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GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR IN A THREE-DIMENSIONAL TWO-SPECIES CHEMOTAXIS-STOKES SYSTEM WITH TENSOR-VALUED SENSITIVITY

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Abstract. In this paper, we deal with a two-species chemotaxis-Stokes system with Lotka-Volterra competitive kinetics under homogeneous Neumann boundary conditions in a general three-dimensional bounded domain with smooth boundary. Under appropriate regularity assumptions on the initial data, by some L^p-estimate techniques, we show that the system possesses at least one global and bounded weak solution, in addition to discussing the asymptotic behavior of the solutions. Our results generalizes and improves partial previously known ones.

1. Introduction

This paper is concerned with the following two-species chemotaxis-Stokes system with Lotka-Volterra competitive kinetics:

(1.1)

$$\begin{cases} (n_1)_t + u\cdot\nabla n_1 = \nabla\cdot(D_1(n_1)\nabla n_1) - \nabla\cdot(n_1 S_1(x,n_1,v)\cdot\nabla v) \\ \qquad\qquad\qquad +\mu_1 n_1(1-n_1-a_1 n_2)\ x\in\Omega,\ t>0, \\ (n_2)_t + u\cdot\nabla n_2 = \nabla\cdot(D_2(n_2)\nabla n_2) - \nabla\cdot(n_2 S_2(x,n_2,v)\cdot\nabla v) \\ \qquad\qquad\qquad +\mu_2 n_2(1-a_2 n_1-n_2)\ x\in\Omega,\ t>0, \\ v_t + u\cdot\nabla v = \triangle v - v + \alpha n_1 + \beta n_2, x\in\Omega,\ t>0, \\ u_t + \nabla P = \triangle u + (n_1+n_2)\nabla\phi,\ \nabla\cdot u = 0,\ x\in\Omega,\ t>0, \\ (D_i(n_i)\nabla n_i - n_i S_i(x,n_i,v)\cdot\nabla v)\cdot\nu = \frac{\partial v}{\partial\nu} = 0,\ u=0,\ x\in\partial\Omega, t>0, \\ n_1(x,0)=n_{10}(x),\ n_2(x,0)=n_{20}(x),\ v(x,0)=v_0(x),\ u(x,0)=u_0(x),\ x\in\Omega, \end{cases}$$

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where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. The system (1.1) is proposed to describe the exercise of two species which impact on a single chemoattractant in fluid, where n_1 and n_2 represent densities of species, v stands for the chemical concentration, u shows the fluid velocity field and P denotes the pressure of the fluid. $\mu_1, \mu_2, \alpha, \beta > 0$ are constants, $n_{10}, n_{20}, v_0, u_0, \phi$ are known functions satisfying

$$\begin{cases} n_{10}, n_{20} \in C^{\vartheta}(\overline{\Omega}) \text{ for certain } \vartheta > 0 \text{ with } n_{10}, n_{20} \geq 0 \text{ in } \Omega, \\ v_0 \in W^{1,\infty}(\Omega) \text{ satisfies } v_0 \geq 0 \ in\ \Omega, \\ u_0 \in D(\mathcal{A}_r^{\epsilon}) \text{ for some } \epsilon \in (\frac{3}{4}, 1) \text{ and any } r \in (1, \infty) \end{cases} \tag{1.2}$$

with $\mathcal{A}_r$ representing the Stokes operator with domain $D(\mathcal{A}_r^{\epsilon}) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L_\sigma^r(\Omega)$, where $L_\sigma^r(\Omega) := \{\varphi \in L^r(\Omega) | \nabla \cdot \varphi = 0\}$ for $r \in (1, \infty)$.

As for the diffusion coefficient in (1.1), we suppose that D satisfies

$$D_1, D_2 \in C_{loc}^{\theta}([0, \infty)) \text{ for some } \theta > 0, \tag{1.3}$$

as well as

$$D_1(n_1) \geq C_{D_1} n_1^{m_1 - 1} \text{ for all } n_1 > 0 \text{ with } m_1 > 1 \text{ and } C_{D_1} > 0 \tag{1.4}$$

and

$$D_2(n_2) \geq C_{D_2} n_2^{m_2 - 1} \text{ for all } n_2 > 0 \text{ with } m_2 > 1 \text{ and } C_{D_2} > 0. \tag{1.5}$$

Under the assumptions of $D_i(n_i)$, the first two equations of system (1.1) may be degenerate at $n_i = 0$, $i = 1, 2$.

Except for this, we assume that the tensor-valued sensitivity S_1, S_2 satisfies

$$S_1, S_2 \in C^2(\overline{\Omega} \times [0, \infty)^2; \mathbb{R}^{3\times 3}) \tag{1.6}$$

as well as

$$|S_1(x, n_1, v)| \leq C_{S_1}(1 + n_1)^{-\alpha_1}, \ |S_2(x, n_2, v)| \leq C_{S_2}(1 + n_2)^{-\alpha_2} \tag{1.7}$$

for all $(x, n_i, v) \in \Omega \times [0, \infty)^2$, $i = 1, 2$, with some $C_{S_i} > 0$ and $\alpha_i > 0$, $i = 1, 2$, $\alpha_i > 0$ means that the magnitude of the chemotactic flux is weakened when the bacterial density increases. The function ϕ is known and fulfills

$$\phi \in W^{1,\infty}(\Omega). \tag{1.8}$$

Processes of directed movement of cells in response to a chemical stimulus, referred to as chemotaxis, play an important role in the interaction of cells with their environment. A typical model describing chemotaxis is the classical Keller-Segel system derived by Keller and Segel in 1970s [13]. The mathematical analysis of classical Keller-Segel system and the variant thereof mainly concentrates on the boundedness and blow-up of the solutions [8,11,25]. As the blow-up has not been observed in the real biological process, many mechanisms, such as nonlinear porous medium diffusion, saturation effect, logistic source, etc., are introduced to avoid the blow-up of solutions [12,15,29]. In the past few decades, Keller-Segel system has attracted extensive attentions. For a helpful overview of many models arising out of this fundamental description we refer to the survey [3]. To better understand the model (1.1), it is necessary to separate study two-species chemotaxis system and chemotaxis-Stokes system with single specie. For two-species chemotaxis model, this has been extensively studied by many authors. When the two-species have influence in each other, namely, the system has Lotka-Volterra competitive kinetics. In the two-dimensional case, Bai and Winkler [1] obtained global existence of solution to the system (1.1) with $D_i(n_i) = 1$, $S_i(x, n_i, v) = \chi_i > 0$, $i = 1, 2$. Moreover, they also taken into account asymptotic behavior of solutions to the system (1.1), when $a_1, a_2 \in (0, 1)$, $n_1(\cdot, t) \to \frac{1-a_1}{1-a_1a_2}$, $n_2(\cdot, t) \to \frac{1-a_2}{1-a_1a_2}$, $v(\cdot, t) \to \frac{\alpha(1-a_1)+\beta(1-a_2)}{1-a_1a_2}$ in $L^\infty(\Omega)$ as $t \to \infty$; when $a_1 \geq 1 > a_2 > 0$, $n_1(\cdot, t) \to 0$, $n_2(\cdot, t) \to 1$, $v(\cdot, t) \to \beta$ in $L^\infty(\Omega)$ as $t \to \infty$, Mizukami [22] improved this result. Recently, Mizukami [23] further modified the result in [22], this is filling up the gap between [1] and [22]. In the three-dimensional case, Lin and Mu [16] supposed that μ_1 and μ_2 are large enough to obtained a similar result. In the high dimensional case, Lin et al. [17] and Zhang et al. [47] are obtained unique global classical bounded solution, respectively. However, the asymptotic behavior of solution is not involved. When the system has a logistic source, but the two-species have not influence in each other, in other wards, the competitive kinetics term $\mu_1 n_1(1 - n_1 - a_1 n_2)$ and $\mu_2 n_2(1 - a_2 n_1 - n_2)$ are replaced by $\mu_1 n_1(1 - n_1)$ and $\mu_2 n_2(1 - n_2)$, Negreanu et al. [27] and [26] separate discussed the system has unique uniformly bounded solution with $v_t = \varepsilon \triangle v + h(n_1, n_2, v)$, $\varepsilon \in [0, 1)$, Mizukami et al. [24] remove the restriction of $\varepsilon \in [0, 1)$ to obtained a similar result. When the third equation degenerate into elliptic equation, Stinner et al. [32] and Lin et al. [18] are obtained global existence of solution to the system (1.1) with $D_i(n_i) = 1$, $S_i(x, n_i, v) = \chi_i > 0$, $i = 1, 2$. Moreover, they also taken into account asymptotic behavior of solutions to the system (1.1).

