_i(\lambda_{i,H}) = 0$ and $T_j(\lambda_{i,H}) \neq 0$ for $j \neq i$. By (42), we need $D_i(\lambda_{i,H}) > 0$ to make $\lambda_{i,H}$ as a possible bifurcation value, which means $\mu_i < \mu_H$ by Lemma 2.1, where μ_H is given in (21). Let $n_0 \in \mathbf{N}_0$ such that $\mu_{n_0} < \mu_H \le \mu_{n_0+1}$, then we can see (42) holds with $\lambda_H = \lambda_{i,H}$ for $i \in \{0, \dots, n_0\}$ (see Fig. 2). Finally, we consider the conditions in (43). Let the eigenvalues close to the pure imaginary one at $\lambda = \lambda_{i,H}$ be $A(\lambda) \pm iB(\lambda)$. Then

$$A'(\lambda_{i,H}) = \frac{T_i'(\lambda_{i,H})}{2} = -\frac{1}{2} < 0, \ B'(\lambda_{i,H}) = \sqrt{D_i(\lambda_{i,H})} > 0.$$

Then all conditions in (HS) are satisfied if $i \in \{0, \dots, n_0\}$. □

Next we calculate the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits bifurcating from $\lambda = \lambda_0$:

Theorem 3.2. *Suppose the assumptions in Theorem 3.1 hold. Then*

(1) *if* $\mathrm{Re}(c_1(\lambda_0)) < 0$*, the Hopf bifurcation at* $\lambda = \lambda_0$ *is subcritical and the bifurcating periodic solutions are orbitally asymptotical stable*;
(2) *if* $\mathrm{Re}(c_1(\lambda_0)) > 0$*, the Hopf bifurcation at* $\lambda = \lambda_0$ *is supercritical and the bifurcating periodic solutions are unstable,*

where $\mathrm{Re}(c_1(\lambda_0))$ *is define in* (46).

Proof. We use the normal form method and center manifold theorem in [16] to prove this theorem. Let $L^*(\lambda)$ be the conjugate operator of $L(\lambda)$ defined as (7), i.e.,

$$L^*(\lambda) = \begin{pmatrix} d\Delta + \lambda_0 & \lambda \\ -cr\alpha & \Delta - \lambda \end{pmatrix}, \tag{44}$$

with domain

$$D(L^*(\lambda)) = D(L(\lambda)) = \mathbf{X} \oplus i\mathbf{X} = \{x_1 + ix_2 : x_1, x_2 \in \mathbf{X}\},$$

where λ_0 is given in (12) and

$$\mathbf{X} := \left\{ (u, v) \in H^2(\Omega) \times H^2(\Omega) : \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

Let

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{cr\alpha}(\lambda_0 - i\sqrt{\lambda_0 A}) \end{pmatrix}, \quad q^* = \begin{pmatrix} q_1^* \\ q_2^* \end{pmatrix} = \frac{1}{2|\Omega|} \begin{pmatrix} 1 + i\sqrt{\frac{\lambda_0}{A}} \\ -i\frac{cr\alpha}{\sqrt{\lambda_0 A}} \end{pmatrix},$$

and A be the constants given in (15). It holds

(1) $\langle L^*(\lambda)\xi, \eta\rangle = \langle \xi, L(\lambda)\eta\rangle$ for $\xi \in D(L^*(\lambda))$ and $\eta \in D(L(\lambda))$,
(2) $L^*(\lambda_0)q^* = -i\sqrt{\lambda_0 A}q^*$ and $L(\lambda_0)q = i\sqrt{\lambda_0 A}q$,
(3) $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle = 0$,

where

$$\langle \xi, \eta \rangle = \int_\Omega \overline{\xi}^T \eta dx$$

denotes the inner product in $L^2(\Omega) \times L^2(\Omega)$.

According to [16], we decompose $\mathbf{X} = \mathbf{X}^C \oplus \mathbf{X}^S$ with

$$\mathbf{X}^C = \{zq + \overline{zq} : z \in \mathbf{C}\}, \ \mathbf{X}^S = \{\omega \in X : \langle q^*, \omega\rangle = 0\}.$$

For any $(u, v) \in \mathbf{X}$, there exist $z \in \mathbf{C}$ and $\omega = (\omega_1, \omega_2) \in \mathbf{X}^S$ such that

$$(u, v)^T = zq + \overline{zq} + (\omega_1, \omega_2)^T, \ z = \left\langle q^*, (u, v)^T \right\rangle.$$

Thus,

$$\begin{aligned} u &= z + \overline{z} + \omega_1, \\ v &= \frac{z}{cr\alpha}\left(\lambda_0 - i\sqrt{\lambda_0 A}\right) + \frac{\overline{z}}{cr\alpha}\left(\lambda_0 + i\sqrt{\lambda_0 A}\right) + \omega_2. \end{aligned}$$

Then system (3) in (z, ω) coordinates become

$$\begin{cases} \dfrac{dz}{dt} = i\sqrt{\lambda_0 A}z + \langle q^*, \mathfrak{F}\rangle, \\ \dfrac{d\omega}{dt} = L(\lambda)\omega + H(z, \overline{z}, \omega), \end{cases} \tag{45}$$

where $H(z, \overline{z}, \omega) = \mathfrak{F} - \langle q^*, \mathfrak{F}\rangle q - \langle \overline{q}^*, \mathfrak{F}\rangle \overline{q}$, $\mathfrak{F} = (f, 0)^T$ and $f = r(-bu^3 + (a - 3b\alpha)u^2 - cuv)$, and so

$$\langle q^*, \mathfrak{F}\rangle = \frac{1}{2}\left(1 - i\sqrt{\frac{\lambda_0}{A}}\right) f, \ \langle \overline{q}^*, \mathfrak{F}\rangle = \frac{1}{2}\left(1 + i\sqrt{\frac{\lambda_0}{A}}\right) f,$$

$$\langle q^*, \mathfrak{F}\rangle q = \frac{1}{2}\begin{pmatrix} 1 - i\sqrt{\frac{\lambda_0}{A}} \\ -\frac{i}{cr\alpha}\left(\sqrt{\lambda_0 A} + \lambda_0\sqrt{\frac{\lambda_0}{A}}\right)\end{pmatrix} f,$$

$$\langle \overline{q}^*, \mathfrak{F}\rangle \overline{q} == \frac{1}{2}\begin{pmatrix} 1 + i\sqrt{\frac{\lambda_0}{A}} \\ \frac{i}{cr\alpha}\left(\sqrt{\lambda_0 A} + \lambda_0\sqrt{\frac{\lambda_0}{A}}\right)\end{pmatrix} f.$$

A direct calculation shows that $H(z, \overline{z}, \omega) = (0, 0)^T$.