For chemotaxis-Stokes system with single specie, this model which was proposed in [36] for the spatio-temporal evolution in populations of oxytactically moving bacteria that interact with a surrounding fluid through transport and buoyancy with the third equation of (1.1) is replaced by $v_t + u \cdot \nabla v = \triangle v - nf(v)$, the corresponding model is given by:

$$
\begin{cases}
n_t + u \cdot \nabla n = \triangle n - \nabla \cdot (n\chi(n, v) \cdot \nabla v) \; x \in \Omega, \; t > 0, \\
v_t + u \cdot \nabla v = \triangle v - nf(v), \; x \in \Omega, \; t > 0, \\
u_t + \kappa(u \cdot \nabla)u = \triangle u + \nabla P + n\nabla\phi, \; x \in \Omega, \; t > 0, \\
\nabla \cdot u = 0 \; x \in \Omega, \; t > 0.
\end{cases}
\tag{1.9}
$$

When $\triangle n$ is replaced by $\triangle n^m$, Winkler [43] recent analysis has revealed that $m > \frac{7}{6}$ for all reasonably regular initial data, a corresponding no-flux Neumann initial-boundary value problem possesses a globally defined weak solution which is bounded in three-dimensional bounded convex domains. Tao et al. [33] which assured global solvability within the large range $m > \frac{8}{7}$, but only in a class of weak solutions locally bounded in $\overline{\Omega} \times [0, \infty)$. Winkler [46] which allows for the construction of global weak solution to an associated initial-boundary value problem under the milder assumption that $m > \frac{9}{8}$. Moreover, the obtained solutions are shown to approach the spatially homogeneous steady state $(\frac{1}{|\Omega|}\int_\Omega n_0, 0, 0)$ in the large time limit. As in the classical Keller-Segel model where the chemoattractant is produced rather than consumed by bacteria, the relevant Keller-Segel-fluid system of the form

$$\begin{cases} n_t + u\cdot\nabla n = \triangle n - \nabla\cdot(nS(x,n,v)\cdot\nabla v) \ x\in\Omega,\ t>0, \\ v_t + u\cdot\nabla v = \triangle v - v + n,\ x\in\Omega,\ t>0, \\ u_t + \kappa(u\cdot\nabla)u = \triangle u + \nabla P + n\nabla\phi,\ x\in\Omega,\ t>0, \\ \nabla\cdot u = 0 \ x\in\Omega,\ t>0. \end{cases} \tag{1.10}$$

Compare with (1.9), the mathematical analysis of (1.10) is quite few. When $S(x, n, v)$ is a tensor-valued sensitivity satisfying some dampening condition in (1.7), Wang et al. [39] obtained global existence and boundedness in a Keller-Segel-Stokes system ($\kappa = 0$) in two-dimensional smoothly bounded domains, to the best of our knowledge, this is the first result on global existence and boundedness in a Keller-Segel-Stokes system with tensor-valued sensitivity. With the same author [40], when $\alpha > \frac{1}{2}$, they also obtained global classical solutions which is uniformly bounded in three-dimensional smoothly bounded domains. Parallel to the case of the corresponding Keller-Segel-Navier-Stokes system, Wang [37] proved the system (1.10) possesses at least one global very weak solution with $\alpha > \frac{1}{3}$ in three-dimensional smoothly bounded domains. Winkler [45] shown that if $\alpha > \frac{1}{3}$, the problem (1.10) with $\kappa = 0$ possesses a global classical solution. When the system (1.10) has a logistic source $rn - \mu n^2$ and external force g in the fluid equation, Tao et al. [34] shown that under the explicit condition $\mu \geq 23$ and suitable regularity assumptions on the initial data, the corresponding initial-boundary problem possesses a global classical solution which is bounded in three-dimensional smoothly bounded domains. Apart from this, it is also proved that if $r = 0$, then both $n(\cdot, t)$ and $v(\cdot, t)$ decay to zero with respect to the norm in L^∞ as $t \to \infty$, and that if moreover $\int_0^\infty \int_\Omega |g|^2 < 0$, then also $u(\cdot, t) \to 0$ in L^∞ as $t \to \infty$. In two-dimensional smoothly bounded domains, Tao et al. [35] obtained the Keller-Segel-Navier-Stokes possesses a global classical bounded solution when $\mu > 0$, and also get the same large time behavior. Jiu et al. [19] shown that under the conditions $m \geq \frac{1}{3}$ and $\alpha > \frac{6}{5} - m$, and proper regularity hypotheses on the initial data, the corresponding initial-boundary problem possesses at least one global bounded weak solution for the Keller-Segel-Stokes system with nonlinear diffusion and logistic source in the three-dimensional bounded domains. When $\triangle n$ is replaced by $\triangle n^m$, $S(x, n, v) \equiv 1$, Black [2] proved that if $m > \frac{5}{3}$, the problem (1.10) admits at least one global weak solution, in addition, if $m > \frac{4}{3}$, the problem (1.10) admits at least one global very weak solution. For the latest progress of other chemotaxis-fluid system, please refer to [4, 16, 21, 38, 44].

For two-species chemotaxis-Stokes system with Lotka-Volterra competitive kinetics, the mathematical analysis of (1.1) is quite fragmentary. When $D_i(n_i) \equiv 1$, $S_i(x, n_i, v) = \chi_i$, $i = 1, 2$, the third equation turned into $v_t + u \cdot \nabla v = \triangle v - (\alpha n_1 + \beta n_2)v$, the fourth equation turn into Navier-Stokes equation, Hirata et al. [9] obtained global classical and bounded solution which is unique in two-dimensional smoothly bounded domains. Moreover, they also taken into account asymptotic behavior of solutions to the system (1.1), when $a_1, a_2 \in (0, 1)$, $n_1(\cdot, t) \to \frac{1-a_1}{1-a_1a_2}$, $n_2(\cdot, t) \to \frac{1-a_2}{1-a_1a_2}$, $v(\cdot, t) \to 0$, $u(\cdot, t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$; when $a_1 \geq 1 > a_2 > 0$, $n_1(\cdot, t) \to 0$, $n_2(\cdot, t) \to 1$, $v(\cdot, t) \to 0$, $u(\cdot, t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$. To the best of our knowledge, this is the first result on global existence and boundedness for two-species chemotaxis-Navier-Stokes system with competitive kinetics. The same authors [10] shown that global existence of weak solutions in three-dimensional smoothly bounded domains, the system is the same as in [9]. When $D_i(n_i) \equiv 1$, $S_i(x, n_i, v) = \chi_i$, $i = 1, 2$, the third equation turned into $v_t + u \cdot \nabla v = \triangle v - (\alpha n_1 + \beta n_2)v$, the fourth equation is the same as in (1.1), Cao et al. [5] obtained the system possesses a classical solution which is unique in the sense that it allows up to addition of spatially constants to the pressure P with $\frac{\chi}{\mu} < \xi_0$, where $\chi := \max\{\chi_1, \chi_2\}$, $\mu := \min\{\mu_1, \mu_2\}$, $\xi_0 > 0$ is a constant in three-dimensional smoothly bounded domains. Moreover, they obtained the similar results for asymptotic behavior of solutions in [9]. When $D_i(n_i) \equiv 1$, $S_i(x, n_i, v) = \chi_i$, $i = 1, 2$, the third equation and fourth equation is the same as in (1.1), Cao et al. [6] proved a similar result with the different conditions.

Throughout above analysis, compared with two-species chemotaxis system and chemotaxis-Stokes system with single specie, it is not so mature that the two-species chemotaxis-Stokes system with Lotka-Volterra competitive kinetics. Inspired by the arguments in previous studies [10, 20, 45], we mainly investigate the global existence and boundedness in a three-dimensional two-species chemotaxis-Stokes system with tensor-valued sensitivity to the system (1.1), in addition to discuss the asymptotic behavior of the solutions. More precisely, we have:

Theorem 3.1. *Let $a_1, a_2 \geq 0$, $\alpha, \beta > 0$, $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Suppose that $D(n_i)$ and S_i, $i = 1, 2$, satisfy* (1.3)-(1.7) *with $m_i \geq \frac{1}{3}$, $i = 1, 2$ and*

$$m_1 + \alpha_1 > \frac{23}{18}, \; m_2 + \alpha_2 > \frac{23}{18}.$$

Then for any choice of the initial data $n_{01}, n_{02}, v_0, u_0, \phi$ fulfill (1.2) *and* (1.8), *system* (1.1) *possesses at least one non-negative global weak solution (n_1, n_2, v, u, P) in the sense of Definition 2.1. Also, this solution is bounded in $\Omega \times (0, \infty)$ in the sense that*

$$\|n_1(\cdot, t)\|_{L^\infty(\Omega)} + \|n_2(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C$$

for all $t > 0$ *with some constant* $C > 0$. *In addition,* v *and* u *are continuous in* $\overline{\Omega} \times (0, \infty)$, *and* n_1, n_2 *as an* $L^\infty(\Omega)$*-valued function is continuous on* $[0, \infty)$ *with respect to the weak-$*$ topology, i.e.,*

$$n_1, n_2 \in C^0_{w-*}([0, \infty); L^\infty(\Omega)).$$

Theorem 4.1. *Let* $D_1(n_1) = D_2(n_2) \equiv 1$, *assume that the condition of Theorem 3.1 holds. Then the solution of* (1.1) *has the following properties*:

(i) *Let* $a_1, a_2 \in (0, 1)$, *under the condition that there exists* γ_1 *such that*

$$4\gamma_1 - (1 + \gamma_1)^2 a_1 a_2 > 0$$

and

$$\frac{c_1 C_{S_1}^2 (1 - a_1)}{4a_1\mu_1(1 - a_1a_2)} + \frac{\gamma_1 c_2 C_{S_2}^2 (1 - a_2)}{4a_2\mu_2(1 - a_1a_2)} < \frac{4\gamma_1 - (1 + \gamma_1)^2 a_1a_2}{a_1\alpha^2\gamma_1 + a_2\beta^2 - a_1a_2\alpha\beta(1 + \gamma_1)},$$

then

$$n_1(\cdot, t) \to N_1,\ n_2(\cdot, t) \to N_2,\ v(\cdot, t) \to V_1,\ u(\cdot, t) \to 0 \text{ in } L^\infty \text{ as } t \to \infty,$$

where

$$N_1 := \frac{1 - a_1}{1 - a_1a_2},\ N_2 := \frac{1 - a_2}{1 - a_1a_2},\ V_1 := \alpha N_1 + \beta N_2$$

as well as

$$c_1 = \max\{1, (1 + \|n_1\|_{L^\infty[0, \|n_{10}\|_{L^\infty(\Omega)}+1]})^{1-\alpha_1}\}$$

and

$$c_2 = \max\{1, (1 + \|n_2\|_{L^\infty[0, \|n_{20}\|_{L^\infty(\Omega)}+1]})^{1-\alpha_2}\}.$$

(ii) *Let* $a_1 \geq 1 > a_2$. *under the condition that there exist* γ_3 *and* $a_1' \in [1, a_1]$ *such that*

$$4\gamma_3 - (1 + \gamma_3)^2 a_1' a_2 > 0$$

and

$$\mu_2 > \frac{C_{S_2}^2 \gamma_3 c_2(\alpha^2 a_1'\gamma_3 + \beta^2 a_2 - \alpha\beta a_1' a_2(1 + \gamma_3))}{4a_2(4\gamma_3 - a_1' a_2(1 + \gamma_3)^2)},$$

then

$$n_1(\cdot, t) \to 0,\ n_2(\cdot, t) \to 1,\ v(\cdot, t) \to \beta,\ u(\cdot, t) \to 0 \text{ in } L^\infty \text{ as } t \to \infty.$$

where

$$c_2 = \max\{1, (1 + \|n_2\|_{L^\infty[0, \|n_{20}\|_{L^\infty(\Omega)}+1]})^{1-\alpha_2}\}.$$

In this paper, we use symbols c and C as some generic positive constants. Sometimes, in order to distinguish them, we use symbols C_i and c_i $(i = 1, 2, \dots)$ which depend on $m, C_{D_1}, C_{D_2}, C_{S_1}, C_{S_2}, p, \Omega$ and the initial data only. Moreover, for simplicity, $u(x, t)$ is written as u, the integral $\int_\Omega u(x)dx$ is written as $\int_\Omega u(x)$.

The rest of this paper is organized as follows. In Section 2, we summarize some basic definitions and some useful lemmas in order to prove the main result. In Section 3, we show the main theorem, firstly, and give some fundamental estimates for the solution to the system (2.1) to prove Theorem 3.1. In Section

4, we start with stating the main theorem, and construct the function to prove Theorem 4.1.

2. Preliminaries

Under the assumptions of $D_i(n_i)$, the first two equations of system (1.1) may be degenerate at $n_i = 0$, $i = 1, 2$. Therefore, system (1.1) does not allow for classical solvability in general. We introduce the following definition of weak solution.

Definition 2.1. Let $T \in (0, \infty)$. A quadruple of nonnegative functions (n_1, n_2, v, u) defined in $\Omega \times (0, T)$ is called a weak solution of system (1.1) if the following conditions are satisfied

$$n_i \in L^1_{loc}(\overline{\Omega} \times [0, T)),\ v \in L^\infty_{loc}(\overline{\Omega} \times [0, T)) \cap L^1_{loc}([0, T); W^{1,1}(\Omega)),$$

$$u \in L^1_{loc}([0, T); W^{1,1}(\Omega)),\ G(n_i),\ n_i|\nabla v|,\ n_i|u| \in L^1_{loc}(\overline{\Omega} \times [0, T));$$

the integral equalities

$$\begin{aligned} &-\int_0^T\int_\Omega n_1\psi_t - \int_\Omega n_{10}\psi(\cdot, 0) \\ =\ &-\int_0^T\int_\Omega G_1(n_1) \triangle \psi + \int_0^T\int_\Omega n_1(S_1(x, n_1, v)\cdot\nabla v)\cdot\nabla\psi \\ &+\int_0^T\int_\Omega n_1 u\cdot\nabla\psi + \mu_1\int_0^T\int_\Omega n_1(1 - n_1 - a_1 n_2)\psi \end{aligned}$$

for any $\psi \in C_0^\infty(\overline{\Omega} \times [0, T))$ satisfying $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega \times (0, T)$, as well as

$$\begin{aligned} &-\int_0^T\int_\Omega n_2\psi_t - \int_\Omega n_{20}\psi(\cdot, 0) \\ =\ &-\int_0^T\int_\Omega G_2(n_2) \triangle \psi + \int_0^T\int_\Omega n_2(S_2(x, n_2, v)\cdot\nabla v)\cdot\nabla\psi \\ &+\int_0^T\int_\Omega n_2 u\cdot\nabla\psi + \mu_2\int_0^T\int_\Omega n_2(1 - a_2 n_1 - n_2)\psi \end{aligned}$$

for any $\psi \in C_0^\infty(\overline{\Omega} \times [0, T))$ satisfying $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega \times (0, T)$, and

$$\begin{aligned} &-\int_0^T\int_\Omega v\psi_t - \int_\Omega v_0\psi(\cdot, 0) \\ =\ &-\int_0^T\int_\Omega \nabla v\cdot\nabla\psi - \int_0^T\int_\Omega v\cdot\psi + \alpha\int_0^T\int_\Omega n_1\cdot\psi \\ &+\beta\int_0^T\int_\Omega n_2\cdot\psi + \int_0^T\int_\Omega uv\cdot\nabla\psi \end{aligned}$$

for any $\psi \in C_0^\infty(\overline{\Omega} \times [0,T))$, as well as

$$-\int_0^T \int_\Omega u\psi_t - \int_\Omega u_0\psi(\cdot,0) = -\int_0^T \int_\Omega \nabla u \cdot \nabla\psi + \int_0^T \int_\Omega (n_1+n_2)\nabla\phi\cdot\psi$$

for all $\psi \in C_0^\infty(\overline{\Omega} \times [0,T);\mathbb{R}^3)$ fulfilling $\nabla\psi \equiv 0$ in $\Omega\times(0,T)$, where we let

$$G_i(s) := \int_0^s D_i(\sigma)d\sigma \text{ for } s\geq 0,\ i=1,2.$$

If (n_1,n_2,v,u) is a weak solution of system (1.1) in $\Omega\times(0,T)$ for all $T\in(0,\infty)$, then we call (n_1,n_2,v,u) a global weak solution.

We will construct solutions of (1.1) as limits of solutions to relevant regularized approximated problems, and give some basic estimates for the solutions to the regularized system. First of all, we approximated the diffusion coefficient function in (1.1) by a family $(D_{i\varepsilon})_{\varepsilon\in(0,1)}$ of functions

$$D_{i\varepsilon} \in C^2([0,\infty)) \text{ such that } D_{i\varepsilon}(n_i) \geq \varepsilon \text{ for all } n_i > 0\ (i=1,2),$$

and

$$D_i(n_i) \leq D_{i\varepsilon}(n_i) \leq D_i(n_i) + 2\varepsilon \text{ for all } n_i>0\ (i=1,2).$$

Then, in order to obtain homogeneous Neumann boundary conditions for both n_i $(i=1,2)$ and v, we let $(\rho_\varepsilon)_{\varepsilon\in(0,1)} \subset C_0^\infty(\Omega)$ be a family of standard cut-off functions satisfying $0\leq\rho_\varepsilon\leq 1$ in Ω and $\rho_\varepsilon\to 1$ in Ω as $\varepsilon\to 0$, and define

$$S_{i\varepsilon}(x,n_i,v) = \rho_\varepsilon S_i(x,n_i,v),\ x\in\overline{\Omega},\ n_i\geq 0,\ (i=1,2),\ v\geq 0$$

for $\varepsilon\in(0,1)$ to approximate the sensitivity tensor S_i, which implies that $S_{i\varepsilon}(x,n_i,v)=0$ on $\partial\Omega$ for each fixed $\varepsilon\in(0,1)$, $i=1,2$. The initial data $n_{i0\varepsilon}\in C^\vartheta(\overline{\Omega})$ for some $\vartheta>0$ with $n_{i0\varepsilon}\geq 0$, $i=1,2$ in Ω, $v_{0\varepsilon}\in W^{1,\infty}(\overline{\Omega})$ fulfills $v_{0\varepsilon}\geq 0$ in Ω, and $u_{0\varepsilon}\in D(\mathcal{A}_r^\epsilon)$ for certain $\epsilon\in(\frac{3}{4},1)$ and

$$\begin{cases} \|n_{10\varepsilon}\|_{L^\infty(\Omega)} \leq \|n_{10}\|_{L^\infty(\Omega)}+1,\ \|n_{20\varepsilon}\|_{L^\infty(\Omega)} \leq \|n_{20}\|_{L^\infty(\Omega)}+1, \\ \|v_{0\varepsilon}\|_{W^{1,\infty}(\Omega)} \leq \|v_0\|_{W^{1,\infty}(\Omega)}+1,\ \|u_{0\varepsilon}\|_{W^{1,\infty}(\Omega)} \leq \|u_0\|_{W^{1,\infty}(\Omega)}+1. \end{cases}$$