Let

$$H(z, \overline{z}, \omega) = \frac{1}{2}H_{20}z^2 + H_{11}z\overline{z} + \frac{1}{2}H_{02}\overline{z}^2 + o(|z|^2).$$

It follows [16, Appendix A] that the system (45) possesses a center manifold, then we can write ω in the form

$$\omega = \frac{1}{2}\omega_{20}z^2 + \omega_{11}z\overline{z} + \frac{1}{2}\omega_{02}\overline{z}^2 + o(|z|^2).$$

Thus we have

$$\overline{\omega}_{02} = w_{20} = \left(2i\sqrt{\lambda_0 A}I - L\right)^{-1} H_{20} = 0, \ \omega_{11} = (-L)^{-1}H_{11} = 0.$$

For later uses, we denote

$$\begin{aligned} c_0 &= f_{uu}q_1^2 + 2f_{u,v}q_1q_2 + f_{vv}q_2^2 = 2r(a - 3b\alpha) - 2rcq_2, \\ d_0 &= f_{uu}|q_1|^2 + f_{uv}(q_1\overline{q}_2 + \overline{q}_1q_2) + f_{vv}|q_2|^2 = 2r(a - 3b\alpha) - rc(\overline{q}_2 + q_2), \end{aligned}$$

$$e_0 = f_{uuu}|q_1|^2 q_1 + f_{uuv}(2|q_1|^2 q_2 + q_1^2 \overline{q}_2) \\ + f_{uvv}(2q_1|q_2|^2 + \overline{q}_1 q_2^2) + f_{vvv}|q_2|^2 q_2 = -6rb,$$

with all the partial derivatives evaluated at the point $(u, v) = (0, 0)$. Therefore, the model (3) restricted the center manifold in $z, \overline{z}$ coordinates is given by

$$\frac{dz}{dt} = i\sqrt{\lambda_0 A} z + \frac{1}{2}\phi_{20} z^2 + \phi_{11} z\overline{z} + \frac{1}{2}\phi_{02}\overline{z}^2 + \frac{1}{2}\phi_{21} z^2 \overline{z} + o(|z|^3),$$

where

$$\phi_{20} = \langle q^*, (c_0, 0)^T \rangle = r(a - 3b\alpha) + i\left(\frac{1}{a}\sqrt{\lambda_0 A} + \sqrt{\frac{\lambda_0}{A}}\left(\frac{\lambda_0}{a} - r(a - 3b\alpha)\right)\right),$$

$$\phi_{11} = \langle q^*, (d_0, 0)^T \rangle = \left(r(a - 3b\alpha) + \frac{\lambda_0}{\alpha}\right)\left(1 - i\sqrt{\frac{\lambda_0}{A}}\right),$$

$$\phi_{21} = \langle q^*, (e_0, 0)^T \rangle = -3rb\left(1 - i\sqrt{\frac{\lambda_0}{A}}\right).$$

According to [16], we have

$$\begin{aligned} \mathrm{Re}(c_1(\lambda_0)) &= \mathrm{Re}\left\{\frac{i}{2\sqrt{\lambda_0 A}}\left(\phi_{20}\phi_{11} - 2|\phi_{11}|^2 - \frac{1}{3}|\phi_{02}|^2\right) + \frac{1}{2}\phi_{21}\right\} \\ &= -\frac{1}{2\sqrt{\lambda_0 A}}\left[\mathrm{Re}(\phi_{20})\mathrm{Im}(\phi_{11}) + \mathrm{Im}(\phi_{20})\mathrm{Re}(\phi_{11})\right] + \frac{1}{2}\mathrm{Re}(\phi_{21}) \\ &= -\frac{1}{2}\left[r(a - 3b\alpha) + \frac{\lambda_0}{2}\right]\left[\frac{1}{a} + \frac{1}{a}\lambda_0^2 - 2r(a - 3b\alpha)\lambda_0\right] - \frac{3}{2}rb. \end{aligned} \tag{46}$$

□

4. Analysis of the steady state solutions

In this section, we consider the existence/nonexistence of nonconstant positive solutions to (4). This section is divided into three parts. In the first part, we give some *a priori* estimates of the solutions of (4), which are useful in the later discussions. In part 2 we study the nonexistence of nonconstant solutions of (4), while in part 3 we study the existence of nonconstant solutions via bifurcation method.

4.1. A priori estimates

To derive a priori estimates, we need the follow lemma given in [24]:

Lemma 4.1. *Suppose that* $g \in (\overline{\Omega} \times \mathbf{R})$.

(i) *Assume that* $w \in C^2(\omega) \cap C^1(\overline{\Omega})$ *and satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0 \ \text{in } \Omega, \quad \frac{\partial w}{\partial \nu} \leq 0 \ \text{on } \partial\Omega. \tag{47}$$

If $w(x_0) = \max_{x \in \overline{\Omega}} w(x)$, *then* $g(x_0, w(x_0)) \geq 0$.

(ii) *Assume that* $w \in C^2(\omega) \cap C^1(\overline{\Omega})$ *and satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0 \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} \geq 0 \text{ on } \partial\Omega. \tag{48}$$

If $w(x_0) = \min_{x \in \overline{\Omega}} w(x)$, *then* $g(x_0, w(x_0)) \leq 0$.

Theorem 4.2. *Let* (u, v) *be a positive solution of* (4). *Assume*

$$c \leq \frac{1}{2}\left(\sqrt{a^2+4b} - a\right), \tag{49}$$

then

$$\underline{M} < u(x),\ v(x) < \overline{M}, \tag{50}$$

where

$$\underline{M} = \frac{a + \sqrt{a^2 + 4b - 2c\left(a + \sqrt{a^2+4b}\right)}}{2b} \in \left(0, \frac{1}{c}\right),$$
$$\overline{M} = \frac{a + \sqrt{a^2 + 4b(1 - c\underline{M})}}{2b} \in \left(\underline{M}, \frac{1}{c}\right).$$

Proof. Let

$$u(x_1) = \max_{\overline{\Omega}} u(x),\ v(x_2) = \max_{\overline{\Omega}} v(x),\ u(y_1) = \min_{\overline{\Omega}} u(x),\ v(y_2) = \min_{\overline{\Omega}} v(x).$$

Then by Lemma 4.1,

$$1 + au(x_1) - bu^2(x_1) - cv(x_1) \geq 0,\ 1 + au(y_1) - bu^2(y_1) - cv(y_1) \leq 0, \tag{51}$$
$$u(x_2) - v(x_2) \geq 0,\ u(y_2) - v(y_2) \leq 0. \tag{52}$$

By the first equation of (4) we get

$$-d\Delta u \leq ru(1 + au - bu^2) = bru\left(u - \frac{a - \sqrt{a^2+4b}}{2b}\right)\left(\frac{a + \sqrt{a^2+4b}}{2b} - u\right).$$

Then we get from Lemma 4.1 that

$$u(x_1) \leq \frac{a + \sqrt{a^2+4b}}{2b}. \tag{53}$$

Then it follows from (51), (52) and (53) that

$$bu^2(x_1) - au(x_1) + cu(y_1) - 1 \leq 0, \tag{54}$$
$$bu^2(y_1) - au(y_1) + cu(x_1) - 1 \geq 0, \tag{55}$$
$$v(x_2) \leq u(x_1),\ v(y_2) \geq u(y_1). \tag{56}$$

By (53) and (55), we have

$$bu^2(y_1) - au(y_1) + \frac{c\left(a + \sqrt{a^2+4b}\right)}{2b} - 1 \geq 0. \tag{57}$$

If

$$c \le \frac{2b}{a+\sqrt{a^2+4b}} = \frac{1}{2}\left(\sqrt{a^2+4b}-a\right),$$

then by (57), we have $u(y_1) > \underline{M}$ and $0 < \underline{M} < 1/c$. The remain conclusions follow from (54) and (56). $\square$

Remark 4.3. We give two remarks about the above theorem.

(1) By the definitions of $\overline{M}$ and $\underline{M}$ in Theorem 4.2, we know that

$$\lim_{c\to 0} \overline{M} = \lim_{c\to 0} \underline{M} = \frac{a+\sqrt{a^2+4b}}{2b} =: \alpha_0.$$

Then it follows from Theorem 4.2 that

$$\lim_{c\to 0}(u,v) = (\alpha_0,\alpha_0),$$

where (u,v) is the positive solution of (4), and it is easy to see (α_0,α_0) is the unique positive constant solution of (4) when $c=0$. These facts intrigue us to consider the nonexistence of nonconstant positive solution of (4) when c is small enough (see part 3 of Remark 4.7).