Therefore, for any such ε, the regularized problems

(2.1)

$$\begin{cases} (n_{1\varepsilon})_t + u_\varepsilon\cdot\nabla n_{1\varepsilon} = \nabla\cdot(D_{1\varepsilon}(n_{1\varepsilon})\nabla n_{1\varepsilon}) - \nabla\cdot(n_{1\varepsilon}S_{1\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)\cdot\nabla v_\varepsilon) \\ \qquad\qquad +\mu_1 n_{1\varepsilon}(1-n_{1\varepsilon}-a_1n_{2\varepsilon}),\ x\in\Omega,\ t>0, \\ (n_{2\varepsilon})_t + u_\varepsilon\cdot\nabla n_{2\varepsilon} = \nabla\cdot(D_{2\varepsilon}(n_{2\varepsilon})\nabla n_{2\varepsilon}) - \nabla\cdot(n_{2\varepsilon}S_{2\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)\cdot\nabla v_\varepsilon) \\ \qquad\qquad +\mu_2 n_{2\varepsilon}(1-a_2n_{1\varepsilon}-n_{2\varepsilon}),\ x\in\Omega,\ t>0, \\ v_{\varepsilon t} + u_\varepsilon\cdot\nabla v_\varepsilon = \triangle v_\varepsilon - v_\varepsilon + \alpha n_{1\varepsilon} + \beta n_{2\varepsilon},\ x\in\Omega,\ t>0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \triangle u_\varepsilon + (n_{1\varepsilon}+n_{2\varepsilon})\nabla\phi,\ \nabla\cdot u_\varepsilon = 0,\ x\in\Omega,\ t>0, \\ \frac{\partial n_{1\varepsilon}}{\partial\nu} = \frac{\partial n_{2\varepsilon}}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0,\ u_\varepsilon = 0,\ x\in\partial\Omega,\ t>0, \\ n_{1\varepsilon}(x,0) = n_{10\varepsilon}(x),\ n_{2\varepsilon}(x,0) = n_{20\varepsilon}(x), \\ v_\varepsilon(x,0) = v_{0\varepsilon}(x),\ u_\varepsilon(x,0) = u_{0\varepsilon}(x),\ x\in\Omega. \end{cases}$$

According to the well-established fixed point arguments, the local solvability of (2.1) can be obtained, the proof is similar to that in [42], so here we omit the proof.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that* (1.2)-(1.8) *holds. Then for each $\varepsilon \in (0,1)$, $q > 3$, there exist $T_{\max} \in (0,\infty)$ and a classical solution $(n_{1\varepsilon}, n_{2\varepsilon}, v_\varepsilon, u_\varepsilon, P_\varepsilon)$ such that*

$$\begin{cases} n_{1\varepsilon}, n_{2\varepsilon} \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \\ v_\varepsilon \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})) \cap L^\infty_{loc}([0,T_{\max}); W^{1,q}(\Omega)), \\ u_\varepsilon \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})) \cap L^\infty_{loc}([0,T_{\max}); D(\mathcal{A}^\epsilon)), \\ P_\varepsilon \in C^{1,0}(\Omega) \times (0,T_{\max})), \end{cases}$$

where $T_{\max}$ denotes the maximal existence time. Also, the above solution is unique up to addition of spatially constants to the pressure P_ε. Moreover, we have $n_{i\varepsilon} > 0$ $i = 1,2$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times (0,T_{\max})$, and if $T_{\max} < +\infty$, then

(2.2)
$$\|n_{1\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|n_{2\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|\mathcal{A}^\epsilon_r u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty$$

as $t \to T_{\max}$, where ϵ, r are taken from (1.2).

Lemma 2.2. *For each $\varepsilon \in (0,1)$, there exists a constant $C := C(\tau) > 0$, the solutions of* (2.1) *satisfies*

$$\int_\Omega n_{i\varepsilon}(\cdot,t) \le C \text{ for all } t \in (0,T_{\max}) \ i = 1,2 \tag{2.3}$$

as well as

$$\int_t^{t+\tau}\int_\Omega n_{i\varepsilon}^2 \le C \text{ for all } t \in (0,T_{\max}-\tau) \ i = 1,2 \tag{2.4}$$

and

$$\int_\Omega v_\varepsilon(\cdot,t) \le \max\Big\{\int_\Omega v_0,\ (\alpha+\beta)C\Big\} \text{ for all } t \in (0,T_{\max}). \tag{2.5}$$

Proof. The proof is similar to [28, Lemma 2.2], so we omitted it. □

Lemma 2.3 ([37, Lemma 3.4])**.** *Let $T > 0$ and $y \in C^0([0,T)) \cap C^1(0,T)$ be such that*

$$y'(t) + ay(t) \le g(t) \text{ for all } t \in (0,T),$$

where $g \in L^1_{loc}(\mathbb{R})$ has the property that

$$\frac{1}{\tau}\int_t^{t+\tau} g(s)ds \le b \text{ for all } t \in (0,T)$$

with some $\tau > 0$ and $b > 0$. Then

$$y(t) \le y(0) + \frac{b\tau}{1-e^{-a\tau}} \text{ for all } t \in [0,T).$$

Lemma 2.4 (Gagliardo-Nirenberg interpolation inequality, [30, Lemma 2.2]). *Let $0 < \theta \le p \le \frac{2N}{N-2}$. There exists a positive constant C_{GN} such that for all $u_\varepsilon \in W^{1,2}(\Omega) \cap L^\theta(\Omega)$,*

$$\|u_\varepsilon\|_{L^p(\Omega)} \le C_{GN}(\|\nabla u_\varepsilon\|^a_{L^2(\Omega)}\|u_\varepsilon\|^{1-a}_{L^\theta(\Omega)} + \|u_\varepsilon\|_{L^\theta(\Omega)}) \tag{2.6}$$

is valid with $a = \frac{\frac{N}{\theta}-\frac{N}{p}}{1-\frac{N}{2}+\frac{N}{\theta}} \in (0,1)$.

Lemma 2.5 ([43, Corollary 3.4]). *Let $p \in [1,\infty)$ and $r \in [1,\infty]$ be such that*

$$\begin{cases} r < \frac{3p}{3-p} \text{ if } p \le 3, \\ r \le \infty \text{ if } p > 3. \end{cases} \tag{2.7}$$

Then for all $K > 0$ exists $C = C(p, r, K)$ such that if for some $\varepsilon \in (0,1)$ and $T_{\max} > 0$ we have

$$\|n_{i\varepsilon}(\cdot,t)\|_{L^p(\Omega)} \le K \text{ for all } t \in (0, T_{\max}), \tag{2.8}$$

then

$$\|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le C \text{ for all } t \in (0, T_{\max}). \tag{2.9}$$

Lemma 2.6 ([19, Lemma 3.3]). *Suppose that $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Let $q > 1$ and $\gamma \in [2, 6q]$. Then there exists $C = C(q,\gamma) > 0$ such that for all $\omega \in C^2(\overline{\Omega})$ the following interpolation inequality*

$$\|\nabla\omega\|^\gamma_{L^\gamma(\Omega)} \le C\Big(\|\nabla|\nabla\omega|^q\|^{\frac{3\gamma-6}{3q-1}}_{L^2(\Omega)}\|\nabla\omega\|^{\frac{6q-\gamma}{3q-1}}_{L^2(\Omega)} + \|\nabla\omega\|^\gamma_{L^2(\Omega)}\Big) \tag{2.10}$$

holds.

Lemma 2.7 ([41, Lemma 1.3]). *Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded domain with smooth boundary and let $(e^{t\triangle})_{t\ge0}$ be the Neumann heat semigroup in Ω. Then there exist constants $C, \lambda_1 > 0$ depending only on Ω such that if $1 \le q \le p \le \infty$, then*

$$\|\nabla e^{t\triangle}\psi\|_{L^p(\Omega)} \le C(1 + t^{-\frac12-\frac{N}{2}(\frac1q-\frac1p)})e^{-\lambda_1 t}\|\psi\|_{L^q(\Omega)}$$

holds for all $t > 0$ and each $\psi \in L^p(\Omega)$.

3. Existence of global and bounded weak solution

In this section, we start with expounding the main theorem, and then establish some a priori estimates for solutions to the approximated system (2.1) with non-degenerate diffusion, it is crucial ingredient for the proof of our main results. As a first step towards this, we show the main theorem.