(2) Let's re-consider the proof of Theorem 4.2 and set $\rho(t) = -bt^2+at+1$. Then we can write (55) as

$$cu(x_1) \ge \rho(u(y_1)).$$

Since $0 \le u(y_1) \le u(x_1)$, we get

$$cu(x_1) \ge \min\{\rho(0), \rho(u(x_1))\} = \min\{1, \rho(u(x_1))\}. \tag{58}$$

If $\rho(u(x_1)) \ge 1$, we get from (58) that $u(x_1) \ge 1/c > 0$. On the other hand, if $\rho(u(x_1)) < 1$, we get form (58) that $u(x_1) \ge \alpha > 0$, where α is the positive constant defined as (5). In all we get the following estimate without the assumptions in Theorem 4.2

$$\max_{x\in\overline{\Omega}} u(x) \ge C_1 := \min\left\{\frac{1}{c}, \alpha\right\}. \tag{59}$$

The following Harnack inequality can be found in [22].

Lemma 4.4. *Let $w \in C^2(\Omega)\cap C^1(\overline{\Omega})$ be a positive solution to*

$$\Delta w(x) + c(x)w(x) = 0 \text{ in } \Omega,\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

where $c(x)$ is a continuous function on $\overline{\Omega}$. Then there exists a positive constant C, depending only on

$$\|c\|_\infty := \max_{x\in\overline{\Omega}} |c(x)|$$

and Ω, such that

$$\max_{x\in\overline{\Omega}} w(x) \le C \min_{x\in\overline{\Omega}} w(x). \tag{60}$$

By using (59) and the above lemma, we can ignore (49) in Theorem 4.2 and achieve the following theorem.

Theorem 4.5. *Assume* (u, v) *is a positive solution of* (4) *with* $d \geq d_1$, *where* d_1 *is any positive constant. Then there exists a positive constant* θ *depending only on* a, b, c, r, d_1 *and* Ω *such that*

$$\theta \leq u(x), v(x) \leq \frac{a+\sqrt{a^2+4b}}{2b}.$$

Proof. It follows from the proof of Theorem 4.2 that

$$u(x) \leq \frac{a+\sqrt{a^2+4b}}{2b},\ v(x) \leq \frac{a+\sqrt{a^2+4b}}{2b},\ x \in \overline{\Omega},$$

and

$$\min_{x\in\overline{\Omega}} u(x) \leq \min_{x\in\overline{\Omega}} v(x). \tag{61}$$

Rewrite the first equation of (31) as $\Delta u(x) + c(x)u(x) = 0$ with

$$c(x) := \frac{r}{d}(1 + au - bu^2 - cv).$$

Then it follows from

$$\begin{aligned} \|c\|_\infty &\leq \frac{r}{d}\left(1 + a\|u\|_\infty + b\|u\|_\infty^2 + c\|v\|_\infty\right) \\ &\leq \frac{r}{d_1}\left(1 + (a+c)\frac{a+\sqrt{a^2+4b}}{2b} + \frac{\left(a+\sqrt{a^2+4b}\right)^2}{4b}\right), \end{aligned}$$

(59) and Lemma 4.4 that there exists a positive constant C, depending only on a, b, c, r, d_1 and Ω such that

$$C_1 \leq \max_{x\in\overline{\Omega}} u(x) \leq C \min_{x\in\overline{\Omega}} u(x).$$

So we have

$$u(x) \geq \frac{C_1}{C},\ x \in \overline{\Omega}.$$

By (61),

$$v(x) \geq \frac{C_1}{C},\ x \in \overline{\Omega}.$$

Then the conclusion holds with

$$\theta = \frac{C_1}{C},$$

which depends only on a, b, c, r, d_1 and Ω. □

4.2. Nonexistence of positive nonconstant steady state solutions

In this part, we will prove that (4) has no positive nonconstant solutions if the diffusion coefficient d is large or the "size" of Ω is small.

Theorem 4.6. *Assume* (49) *holds. Let* $\overline{M}$ *and* $\underline{M}$ *be the two constants defined as Theorem 4.2. If* μ_1, d *satisfies*

$$d\mu_1 > 1 + 2a\overline{M} - 3b\underline{M}^2 - c\underline{M}, \tag{62}$$

then (4) *has no positive nonconstant solutions.*

Proof. Let $\overline{\xi} := |\Omega|^{-1}\int_\Omega \xi(x)dx$, $\phi := u - \overline{u}$ and $\psi := v - \overline{v}$. Assume (u, v) be a positive solution of (4). It is obvious that $\int_\Omega \phi dx = \int_\Omega \psi dx = 0$. Multiplying the second equation of (4) by ϕ and integrating over Ω. By Poincaré's inequality, we obtain

$$\mu_1 \int_\Omega \psi^2 dx \le \int_\Omega |\nabla\psi|^2 dx = \lambda \int_\Omega \phi\psi dx - \lambda \int_\Omega \psi^2 dx,$$

which implies $\int_\Omega \phi\psi \ge 0$. Multiplying the first equation of (4) by ϕ and integrating over Ω. By Poincaré's inequality, Theorem 4.2 and $\int_\Omega \phi\psi \ge 0$, we obtain

$$\begin{aligned}
\frac{\mu_1 d}{r}\int_\Omega |\phi|^2 dx &\le \frac{d}{r}\int_\Omega |\nabla\phi|^2 dx \\
&= \int_\Omega (1 + a(u+\overline{u}) - b(u^2 + u\overline{u} + \overline{u}^2) - cv)\phi^2 dx - \overline{u}\int_\Omega \phi\psi dx \\
&\le (1 + 2a\overline{M} - 3b\underline{M}^2 - c\underline{M})\int_\Omega \phi^2.
\end{aligned}$$

Under our hypothesis, the above equality implies $u \equiv \overline{u}$. Then v satisfies

$$\begin{cases} -\Delta v + \lambda v = \lambda\overline{u}, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{63}$$

Since $\lambda > 0$, the above problem has a unique solution $v = \lambda(-\Delta + \lambda)^{-1}\overline{u} \equiv \overline{u}$. □

Remark 4.7. We give some remarks of Theorem 4.6.

(1) It is clear that Theorem 4.6 holds if (49) is satisfied and μ_1 is large enough. Note that large μ_1 is reflected by small "size" of the domain Ω (see [7, 30] for precise explanation of "size").
(2) Obviously, Theorem 4.6 holds if (49) is satisfied and d is large enough.
(3) We choose c is small enough such that (49) holds. Let $\varrho(t) = 1 + 2at - 3bt^2$. Then $\varrho(t)$ is strictly decreasing on $(a/(3b), \infty)$. By remark 4.3, we know that

$$\lim_{c\to 0}\overline{M} = \lim_{c\to 0}\underline{M} = \frac{a + \sqrt{a^2 + 4b}}{2b}.$$

Since

$$\frac{a+\sqrt{a^2+4b}}{2b} > \frac{a+\sqrt{a^2+3b}}{3b} > \frac{a}{3b},$$

then

$$\lim_{c\to 0} 1+2a\overline{M}-3b\underline{M}^2-c\underline{M} = \varrho\left(\frac{a+\sqrt{a^2+4b}}{2b}\right) < \varrho\left(\frac{a+\sqrt{a^2+3b}}{3b}\right) = 0.$$

So Theorem 4.6 holds if c is small enough.

Based on part 3 of Remark 4.7, we can get:

Proposition 4.8. *Let a, b, d, r, λ be fixed positive constants. Then there exists a positive constant c_0 depending on a, b, d, r, λ and Ω such that (4) has no positive nonconstant solutions when $c < c_0$.*

Next we want to discard the condition (49) in part 2 of above remark. To this end, we firstly introduce the following lemma.

Lemma 4.9. *Assume (u_i, v_i) is the positive solution of (4) with $d = d_i$, where $d_i \to \infty$ as $i \to \infty$, then $(u_i, v_i) \to (\alpha, \alpha)$ in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ as $i \to \infty$, where (α, α) is the unique constant equilibrium of (4) defined as (5).*

Proof. By Theorem 4.5, Sobolev embedding theory and standard regularity theory of elliptic equations, there exists a subsequence of (u_i, v_i), relabeled as itself, and $(u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ such that $(u_i, v_i) \to (u, v)$ in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ as $i \to \infty$. Furthermore, it follows from Theorem 4.5 that $u, v \geq \theta > 0$, where θ is a positive constants given in Theorem 4.5, and (u, v) satisfies

$$\begin{cases} -\Delta u = 0, & x \in \Omega, \\ -\Delta v = \lambda(u - v), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega u(1 + au - bu^2 - cv)dx = 0. \end{cases} \tag{64}$$

From the equations for u in (64), we know there exists a positive constant c such that $u \equiv c$. Then v satisfies

$$\begin{cases} -\Delta v + \lambda v = \lambda c, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{65}$$

Then $v = \lambda(-\Delta + \lambda)^{-1}c = c$ and it follows form $u = v \equiv c > 0$ and the forth relation of (64) that $1 + ac - bc^2 - c^2 = 0$, that is $c = \alpha$. □

Based on Lemma 4.9, we can obtain the following result by using Implicit Function Theorem.

Theorem 4.10. *Let a, b, c, r, λ be fixed positive constants. Then there exists a positive constant D depending on a, b, c, r, λ and Ω such that (4) has no positive nonconstant solutions when $d > D$.*

Proof. Let $u = w + \xi$, where $\xi = |\Omega|^{-1}\int_\Omega u dx$. Then $\int_\Omega w dx = 0$. We observe that finding solutions of (4) is equivalent to solving the following problem ($\sigma = 1/d$)

$$\begin{cases} \Delta w + \sigma r(w+\xi)(1 + a(w+\xi) - b(w+\xi)^2 - cv) = 0, & x \in \Omega, \\ \Delta v + \lambda(w + \xi - v) = 0, & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega (w+\xi)(1 + a(w+\xi) - b(w+\xi)^2 - cv) dx = 0. \end{cases} \tag{66}$$

Obviously, for all $\sigma > 0$, $(w, v, \xi) = (0, \alpha, \alpha)$ is a solution of (66), where α is defined as (5).

So, to complete the proof, we only need to prove there exists a positive constant σ_0 depending on a, b, c, r, λ and Ω such that when $\sigma < \sigma_0$, $(0, \alpha, \alpha)$ is the unique solution of (66).

Let

$$\begin{aligned} f_1(\sigma, w, v, \xi) &= \Delta w + \sigma r(w+\xi)(1 + a(w+\xi) - b(w+\xi)^2 - cv), \\ f_2(\sigma, w, v, \xi) &= \Delta v + \lambda(w + \xi - v), \\ f_3(\sigma, w, v, \xi) &= \int_\Omega (w+\xi)(1 + a(w+\xi) - b(w+\xi)^2 - cv) dx. \end{aligned}$$

Then we define

$$\begin{aligned} W_\nu^{2,2} &= \left\{ \omega \in W^{2,2}(\Omega) : \left.\frac{\partial \omega}{\partial \nu}\right|_{\partial\Omega} = 0 \right\}, \\ L_0^2 &= \left\{ \omega \in L^2(\Omega) : \int_\Omega \omega dx = 0 \right\}. \end{aligned}$$

So,

$$\begin{aligned} &F(\sigma, w, v, \xi) \\ &= (f_1, f_2, f_3)(\sigma, w, v, \xi) : \mathbf{R}_+^1 \times \left(L_0^2 \cap W_\nu^{2,2}\right) \times W_\nu^{2,2} \times \mathbf{R}_+^1 \to L_0^2 \times L^2 \times \mathbf{R}^1, \end{aligned}$$

and (66) is equivalent to solving $F(\sigma, w, v, \xi) = 0$. Moreover, similar to the proof of Lemma 4.9, (66) has a unique solution $(w, v, \xi) = (0, \alpha, \alpha)$ when $\sigma = 0$. By simple computations, we have

$$\begin{aligned} \Phi(y, z, \tau) :&= D_{(w,v,\xi)} F(0, 0, \alpha, \alpha)(y, z, \tau) \\ &= \begin{pmatrix} \Delta y \\ \Delta z + \lambda(y - z + \tau) \\ \int_\Omega \left[\left(1 + (2a - c)\alpha - 3b\alpha^2\right)(y + \tau) - c\alpha z\right] dx \end{pmatrix}, \end{aligned}$$

then

$$\Phi : \left(L_0^2 \cap W_\nu^{2,2}\right) \times W_\nu^{2,2} \times \mathbf{R}_+^1 \to L_0^2 \times L^2 \times \mathbf{R}^1.$$

In order to use Implicit Function Theorem, we need to prove Φ is bijective. Obviously, Φ is surjective. So we only need to prove the homogeneous equation $\Phi(y, z, \tau) = 0$ admits unique solution $y = z = \tau = 0$.

Firstly, it follows from $\Phi(y, z, \tau) = 0$ that y satisfies

$$\begin{cases} \Delta y = 0, & x \in \Omega, \\ \dfrac{\partial y}{\partial \nu} = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega y dx = 0. \end{cases}$$

Then $y \equiv 0$.

Secondly, it follows from $\Phi(y, z, \tau) = 0$ and $y \equiv 0$ that z satisfies

$$\begin{cases} -\Delta z + \lambda z = \lambda\tau, & x \in \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Since $\lambda > 0$, the operator $-\Delta + \lambda$ is invertible, which together with τ is a constant, imply $z = \lambda(-\Delta + \lambda)^{-1}\tau$ is a constant.

Finally, it follows $\Phi(y, z, \tau) = 0$, $y \equiv 0$ and y, τ are both constant that

$$z = \tau \text{ and } \left(1 + (2a - c)\alpha - 3b\alpha^2\right)\tau - c\alpha z = 0. \tag{67}$$

Recall (α, α) is the positive constant solution of (4), we know α satisfies $1 + (a - c)\alpha = b\alpha^2$. Then we get

$$((c - a)\alpha - 2)z = 0. \tag{68}$$

Since $(c - a)\alpha - 2 = (c - a)\alpha - 1 - 1 = -b\alpha^2 - 1 < 0$, we get from (67) and (68) that $z = \tau = 0$.

By the Implicit function Theorem, there exists positive constants σ_0 and ϵ_0 such that for each $\sigma \in (0, \sigma_0)$, $(0, \alpha, \alpha)$ is the unique solution of $F(\sigma, w, v, \xi) = 0$ in $B_{\epsilon_0}(0, \alpha, \alpha)$, where $B_{\epsilon_0}(0, \alpha, \alpha)$ is the ball in $\left(L_0^2 \cap W_\nu^{2,2}\right) \times W_\nu^{2,2} \times \mathbf{R}^1$ centered at $(0, \alpha, \alpha)$ with radius ϵ_0. Taking smaller σ_0 and ϵ_0 smaller if necessary, we can conclude the proof by using Lemma 4.9. □

4.3. Existence of positive nonconstant steady state solutions

In this part, we analyze model (4) by bifurcation theory with λ as the bifurcation parameter.