Theorem 3.1. *Let $a_1, a_2 \ge 0$, $\alpha, \beta > 0$, $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Suppose that $D(n_i)$ and S_i, $i = 1, 2$, satisfy (1.3)-(1.7) with $m_i \ge \frac13$, $i = 1, 2$ and*

$$m_1 + \alpha_1 > \frac{23}{18},\ m_2 + \alpha_2 > \frac{23}{18}.$$

Then for any choice of the initial data $n_{01}, n_{02}, v_0, u_0, \phi$ fulfill (1.2) *and* (1.8), *system* (1.1) *possesses at least one non-negative global weak solution* (n_1, n_2, v, u, P) *in the sense of Definition 2.1. Also, this solution is bounded in* $\Omega \times (0,\infty)$ *in the sense that*

$$\|n_1(\cdot,t)\|_{L^\infty(\Omega)} + \|n_2(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C$$

for all $t > 0$ with some constant $C > 0$. In addition, v and u are continuous in $\overline{\Omega} \times (0,\infty)$, and n_1, n_2 as an $L^\infty(\Omega)$-valued function is continuous on $[0,\infty)$ with respect to the weak-$*$ topology, i.e.,

$$n_1, n_2 \in C^0_{w-*}([0,\infty); L^\infty(\Omega)).$$

Remark 3.1. Theorem 3.1 show that the system (1.1) admit a global bounded weak solution nothing to do with the size of $\frac{\chi}{\mu}$ and the dampening intensity of logistic source, this is very important in the proof of [5], [6] and [10]. We now have to leave an open question on existence of global bounded weak solution when $m_i \ge \frac{1}{3}$, $i = 1, 2$, $m_1 + \alpha_1 > \frac{23}{18}, m_2 + \alpha_2 \le \frac{23}{18}$ or $m_1 + \alpha_1 \le \frac{23}{18}, m_2 + \alpha_2 > \frac{23}{18}$ and $m_i < \frac{1}{3}$, $i = 1, 2$, $m_1 + \alpha_1 > \frac{23}{18}, m_2 + \alpha_2 > \frac{23}{18}$.

Then, we will give some priori estimates and prove the main theorem.

Lemma 3.1. *There exists a constant $C > 0$ depending on $\varepsilon \in (0,1)$ such that*

$$\int_\Omega |\nabla u_\varepsilon|^2 \le C \textit{ for all } t \in (0, T_{\max}) \tag{3.1}$$

and

$$\int_t^{t+\tau}\int_\Omega |\mathcal{A}u_\varepsilon|^2 \le C \textit{ for all } t \in (0, T_{\max} - \tau), \tag{3.2}$$

where $\tau = \min\{1, \frac{1}{2}T_{\max}\}$.

Proof. The proof is similar to that in [34, Lemma 2.4], so we omitted it. □

Lemma 3.2. *There exists $C > 0$ depending on $\varepsilon \in (0,1)$ such that*

$$\int_\Omega v_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^2 \le C \textit{ for all } t \in (0, T_{\max}). \tag{3.3}$$

Proof. The proof boundedness of $\|\nabla v_\varepsilon\|_{L^2(\Omega)}$ is similar to that in [34, Lemma 2.6], so we omitted it, by the Poincaré inequality, the boundedness of $\|v_\varepsilon\|_{L^2(\Omega)}$ is obtained. This completes the proof. □

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $p > 1$, suppose that the assumptions in Lemma 2.1 holds. Then for all $\varepsilon \in (0,1)$, we can find a constant $C > 0$ independent of ε and obtain the following inequality*

(3.4)

$$\frac{d}{dt}\Big(\int_\Omega n_{1\varepsilon}^p + \int_\Omega n_{2\varepsilon}^p\Big)$$

$$+\frac{p(p-1)}{2}\Big(C_{D_1}\int_\Omega n_{1\varepsilon}^{m_1+p-3}|\nabla n_{1\varepsilon}|^2+C_{D_2}\int_\Omega n_{2\varepsilon}^{m_2+p-3}|\nabla n_{2\varepsilon}|^2\Big)$$
$$\leq \frac{p(p-1)}{2}\Big(\frac{C_{S_1}^2}{C_{D_1}}\int_\Omega n_{1\varepsilon}^{p+1-m_1-2\alpha_1}|\nabla v_\varepsilon|^2+\frac{C_{S_2}^2}{C_{D_2}}\int_\Omega n_{2\varepsilon}^{p+1-m_2-2\alpha_2}|\nabla v_\varepsilon|^2\Big)+C$$

for all $t\in(0,T_{\max})$*, where* C_{D_1}*,* C_{D_2}*,* C_{S_1} *and* C_{S_2} *are as in* (1.4)*,* (1.5) *and* (1.7)*, respectively.*

Proof. We multiply the first two equation in (2.1) by $pn_{1\varepsilon}^{p-1}$ and use the fact $S_{1\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)=0$ on $\partial\Omega$, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega n_{1\varepsilon}^p+p(p-1)\int_\Omega n_{1\varepsilon}^{p-2}D_{1\varepsilon}(n_{1\varepsilon})|\nabla n_{1\varepsilon}|^2\\ &=p(p-1)\int_\Omega n_{1\varepsilon}^{p-1}\nabla n_{1\varepsilon}\cdot(S_{1\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)\cdot\nabla v_\varepsilon)\\ &\quad+\mu_1p\int_\Omega n_{1\varepsilon}^p-\mu_1p\int_\Omega n_{1\varepsilon}^{p+1}-a_1\mu_1p\int_\Omega n_{1\varepsilon}^pn_{2\varepsilon}\\ &\leq p(p-1)\int_\Omega n_{1\varepsilon}^{p-1}\nabla n_{1\varepsilon}\cdot(S_{1\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)\cdot\nabla v_\varepsilon)\\ &\quad+\mu_1p\int_\Omega n_{1\varepsilon}^p-\mu_1p\int_\Omega n_{1\varepsilon}^{p+1}\end{aligned}\tag{3.5}$$

for all $t\in(0,T_{\max})$. Here we use the definition of D_ε and (1.4), (1.5) to get that

$$p(p-1)\int_\Omega n_{1\varepsilon}^{p-2}D_\varepsilon(n_{1\varepsilon})|\nabla n_{1\varepsilon}|^2\geq C_{D_1}p(p-1)\int_\Omega n_{1\varepsilon}^{m_1+p-3}|\nabla n_{1\varepsilon}|^2\tag{3.6}$$

for all $t\in(0,T_{\max})$. Then, due to (1.7), we obtain

$$\begin{aligned}&p(p-1)\int_\Omega n_{1\varepsilon}^{p-1}\nabla n_{1\varepsilon}\cdot(S_{1\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)\cdot\nabla v_\varepsilon)\\ &\leq C_{S_1}p(p-1)\int_\Omega n_{1\varepsilon}^{p-\alpha_1-1}|\nabla n_{1\varepsilon}||\nabla v_\varepsilon|\\ &\leq\frac{C_{D_1}p(p-1)}{2}\int_\Omega n_{1\varepsilon}^{m_1+p-3}|\nabla n_{1\varepsilon}|^2+\frac{C_{S_1}^2p(p-1)}{2C_{D_1}}\int_\Omega n_{1\varepsilon}^{p+1-m_1-2\alpha_1}|\nabla v_\varepsilon|^2\end{aligned}\tag{3.7}$$

for all $t\in(0,T_{\max})$, here we use the Young's inequality. Dealing with the last two items, we also use the Young's inequality to see that

$$\mu_1p\int_\Omega n_{1\varepsilon}^p-\mu_1p\int_\Omega n_{1\varepsilon}^{p+1}\leq p(\frac{\mu_1}{p+1})^{p+1}(\frac{p}{\mu_1})^p=:C_1\tag{3.8}$$

for all $t\in(0,T_{\max})$, where C_1 is a positive constant. Similar to $n_{1\varepsilon}$, $n_{2\varepsilon}$ has similar inequality as above. Thus, we can obtain (3.4). This completes the proof. □

Lemma 3.4. *Let $q > 1$, it holds for all $\varepsilon \in (0,1)$ that*

$$\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^{2q} + \frac{2(q-1)}{q}\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + q\int_\Omega |\nabla v_\varepsilon|^{2q-2}|D^2 v_\varepsilon|^2 \le 2q(2q-2+\sqrt{3})^2\int_\Omega (\alpha^2 n_{1\varepsilon}^2 + \beta^2 n_{2\varepsilon}^2)|\nabla v_\varepsilon|^{2q-2} + 2q\int_\Omega |\nabla v_\varepsilon|^{2q}|Du_\varepsilon| + C \tag{3.9}$$

on $(0, T_{\max})$ with some positive constant C determined by q.

Proof. The proof is similar to that in [19, Lemma 3.2], so we omitted it. □

Lemma 3.5. *Let $m_i > \frac{23}{18} - \alpha_i$ $(i = 1,2)$ and suppose that $p > 1$ and $q \ge 2$ satisfies*

$$\max\{3 - 3m_i - 2\alpha_i,\ m_i + 2\alpha_i - \frac{1}{3}\} \le p < (m_i + \alpha_i - \frac{5}{6})(3q-1) + \frac{4}{3} - m_i, \tag{3.10}$$

$i = 1,2$. Then for all $\eta > 0$ there exists $C = C(p,q,\eta) > 0$ such that for all $\varepsilon \in (0,1)$

$$\int_\Omega n_{i\varepsilon}^{p+1-m_i-2\alpha_i}|\nabla v_\varepsilon|^2 \le \eta\int_\Omega n_{i\varepsilon}^{m_i+p-3}|\nabla n_{i\varepsilon}|^2 + \eta\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + C \tag{3.11}$$

for all $t \in (0, T_{\max})$, $i = 1,2$.

Proof. By the Hölder inequality, we have

$$\int_\Omega n_{i\varepsilon}^{p+1-m_i-2\alpha_i}|\nabla v_\varepsilon|^2 \le \Big(\int_\Omega n_{i\varepsilon}^{\frac{3}{2}(p+1-m_i-2\alpha_i)}\Big)^{\frac{2}{3}}\Big(\int_\Omega |\nabla v_\varepsilon|^6\Big)^{\frac{1}{3}} = \|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|_{L^{\frac{3(p+1-m_i-2\alpha_i)}{p+m_i-1}}(\Omega)}^{\frac{2(p+1-m_i-2\alpha_i)}{p+m_i-1}}\|\nabla v_\varepsilon\|_{L^6(\Omega)}^2 \tag{3.12}$$

for all $t \in (0, T_{\max})$. Due to $p \ge 3 - 3m_i - 2\alpha_i$, we have $\frac{3(p+1-m_i-2\alpha_i)}{p+m_i-1} \le 6$, $i = 1,2$. Thus, by the Gagliardo-Nirenberg interpolation inequality there exist some positive constants c_1, c_2 may be determined by p such that

$$\begin{aligned}
&\|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|_{L^{\frac{3(p+1-m_i-2\alpha_i)}{p+m_i-1}}(\Omega)}^{\frac{2(p+1-m_i-2\alpha_i)}{p+m_i-1}} \\
&\le c_1\|\nabla n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+1-m_i-2\alpha_i)}{p+m_i-1}\sigma_1}\|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|_{L^{\frac{2}{p+m_i-1}}(\Omega)}^{\frac{2(p+1-m_i-2\alpha_i)}{p+m_i-1}(1-\sigma_1)} \\
&\quad + \|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|_{L^{\frac{2}{p+m_i-1}}(\Omega)}^{\frac{2(p+1-m_i-2\alpha_i)}{p+m_i-1}} \\
&\le c_2\Big(\int_\Omega n_{i\varepsilon}^{m_i+p-3}|\nabla n_{i\varepsilon}|^2 + 1\Big)^{\frac{3(p+1-m_i-2\alpha_i)-2}{3(p+m_i-1)-1}}
\end{aligned} \tag{3.13}$$

for all $t \in (0, T_{\max})$ with

$$\sigma_1 = \frac{\frac{3(p+m_i-1)}{2} - \frac{p+m_i-1}{p+1-m_i-2\alpha_i}}{1-\frac{3}{2}+\frac{3(p+m_i-1)}{2}} \in [0,1]$$

due to $p > 3 - 3m_i - 2\alpha_i$ and $p > m_i + 2\alpha_i - \frac{1}{3}$, $i = 1, 2$. Due to $6 \in [2, 6q]$, from Lemma 2.6, there exist $c_3, c_4 > 0$ such that

$$\begin{aligned} \|\nabla v_\varepsilon\|^2_{L^6(\Omega)} &\le c_3 \|\nabla|\nabla v_\varepsilon|^q\|^{\frac{4}{3q-1}}_{L^2(\Omega)} \|\nabla v_\varepsilon\|^{\frac{2q-2}{3q-1}}_{L^2(\Omega)} + c_3\|\nabla v_\varepsilon\|^2_{L^2(\Omega)} \\ &\le c_4\Big(\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + 1\Big)^{\frac{2}{3q-1}} \text{ for all } t \in (0, T_{\max}). \end{aligned} \tag{3.14}$$

Since $p < (m_i + \alpha_i - \frac{5}{6})(3q-1) + \frac{4}{3} - m_i$, $i = 1, 2$, we have

$$\frac{3(p+1-m_i-2\alpha_i)-2}{3(p+m_i-1)-1} + \frac{2}{3q-1} < 1.$$

Combining with (3.12)-(3.14) and using Young's inequality, the desired results are obtained. This completes the proof. □

Lemma 3.6. *Let $m_i > \frac{23}{18} - \alpha_i$ $(i = 1, 2)$ and suppose that $p > 1$ and $q \ge 2$ satisfies*

$$p > \max\{2 - m_i,\ \frac{4}{9}(3q-1) + \frac{4}{3} - m_i\},\ i = 1, 2. \tag{3.15}$$

Then for all $\eta > 0$ there exists $C = C(p, q, \eta) > 0$ such that for all $\varepsilon \in (0, 1)$

$$\int_\Omega n^2_{i\varepsilon}|\nabla v_\varepsilon|^{2q-2} + \int_\Omega n^2_{i\varepsilon} \le \eta\int_\Omega n^{p+m_i-3}_{i\varepsilon}|\nabla n_{i\varepsilon}|^2 + \eta\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + C \tag{3.16}$$

for all $t \in (0, T_{\max})$, $i = 1, 2$.

Proof. By the Hölder inequality, we have

$$\begin{aligned} \int_\Omega n^2_{i\varepsilon}|\nabla v_\varepsilon|^{2q-2} &\le \Big(\int_\Omega n^3_{i\varepsilon}\Big)^{\frac{2}{3}}\Big(\int_\Omega |\nabla v_\varepsilon|^{6q-6}\Big)^{\frac{1}{3}} \\ &= \|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{4}{p+m_i-1}}_{L^{\frac{6}{p+m_i-1}}(\Omega)} \|\nabla v_\varepsilon\|^{2(q-1)}_{L^{6(q-1)}(\Omega)} \end{aligned} \tag{3.17}$$

for all $t \in (0, T_{\max})$, $i = 1, 2$. Since $p > 2 - m_i$, $i = 1, 2$, we have $\frac{6}{p+m_i-1} \le 6$, by the Gagliardo-Nirenberg interpolation inequality there exist some positive constants c_1, c_2 may be determined by p such that

$$\begin{aligned} \|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{4}{p+m_i-1}}_{L^{\frac{6}{p+m_i-1}}(\Omega)} &\le c_1\|\nabla n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{4}{p+m_i-1}\sigma_2}_{L^2(\Omega)} \|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{4}{p+m_i-1}(1-\sigma_2)}_{L^{\frac{2}{p+m_i-1}}(\Omega)} \\ &\quad + c_1\|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{4}{p+m_i-1}}_{L^{\frac{2}{p+m_i-1}}(\Omega)} \\ &\le c_2\Big(\int_\Omega n^{p+m_i-3}_{i\varepsilon}|\nabla n_{i\varepsilon}|^2 + 1\Big)^{\frac{4}{3(p+m_i-1)-1}} \end{aligned} \tag{3.18}$$

for all $t \in (0, T_{\max})$, with

$$\sigma_2 = \frac{\frac{3(p+m_i-1)}{2} - \frac{3(p+m_i-1)}{6}}{1 - \frac{3}{2} + \frac{3(p+m_i-1)}{2}} \in [0,1]$$

due to $p > 2 - m_i$. Since $6(q-1) \in [2, 6q]$, from Lemma 2.6, we have

$$\begin{aligned}\|\nabla v_\varepsilon\|^{2(q-1)}_{L^{6(q-1)}(\Omega)} &\leq c_3 \|\nabla|\nabla v_\varepsilon|^q\|^{\frac{2[3(q-1)-1]}{3q-1}}_{L^2(\Omega)} \|\nabla v_\varepsilon\|^{\frac{2}{3q-1}}_{L^2(\Omega)} + c_3 \|\nabla v_\varepsilon\|^{2(q-1)}_{L^2(\Omega)} \\ &\leq c_4 \Big(\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + 1\Big)^{\frac{3(q-1)-1}{3q-1}} \text{ for all } t \in (0, T_{\max}). \end{aligned} \tag{3.19}$$

Since $p > \frac{4}{9}(3q-1) + \frac{4}{3} - m_i$, we have

$$\frac{4}{3(p+m_i-1)-1} + \frac{3(q-1)-1}{3q-1} < 1.$$

By the Young's inequality, there exists a positive constant c_5 such that

$$\int_\Omega n_{i\varepsilon}^2 |\nabla v_\varepsilon|^{2q-2} \leq \frac{\eta}{2}\int_\Omega n_{i\varepsilon}^{p+m_i-3}|\nabla n_{i\varepsilon}|^2 + \frac{\eta}{2}\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + c_5 \tag{3.20}$$

for all $t \in (0, T_{\max})$. Similarly, by the Gagliardo-Nirenberg interpolation inequality there exists a positive constant c_6 such that

$$\int_\Omega n_{i\varepsilon}^2 \leq \frac{\eta}{2}\int_\Omega n_{i\varepsilon}^{p+m_i-3}|\nabla n_{i\varepsilon}|^2 + c_6 \tag{3.21}$$

for all $t \in (0, T_{\max})$. Combining with (3.20) and (3.21), the desired results are obtained. This completes the proof. □

Lemma 3.7. *For any $q > 1$, then for all $\eta > 0$, there exists a positive constant C such that*

$$\int_\Omega |\nabla v_\varepsilon|^{2q}|Du_\varepsilon| \leq \eta \int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + C \text{ for all } t \in (0, T_{\max}), \tag{3.22}$$

where C determined by q and λ_4.