Theorem 4.11. *Assume* (10) *or* (11) *holds. Let Ω be a bounded smooth domain so that all eigenvalues μ_i, $i \in \mathbf{N}_0$, are simple, $\phi_i(x)$ is the corresponding eigenfunction. Moreover,*

(SP) *There exist $p, q \in \mathbf{N}$ such that $\mu_{p-1} \le \mu_H < \mu_p \le \mu_q < \mu_3^* \le \mu_{q+1}$, where μ_3^* and μ_H are given in* (20) *and* (21) *respectively.*

Then for any $i \in \lceil p, q \rceil$, *which is defined as*

$$\lceil p,q \rceil = \begin{cases} [p,q] \cap \mathbf{N}, & \text{if } p < q; \\ \{p\}, & \text{if } p = q, \end{cases} \tag{69}$$
$$\lambda_{i,S} = \lambda_S(\mu_i) \text{ for } i \in \lceil p,q \rceil,$$

there exists a unique $\lambda_{i,S}$ *such that* $D_i(\lambda_{i,S}) = 0$ *and* $T_i(\lambda_{i,S}) \neq 0$. *If in addition, we assume that*

$$\lambda_{i,S} \neq \lambda_{j,S}, \ \lambda_{i,S} \neq \lambda_{j,H} \text{ for any } j \in \lceil p,q \rceil \text{ and } i \neq j, \tag{70}$$

where $\lambda_{j,H}$ *is defined as* (41), *then the following conclusions hold.*

(1) *There is a smooth curve* Γ_i *of positive solutions of* (4) *bifurcating from* $(\lambda, u, v) = (\lambda_{i,S}, \alpha, \alpha)$. *Near* $(\lambda, u, v) = (\lambda_{i,S}, \alpha, \alpha)$, *we have* $\Gamma_i = \{(\lambda_i(s)), u_i(s), v_i(s) : |s| < \epsilon\}$, *where*

$$\begin{cases} u_i(s) = \alpha + s l_i \phi_i(x) + s\psi_{1,i}(s), \\ v_i(s) = \alpha + s m_i \phi_i(x) + s\psi_{2,i}(s), \end{cases}$$

for some smooth functions λ_i, $\psi_{1,i}$ *and* $\psi_{2,i}$ *such that* $\lambda_i(0) = \lambda_{i,S}$ *and* $\psi_{1,i}(0) = \psi_{2,i}(0) = 0$. *Here* l_i, m_i *satisfies*

$$L(\lambda_{i,S}) \left[(l_i, m_i)^T \phi_i(x) \right] = (0,0)^T,$$

where ϵ *is a small positive constant,* α *is the positive constant given in* (5), L *is the operator defined as* (7).

(2) *In addition,* Γ_i *contained in a global branch* Σ_i *of positive nontrivial solutions of* (4), *and either* $\overline{\Sigma}_i$ *contains another* $(\lambda_{j,S}, \alpha, \alpha)$ *or the projection of* $\overline{\Sigma}_i$ *onto* λ*-axis contains the interval* $(0, \lambda_{i,S})$, *or the projection of* $\overline{\Sigma}_i$ *onto* λ*-axis contains the interval* $(\lambda_{i,S}, \infty)$.

Proof. We identify state bifurcation value λ_S of (4), which satisfies the following conditions [50].

(SS) There exists $i \in \mathbf{N}_0$ such that

$$D_i(\lambda_S) = 0, \ D_i'(\lambda_S) \neq 0, \ T_i(\lambda_S) \neq 0, D_j(\lambda_S) \neq 0 \text{ and } T_j(\lambda_S) \neq 0$$

for $j \in \mathbf{N}_0 \setminus \{i\}$, where $D_i(\lambda)$ and $T_i(\lambda)$ are given in (9).

Since $D_0(\lambda) = \lambda A > 0$, where A is defined as (15), we only consider $i \in \mathbf{N}$. In the following, we determine λ-values satisfying (SS). We notice that $D_i(\lambda) = 0$ is equivalent to $\lambda = \lambda_S(\mu_i)$, where $\lambda_S(\mu)$ is defined as (17). Hence we make the following additional assumption on the spectral set $\{\mu_n\}_{n \in \mathbf{N}_0}$ according to Lemma 2.1.

In the following, for p, q satisfy (SP), the points $\lambda_{i,S}$ defined in (69) are potential steady state bifurcation points. In follows from Lemma 2.1 that for each $i \in \lceil p,q \rceil$, there exists only one point $\lambda = \lambda_{i,S}$ such that $D_i(\lambda_{i,S}) = 0$ and $T_i(\lambda_{i,S}) \neq 0$. On the other hand, it is possible that for some $\lambda \in (\lambda_S(\mu_H), \lambda_S^*)$ with λ_S^* defined as (26) such that

(SQ) $\lambda_{i,S} = \lambda_{j,S} = \tilde{\lambda}$ for some $i, j \in \lceil p,q \rceil$ and $i \neq j$, i.e., $D_i(\tilde{\lambda}) = D_j(\tilde{\lambda})$.

(SS) is not satisfied for i if (SQ) holds, and we shall not consider bifurcations at such a point. On the other hand, it is also possible that

(SR) $\lambda_{i,S} = \lambda_{j,H}$ for some $i, j \in \lceil p, q \rceil$ and $i \neq j$, where $\lambda_{j,H}$ is a Hopf bifurcation point defined as (41).

However, from an argument in [50], for $\mathbf{N} = 1$ and $\Omega = (\ell\pi)$, there are only countably many ℓ, such that (SQ) or (SP) occurs. One also can show that (SQ) or (SP) does not occur for generic domains in $\mathbf{R^N}$ (see [39]). Finally, since $D_i'(\lambda_{i,S}) = d\mu_i + A > 0$, we get $D_i'(\lambda_{i,S}) \neq 0$.

The condition (SS) has been proved in the previous paragraphs, and the bifurcation solutions to (4) occur at $\lambda = \lambda_{i,S}$. Note that we assume (70) holds, so $\lambda = \lambda_{i,S}$ is always a bifurcation from simple eigenvalue point. Then the first conclusions follows from [50]. By Theorem 4.5, we know that the positive solution $(u, v) \in \Gamma_i$ has positive upper and lower bounds independent of λ if (49) holds. From the global bifurcation in [37], Γ_i is contained in a global branch Σ_i of positive solutions, and either $\overline{\Sigma}_i$ contains another $(\lambda_{j,S}, \alpha, \alpha)$ or $\overline{\Sigma}_i$ is not compact. Furthermore, if $\overline{\Sigma}_i$ is not compact, then $\overline{\Sigma}_i$ contains a boundary point $(\tilde{\lambda}, \tilde{u}, \tilde{v})$, and since $(\tilde{u}, \tilde{v})$ satisfies (50), it follows $\tilde{\lambda} = 0$ or $\tilde{\lambda} = \infty$ and the second conclusion follows. □

5. Numerical simulation

To visualize the cascade of asymptotically stability, Turing instability, Hopf bifurcation and steady state bifurcations described in Theorems 2.2, 2.3, 3.1 and 4.11, we consider serval numerical examples and assume the spatial dimension $N = 1$ and $\Omega = (0, 3\pi)$. Then $\mu_i = i^2/9$, $i \in \mathbf{N}_0$. In the following we use the notations in Lemma 2.1, Theorems 2.2, 2.3, 3.1 and 4.11.

Example 5.1. We choose $a = 5$, $b = 2$, $c = 4$, $r = 1$ such that (10) hold. Then $\alpha = 1$, $\lambda_0 = 1$, $A = 3$, $D_2^* = -1 + 2/\sqrt{3} \approx 0.1547$, $D_3^* = (2 - \sqrt{3})^2 \approx 0.0718$, $D_4^* = (2 + \sqrt{3})^2 \approx 13.9282$.

Firstly, We choose $d = 0.05$ to satisfy $d < D_2^*$ and $d < D_3^*$ such that $\lambda_S^* > \lambda_0$, then we can compute $\mu_L = 4$ and $\mu_R = 15$. So we have

$$\begin{aligned} \mu_6 = \mu_L < \mu_7 = \frac{49}{9} < \mu_8 = \frac{64}{9} < \mu_9 = \frac{81}{9} < \mu_{10} \\ = \frac{100}{9} < \mu_{11} = \frac{121}{9} < \mu_R < \mu_{12} = 16. \end{aligned}$$

Then

$$\overline{\lambda} = \max\{\lambda_S(\mu_7),\ \lambda_S(\mu_8),\ \lambda_S(\mu_9),\ \lambda_S(\mu_{10}),\ \lambda_S(\mu_{11})\} = \lambda_S(\mu_{10}) \approx 1.3889.$$

Furthermore, we can compute $\mu_H \approx 0.7314$ and $\mu_3^* = 20$, then by (69), μ_i, $i = 4, \ldots, 13$, are possible state bifurcation values. Firstly, if we choose $\lambda > \overline{\lambda}$, then it follows from Theorem 2.2 that $(\alpha, \alpha) = (1, 1)$ is asymptotically stable (see Fig. 3). Secondly, when λ decreases, the first bifurcation point encountered is $\lambda_S(\mu_{10}) \approx 1.3889$, so if we choose $\lambda \in (\lambda_0, \overline{\lambda})$, which implies

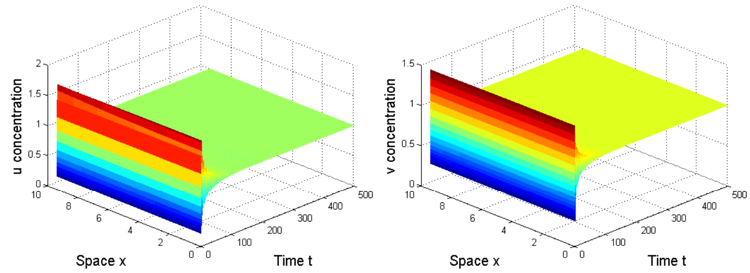

FIGURE 3. Numerical simulation of the system (3) with $a = 5$, $b = 2$, $c = 4$, $r = 1$ such that (10) hold. We choose $d = 0.05$, $\lambda = 1.5$ and the initial values $u_0(x) = v_0(x) = 1 + 0.5\cos(0.2/(3\pi)x)$, then the solution convergence to the unique positive equilibrium $(1,1)$.

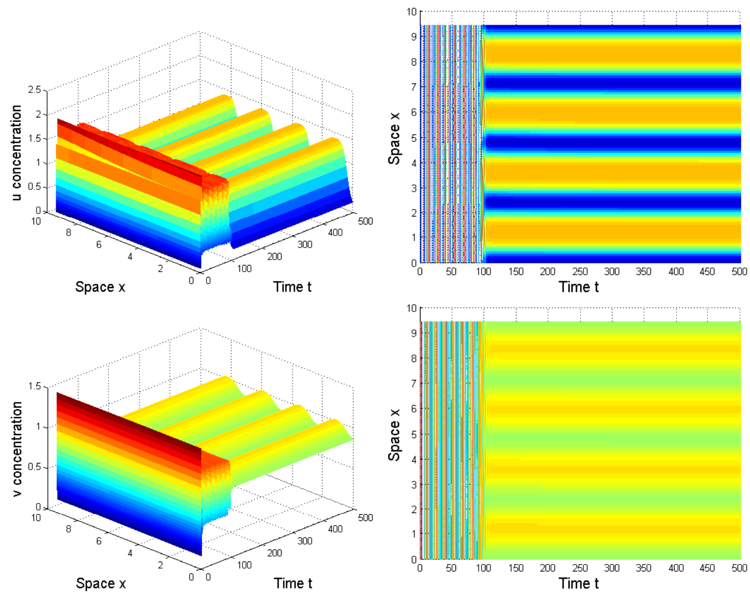

FIGURE 4. Numerical simulation of the system (3) with $a = 5$, $b = 2$, $c = 4$, $r = 1$ such that (10) hold. We choose $d = 0.05$, $\lambda = 1.1$ and the initial values $u_0(x) = v_0(x) = 1 + 0.5\cos(0.2/(3\pi)x)$, then the solution convergence to a spatially nonhomogeneous steady state solution. The upper two graphs are for u, the lower two graphs are for v.

Hopf bifurcation cannot happen since the largest Hopf bifurcation value is λ_0, then steady state bifurcation (Turing bifurcation) happens (see Fig. 4).

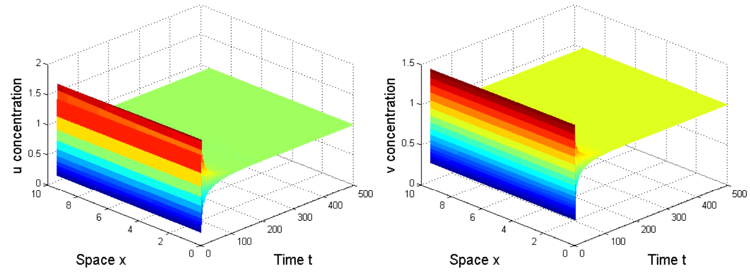

FIGURE 5. Numerical simulation of the system (3) with $a = 5$, $b = 2$, $c = 4$, $r = 1$ such that (10) hold. We choose $d = 0.2$, $\lambda = 1.1$ and the initial values $u_0(x) = v_0(x) = 1 + 0.5\cos(0.2/(3\pi)x)$, then the solution convergence to the unique positive equilibrium $(1, 1)$.

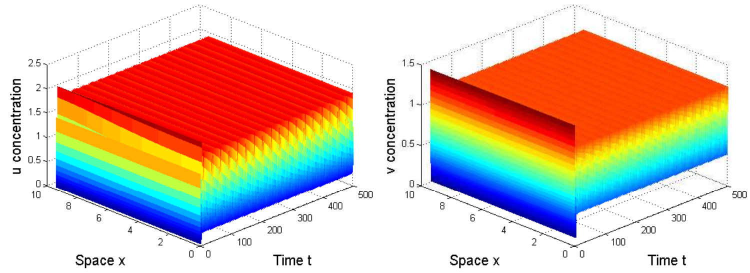

FIGURE 6. Numerical simulation of the system (3) with $a = 5$, $b = 2$, $c = 4$, $r = 1$ such that (10) hold. We choose $d = 0.2$, $\lambda = 0.9$ and the initial values $u_0(x) = v_0(x) = 1+0.5\cos(0.2/(3\pi)x)$, then the solution convergence to a spatially homogeneous periodic orbit.

Secondly, we choose $d = 0.2 > D_2^*$ such that $\lambda_S^* \approx 0.3590 < \lambda_0$, then $\overline{\lambda} = \lambda_0$. Since

$$\begin{aligned}\lambda_0 = \lambda_H(\mu_0) > \lambda_1 = \lambda_H(\mu_1) &\approx 0.8667 \\ > \lambda_H(\mu_2) \approx 0.4667 > \lambda_H(\mu_2^*) &= 0 > \lambda_H(\mu_3) = -0.2,\end{aligned}$$

then the possible Hopf bifurcation values are $\lambda_H(\mu_i)$, $i = 0, 1, 2$. Firstly, if we choose $\lambda > \overline{\lambda} = \lambda_0$, then it follows from Theorem 2.2 that $(\alpha, \alpha) = (1, 1)$ is asymptotically stable (see Fig. 5). Secondly, when λ decreases, the first bifurcation point encountered is $\lambda_0 = 1$, so if we choose $\lambda \in (\lambda_S^*, \lambda_0)$, which implies steady state cannot happen since the largest steady state bifurcation value is no larger than λ_S^*, then Hopf bifurcation happens (see Fig. 6).