Proof. We invoke the Hölder inequality with same exponents 2 to see that

$$\int_\Omega |\nabla v_\varepsilon|^{2q}|Du_\varepsilon| \leq \Big(\int_\Omega |\nabla v_\varepsilon|^{4q}\Big)^{\frac{1}{2}} \Big(\int_\Omega |Du_\varepsilon|^2\Big)^{\frac{1}{2}} \leq M\Big(\int_\Omega |\nabla v_\varepsilon|^{4q}\Big)^{\frac{1}{2}} \tag{3.23}$$

for all $t \in (0, T_{\max})$, here we used the result of Lemma 3.1. Owing to $2 \leq 4q \leq 6q$ satisfying the condition of Lemma 2.6, thus exists a positive constant c_1 determined by q such that

$$\begin{aligned}\Big(\int_\Omega |\nabla v_\varepsilon|^{4q}\Big)^{\frac{1}{2}} &= \|\nabla v_\varepsilon\|^{2q}_{L^{4q}(\Omega)} \\ &\leq c_1\Big(\|\nabla|\nabla v_\varepsilon|^q\|^{\frac{3(2q-1)}{3q-1}}_{L^2(\Omega)} \|\nabla v_\varepsilon\|^{\frac{q}{3q-1}}_{L^2(\Omega)} + \|\nabla v_\varepsilon\|^{2q}_{L^2(\Omega)}\Big) \\ &\leq \frac{\eta}{M}\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + \frac{C}{M} \end{aligned} \tag{3.24}$$

for all $t \in (0, T_{\max})$, here we use the Young's inequality, combine with (3.23) and (3.24), the desired result can be obtained. This completes the proof. $\square$

Lemma 3.8. *Let $m_i > 1$ and $m_i > \frac{23}{18} - \alpha_i$, $(i = 1, 2)$. Then for sufficiently large $p > 1$ and $q \geq 2$ in Lemmas 3.5 and 3.6, there exists a constant $C = C(p, q) > 0$ such that*

$$\|n_{1\varepsilon}\|_{L^p(\Omega)} + \|n_{2\varepsilon}\|_{L^p(\Omega)} + \|\nabla v_\varepsilon\|_{L^{2q}(\Omega)} \leq C \text{ for all } t \in (0, T_{\max}). \tag{3.25}$$

Proof. Since $m_i > \frac{23}{18} - \alpha_i$, we have $m_i + \alpha_i - \frac{5}{6} > \frac{4}{9}$, so

$$\frac{4}{9}(3q-1) + \frac{4}{3} - m_i < (m_i + \alpha_i - \frac{5}{6})(3q-1) + \frac{4}{3} - m_i, \ i = 1, 2$$

for any $q > 1$, there exist sufficiently large $p \geq 1$ such that

$$\frac{4}{9}(3q-1) + \frac{4}{3} - m_i < p < (m_i + \alpha_i - \frac{5}{6})(3q-1) + \frac{4}{3} - m_i, \ i = 1, 2.$$

Combining with Lemmas 3.5-3.7, choosing properly small $\eta > 0$, there exist some constant $c_1 = c_1(p, q), c_2 = c_2(p, q) > 0$ such that

$$\begin{aligned} &\frac{d}{dt}\Big(\int_\Omega n_{1\varepsilon} + \int_\Omega n_{2\varepsilon} + \int_\Omega |\nabla v_\varepsilon|^{2q}\Big) \\ &+ c_1\Big(\int_\Omega |\nabla n_{1\varepsilon}^{\frac{p+m_1-1}{2}}|^2 + \int_\Omega |\nabla n_{2\varepsilon}^{\frac{p+m_i-1}{2}}|^2 + \int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2\Big) \leq c_2 \end{aligned} \tag{3.26}$$

for all $t \in (0, T_{\max})$. By the Gagliardo-Nirenberg interpolation inequality, there exist constants $c_1, c_2 > 0$ may be determined by p such that

$$\begin{aligned} \|n_{i\varepsilon}\|^p_{L^p(\Omega)} &= \|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{2p}{p+m_i-1}}_{L^{\frac{2p}{p+m_i-1}}(\Omega)} \\ &\leq c_1\|\nabla n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{2p}{p+m_i-1}\varrho_1}_{L^2(\Omega)}\|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{2p}{p+m_i-1}(1-\varrho_1)}_{L^{\frac{2}{p+m_i-1}}(\Omega)} \\ &\quad + c_1\|n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{2p}{p+m_i-1}}_{L^2(\Omega)} \\ &\leq c_2\|\nabla n_{i\varepsilon}^{\frac{p+m_i-1}{2}}\|^{\frac{2p}{p+m_i-1}\varrho_2} + c_9 \end{aligned} \tag{3.27}$$

for all $t \in (0, T_{\max})$, $i = 1, 2$, where

$$\varrho_1 = \frac{\frac{1}{2} - \frac{1}{2p}}{\frac{p+m_i-1}{2} - \frac{1}{6}}(p + m_i - 1) \in [0, 1]$$

and

$$\frac{2p}{p + m_i - 1}\varrho_1 = \frac{p-1}{\frac{p+m_i-1}{2} - \frac{1}{6}} \leq 2,$$

here we use the fact that $m_i \geq \frac{1}{3}$. Using the Young's inequality to (3.27), there exists a constant $c_3 > 0$ independent of $\varepsilon \in (0, 1)$ such that

$$\int_\Omega n_{i\varepsilon}^p \leq c_3 \int_\Omega |\nabla n_{i\varepsilon}^{\frac{p+m_i-1}{2}}|^2 + c_3, \ i = 1, 2. \tag{3.28}$$

By Lemma 2.6 and Young's inequality, there exist two constants $c_4, c_5 > 0$ may be determined by p, such that

$$\|\nabla v_\varepsilon\|^{2q}_{L^{2q}(\Omega)} \le c_4\Big(\|\nabla|\nabla v_\varepsilon|^q\|^{\frac{6(q-1)}{3q-1}}_{L^2(\Omega)}\|\nabla v_\varepsilon\|^{\frac{4q}{3q-1}}_{L^2(\Omega)} + \|\nabla v_\varepsilon\|^{2q}_{L^2(\Omega)}\Big)$$

$$\le c_5\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + c_5 \tag{3.29}$$

for all $t \in (0, T_{\max})$. Therefore, together with (3.28) and (3.29), we have

$$\int_\Omega n^p_{1\varepsilon} + \int_\Omega n^p_{2\varepsilon} + \int_\Omega |\nabla v_\varepsilon|^{2q} \tag{3.30}$$

$$\le c_6\int_\Omega |\nabla n_{1\varepsilon}^{\frac{p+m_1-1}{2}}|^2 + \int_\Omega |\nabla n_{2\varepsilon}^{\frac{p+m_2-1}{2}}|^2 + \int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2$$

for all $t \in (0, T_{\max})$, with $c_6 = c_3 + c_5$. Now, let $y(t) := \int_\Omega n^p_{1\varepsilon} + \int_\Omega n^p_{2\varepsilon} + \int_\Omega |\nabla v_\varepsilon|^{2q}$, $h(t) := \int_\Omega |\nabla n_{1\varepsilon}^{\frac{p+m_1-1}{2}}|^2 + \int_\Omega |\nabla n_{2\varepsilon}^{\frac{p+m_2-1}{2}}|^2 + \int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2$, from (3.28) and (3.32), there exist some positive constants $c_7, c_8 > 0$ such that

$$y'(t) + c_7 y(t) + c_7 h(t) \le c_8 \text{ for all } t \in (0, T_{\max}).$$

Following from a comparison argument, we have

$$y(t) \le c_9 := \max\{y(0),\ \frac{c_8}{c_7}\} \text{ for all } t \in (0, T_{\max}).$$

This completes the proof. □

Corollary 3.1. *Let $m_1 > 1$, $m_2 > 1$ and $m_1 > \frac{23}{18} - \alpha_1$, $m_2 > \frac{23}{18} - \alpha_2$. Then for any $p > 1$, $q > 1$ and $r \ge 1$, there exists some constat $C > 0$ such that for any $\varepsilon \in (0, 1)$*

$$\|n_{1\varepsilon}\|_{L^p(\Omega)} + \|n_{2\varepsilon}\|_{L^p(\Omega)} + \|\nabla v_\varepsilon\|_{L^q(\Omega)} + \|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le C \textit{ for all } t \in (0, T_{\max}). \tag{3.31}$$

Proof. This proof is almost similar to that of [40, Corollary 3.2], to avoid repetition, so we omitted it. □