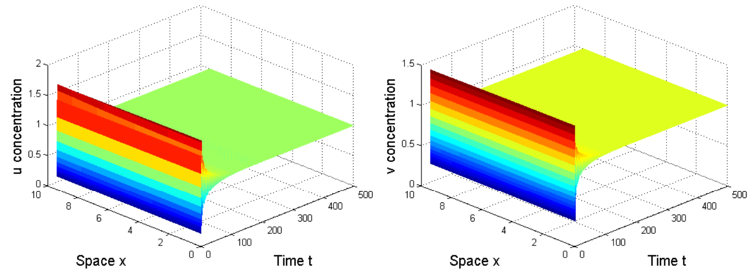

FIGURE 7. Numerical simulation of the system (3) with $a = 5$, $b = 2$, $c = 4$, $r = 1$ such that (10) hold. We choose $d = 0.1$, $\lambda = 1.1$ and the initial values $u_0(x) = v_0(x) = 1 + 0.5\cos(0.2/(3\pi)x)$, then the solution convergence to the unique positive equilibrium $(1, 1)$.

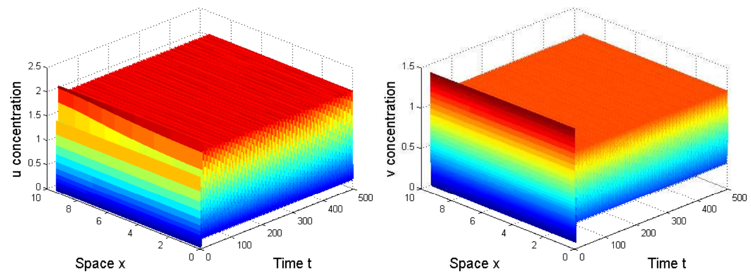

FIGURE 8. Numerical simulation of the system (3) with $a = 5$, $b = 2$, $c = 4$, $r = 1$ such that (10) hold. We choose $d = 0.1$, $\lambda = 0.9$ and the initial values $u_0(x) = v_0(x) = 1 + 0.5\cos(0.2/(3\pi)x)$, then the solution convergence to a spatially homogeneous periodic orbit.

Thirdly, we choose $d = 0.1 \in (D_3^*, D_2^*)$ such that $\lambda_S^* \approx 0.7180 < \lambda_0$, then $\overline{\lambda} = \lambda_0$. Since

$$\begin{aligned}\lambda_0 = \lambda_H(\mu_0) > \lambda_1 = \lambda_H(\mu_1) \approx 0.8778 \\ > \lambda_H(\mu_2) \approx 0.5111 > \lambda_H(\mu_2^*) = 0 > \lambda_H(\mu_3) = -0.1,\end{aligned}$$

then the possible Hopf bifurcation values are $\lambda_H(\mu_i)$, $i = 0, 1, 2$. Firstly, if we choose $\lambda > \overline{\lambda} = \lambda_0$, then it follows from Theorem 2.2 that $(\alpha, \alpha) = (1, 1)$ is asymptotically stable (see Fig. 7). Secondly, when λ decreases, the first bifurcation point encountered is $\lambda_0 = 1$, so if we choose $\lambda \in (\lambda_S^*, \lambda_0)$, which implies steady state cannot happen since the largest steady state bifurcation value is no larger than λ_S^*, then Hopf bifurcation happens (see Fig. 8).

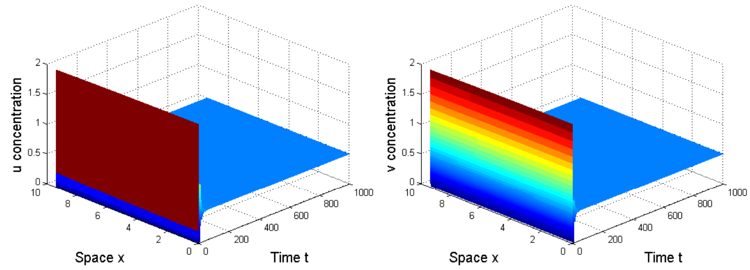

FIGURE 9. Numerical simulation of the system (3) with $a = 3$, $b = 2$, $c = 4$, $r = 1$ such that (11) hold. We choose $d = 0.05$, $\lambda = 0.5$ and the initial values $u_0(x) = v_0(x) = 0.5 + 1.5\cos(0.2/(3\pi)x)$, then the solution convergence to the unique positive equilibrium $(0.5, 0.5)$.

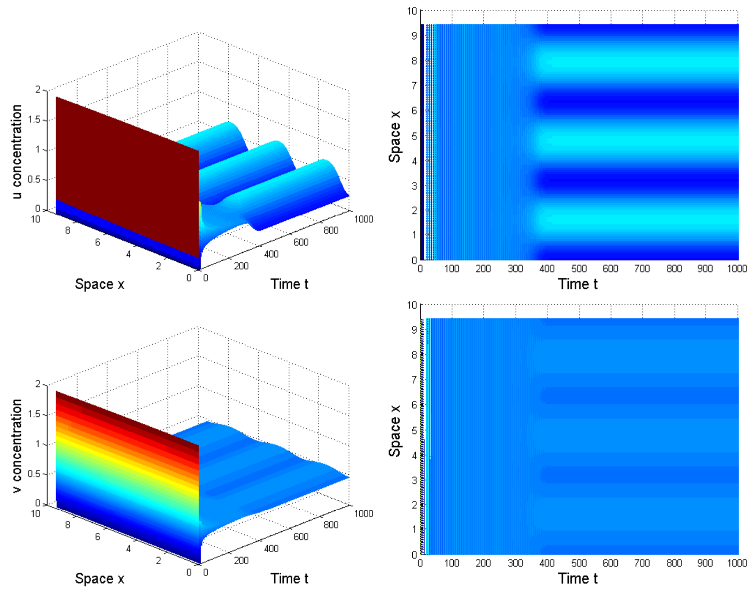

FIGURE 10. Numerical simulation of the system (3) with $a = 3$, $b = 2$, $c = 4$, $r = 1$ such that (11) hold. We choose $d = 0.05$, $\lambda = 0.6$ and the initial values $u_0(x) = v_0(x) = 0.5 + 1.5\cos(0.2/(3\pi)x)$, then the solution convergence to a spatially nonhomogeneous steady state solution. The upper two graphs are for u, the lower two graphs are for v.

Example 5.2. We choose $a = 3$, $b = 2$, $c = 4$, $r = 1$ such that (11) hold. Then $\alpha = 0.5$, $\lambda_0 = 0.5$, $A = 1.5$, $D_2^* = -1 + 2/\sqrt{3} \approx 0.1547$, $D_3^* = (2 - \sqrt{3})^2 \approx 0.0718$, $D_4^* = (2 + \sqrt{3})^2 \approx 13.9282$.

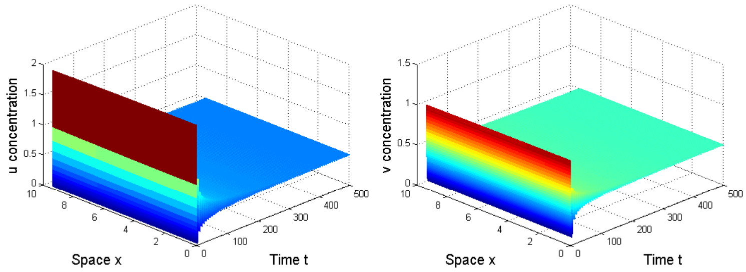

FIGURE 11. Numerical simulation of the system (3) with $a = 3$, $b = 2$, $c = 4$, $r = 1$ such that (11) hold. We choose $d = 0.2$, $\lambda = 0.6$ and the initial values $u_0(x) = v_0(x) = 0.5 + 1.5\cos(0.2/(3\pi)x)$, then the solution convergence to the unique positive equilibrium $(0.5, 0.5)$.

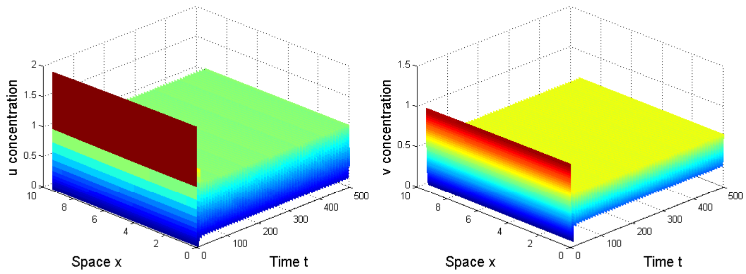

FIGURE 12. Numerical simulation of the system (3) with $a = 3$, $b = 2$, $c = 4$, $r = 1$ such that (11) hold. We choose $d = 0.2$, $\lambda = 0.4$ and the initial values $u_0(x) = v_0(x) = 0.5 + 1.5\cos(0.2/(3\pi)x)$, then the solution convergence to a spatially homogeneous periodic orbit.

Firstly, We choose $d = 0.05$ to satisfy $d < D_2^*$ and $d < D_3^*$ such that $\lambda_S^* > \lambda_0$, then we can compute $\mu_L = 2$ and $\mu_R = 7.5$. So we have

$$\mu_4 = \frac{16}{9} < \mu_L < \mu_5 = \frac{25}{9} < \mu_6 = \frac{36}{9} < \mu_7 = \frac{49}{9} < \mu_8 = \frac{64}{9} < \mu_R < \mu_9 = 9.$$

Then

$$\overline{\lambda} = \max\{\lambda_S(\mu_5),\ \lambda_S(\mu_6),\ \lambda_S(\mu_7),\ \lambda_S(\mu_8)\} = \lambda_S(\mu_6) \approx 0.7059.$$

Furthermore, we can compute $\mu_H \approx 0.3657$ and $\mu_3^* = 10$, then by (69), μ_i, $i = 2, \ldots, 9$, are possible state bifurcation values. Firstly, if we choose $\lambda > \overline{\lambda}$, then it follows from Theorem 2.2 that $(\alpha, \alpha) = (0.5, 0.5)$ is asymptotically stable (see Fig. 9). Secondly, when λ decreases, the first bifurcation point encountered is $\lambda_S(\mu_6) \approx 0.7059$, so if we choose $\lambda \in (\lambda_0, \overline{\lambda})$, which implies

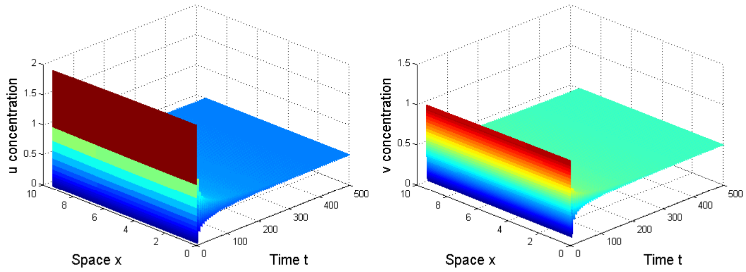

FIGURE 13. Numerical simulation of the system (3) with $a = 3$, $b = 2$, $c = 4$, $r = 1$ such that (11) hold. We choose $d = 0.1$, $\lambda = 0.6$ and the initial values $u_0(x) = v_0(x) = 0.5 + 1.5\cos(0.2/(3\pi)x)$, then the solution convergence to the unique positive equilibrium $(0.5, 0.5)$.

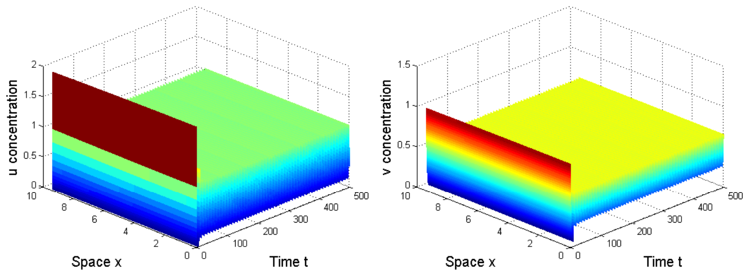

FIGURE 14. Numerical simulation of the system (3) with $a = 3$, $b = 2$, $c = 4$, $r = 1$ such that (11) hold. We choose $d = 0.1$, $\lambda = 0.4$ and the initial values $u_0(x) = v_0(x) = 0.5 + 1.5\cos(0.2/(3\pi)x)$, then the solution convergence to a spatially homogeneous periodic orbit.

Hopf bifurcation cannot happen since the largest Hopf bifurcation value is λ_0, then steady state bifurcation (Turing bifurcation) happens (see Fig. 10).

Secondly, we choose $d = 0.2 > D_2^*$ such that $\lambda_S^* \approx 0.1795 < \lambda_0$, then $\overline{\lambda} = \lambda_0$. Since

$$\lambda_0 = \lambda_H(\mu_0) > \lambda_1 = \lambda_H(\mu_1) \approx 0.3667 > \lambda_H(\mu_2^*) = 0 > \lambda_H(\mu_2) = -0.0333,$$

then the possible Hopf bifurcation values are $\lambda_H(\mu_i)$, $i = 0, 1$. Firstly, if we choose $\lambda > \overline{\lambda} = \lambda_0$, then it follows from Theorem 2.2 that $(\alpha, \alpha) = (0.5, 0.5)$ is asymptotically stable (see Fig. 11). Secondly, when λ decreases, the first bifurcation point encountered is $\lambda_0 = 0.5$, so if we choose $\lambda \in (\lambda_S^*, \lambda_0)$, which implies steady state cannot happen since the largest steady state bifurcation value is no larger than λ_S^*, then Hopf bifurcation happens (see Fig. 12).

Thirdly, we choose $d = 0.1 \in (D_3^*, D_2^*)$ such that $\lambda_S^* \approx 0.3590 < \lambda_0$, then $\overline{\lambda} = \lambda_0$. Since

$$\begin{aligned}\lambda_0 = \lambda_H(\mu_0) > \lambda_1 = \lambda_H(\mu_1) \approx 0.3778 \\ > \lambda_H(\mu_2) \approx 0.0111 > \lambda_H(\mu_2^*) = 0 > \lambda_H(\mu_3) = -0.6,\end{aligned}$$

then the possible Hopf bifurcation values are $\lambda_H(\mu_i)$, $i = 0, 1, 2$. Firstly, if we choose $\lambda > \overline{\lambda} = \lambda_0$, then it follows from Theorem 2.2 that $(\alpha, \alpha) = (0.5, 0.5)$ is asymptotically stable (see Fig. 13). Secondly, when λ decreases, the first bifurcation point encountered is $\lambda_0 = 0.5$, so if we choose $\lambda \in (\lambda_S^*, \lambda_0)$, which implies steady state cannot happen since the largest steady state bifurcation value is no larger than λ_S^*, then Hopf bifurcation happens (see Fig. 14).

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