Proposition 3.1. *Suppose the assumptions of Lemma 2.1 hold with $m_1 > 1$, $m_2 > 1$ and $m_1 > \frac{23}{18} - \alpha_1$, $m_2 > \frac{23}{18} - \alpha_2$. Then system (2.10) admits a global classical solution $(n_{1\varepsilon}, n_{2\varepsilon}, v_\varepsilon, u_\varepsilon, P_\varepsilon)$, which is uniformly bounded for all $\varepsilon \in (0, 1)$,*

$$\|n_{1\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|n_{2\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \tag{3.32}$$

for all $t \in (0, \infty)$, with some constant $C > 0$. Moreover, we also have

$$\|\mathcal{A}^\epsilon u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \textit{ for all } t \in (0, \infty). \tag{3.33}$$

Proof. This proof is relying on the properties for the Neumann heat semigroup and Stokes semigroup, we can find in [7,31,41]. In accordance with (3.27) with $p > 3$, we can apply Lemma 2.5 to obtain (2.9) with $r = \infty$, and therefore

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \text{ for all } t \in (0, T_{\max}) \tag{3.34}$$

is valid. Taking the results of Corollary 3.1 with appropriately large p and q as a initial point, we make use of Moser-type iteration to the first two equations in (2.1) and then obtain

$$\|n_{i\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} \le C \ (i=1,2) \text{ for all } t\in(0,T_{\max}). \tag{3.35}$$

On account of (3.34) and (3.35), we apply the parabolic regularity theory to the third equation in (2.1) to get

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \text{ for all } t\in(0,T_{\max}). \tag{3.36}$$

Finally, we prove (3.33), let $\epsilon\in(\frac{3}{4},1)$, we apply the Helmholtz projection and the fractional power $\mathcal{A}_r^\epsilon$ to the fourth equation in (2.1), it follows from the variation-of-constants formula that

$$u_\varepsilon(\cdot,t)=e^{-t\mathcal{A}_r}u_{0\varepsilon}+\int_0^t e^{-(t-s)\mathcal{A}_r}\mathcal{P}[(n_{1\varepsilon}+n_{2\varepsilon})\nabla\phi](\cdot,s)ds,\ t\in(0,T_{\max}),$$

we can obtain

$$\begin{aligned}
\|\mathcal{A}_r^\epsilon u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &\le \|\mathcal{A}_r^\epsilon e^{-t\mathcal{A}_r}u_{0\varepsilon}\|_{L^2(\Omega)} \\
&\quad+\int_0^t \|\mathcal{A}_r^\epsilon e^{-(t-s)\mathcal{A}_r}\mathcal{P}[(n_{1\varepsilon}+n_{2\varepsilon})\nabla\phi](\cdot,s)\|_{L^2(\Omega)}ds \\
&\le c_1t^{-\epsilon}\|u_{0\varepsilon}\|_{L^2(\Omega)}+c_{18}(\|n_{1\varepsilon}\|_{L^\infty(\Omega)}+\|n_{2\varepsilon}\|_{L^\infty(\Omega)})\|\nabla\phi\|_{L^2(\Omega)} \\
&\quad\times\int_0^t (t-s)^{-\epsilon}e^{-\lambda(t-s)}ds \\
&\le c_2
\end{aligned}$$

for all $t\in(0,T_{\max})$, with some constants $c_1,c_2,\lambda>0$. Along with (3.34)-(3.36) and blow-up criterion (2.2), we infer that $T_{\max}$=∞ and prove the proposition. This completes the proof. □

Lemma 3.9. *Suppose the assumptions of Lemma 2.1 hold with $m_1>1$, $m_2>1$ and $m_1>\frac{23}{18}-\alpha_1$, $m_2>\frac{23}{18}-\alpha_2$. Then one can find $\theta\in(0,1)$ such that for some $C>0$*

$$\|v_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])}\le C \text{ for all } t\in(0,\infty) \tag{3.37}$$

as well as

$$\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])}\le C \text{ for all } t\in(0,\infty) \tag{3.38}$$

and such that for any $\varsigma>0$ there exists $C(\varsigma)>0$ fulfilling

$$\|\nabla v_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])}\le C \text{ for all } t\in[\varsigma,\infty). \tag{3.39}$$

Proof. This proof can find in [19, Lemma 3.12], see also [14]. so we omitted it. □

Finally, we prove the main theorem.

The proof of Theorem 3.1. Firstly, with the help of Proposition 3.1 we derive that for each $\varepsilon \in (0,1)$, system (2.1) admits a classical solution $(n_{1\varepsilon}, n_{2\varepsilon}, v_\varepsilon, u_\varepsilon, P_\varepsilon)$ which is defined for all $t > 0$. Let $\varphi \in W_0^{3,2}(\Omega)$, it is known by the embedding theorem, we see that $W_0^{3,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ with $N = 3$. Thus, $\varphi \in L^\infty(\Omega)$ and $\|\varphi\|_{W^{1,\infty}(\Omega)} \le c_1\|\varphi\|_{W_0^{3,2}(\Omega)}$ with some positive constant c_1. Let $\kappa > m_1$ satisfy $\kappa \ge 2(m_1 - 1)$. Multiplying both sides of the first equation in (2.10) by $\kappa n_{1\varepsilon}^{\kappa-1}\varphi$ and integrating by parts on Ω, we obtain

$$
\begin{aligned}
&\frac{1}{\kappa}\int_\Omega \Big(\frac{\partial}{\partial t} n_{1\varepsilon}^\kappa\Big)\varphi \\
=\ & \int_\Omega n_{1\varepsilon}^{\kappa-1}\Big[\nabla\cdot\Big(D_{1\varepsilon}(n_{1\varepsilon})\nabla n_{1\varepsilon} - n_{1\varepsilon}S_{1\varepsilon}(x, n_{1\varepsilon}, v_\varepsilon)\cdot\nabla v_\varepsilon\Big) - u_\varepsilon\nabla n_{1\varepsilon}\Big]\cdot\varphi \\
&+ \mu_1\int_\Omega n_{1\varepsilon}^\kappa(1 - n_{1\varepsilon} - a_1 n_{2\varepsilon})\varphi \\
=\ & -(\kappa-1)\int_\Omega n_{1\varepsilon}^{\kappa-2}D_{1\varepsilon}(n_{1\varepsilon})|\nabla n_{1\varepsilon}|^2\varphi - \int_\Omega n_{1\varepsilon}^{\kappa-1}D_{1\varepsilon}(n_{1\varepsilon})\nabla n_{1\varepsilon}\cdot\nabla\varphi \\
&+ (\kappa-1)\int_\Omega n_{1\varepsilon}^{\kappa-1}\nabla n_{1\varepsilon}\cdot\Big(S_{1\varepsilon}(x, n_{1\varepsilon}, v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\varphi \\
&+ \int_\Omega n_{1\varepsilon}^\kappa\Big(S_{1\varepsilon}(x, n_{1\varepsilon}, v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\cdot\nabla\varphi + \frac{1}{\kappa}\int_\Omega n_{1\varepsilon}^\kappa u_\varepsilon\cdot\nabla\varphi \\
&+ \mu_1\int_\Omega n_{1\varepsilon}^\kappa(1 - n_{1\varepsilon} - a_1 n_{2\varepsilon})\varphi
\end{aligned} \tag{3.40}
$$

for all $t \in (0,\infty)$. From Lemma 3.9, we can fix positive constants c_2, c_3 and c_4 such that

$$|n_{i\varepsilon}| \le c_2,\ |\nabla v_\varepsilon| \le c_3 \text{ and } |u_\varepsilon| \le c_4 \text{ in } \Omega\times(0,\infty) \text{ for all } \varepsilon\in(0,1), \tag{3.41}$$

$i = 1, 2$, on account of the fact that $D_{1\varepsilon} < D_1 + 2\varepsilon$ in $(0,\infty)$ for all $\varepsilon \in (0,1)$, we have

$$D_{1\varepsilon}(n_{1\varepsilon}) \le c_5 := \|D_1\|_{L^\infty(0,c_2)} + 2 \text{ in } \Omega\times(0,\infty) \text{ for all } \varepsilon\in(0,1). \tag{3.42}$$

Let $p := \kappa - m_1 + 1$ satisfies $p > 1$ and $p \ge m - 1$, by (3.26), yield that

$$\int_0^\infty\int_\Omega n_{1\varepsilon}^{\kappa-2}|\nabla n_{1\varepsilon}|^2 = \int_0^\infty\int_\Omega n_{1\varepsilon}^{p+m_1-3}|\nabla n_{1\varepsilon}|^2 \le c_6 \tag{3.43}$$

for all $t \in (0,\infty)$ with certain constant $c_6 > 0$. Using (3.41), (3.42) and Young's inequality, we have

$$\Big|-(\kappa-1)\int_\Omega n_{1\varepsilon}^{\kappa-2}D_{1\varepsilon}(n_{1\varepsilon})|\nabla n_{1\varepsilon}|^2\varphi\Big| \le (\kappa-1)c_5\cdot\Big(\int_\Omega n_{1\varepsilon}^{\kappa-2}|\nabla n_{1\varepsilon}|^2\Big)\cdot\|\varphi\|_{L^\infty(\Omega)} \tag{3.44}$$

and

$$\Big|-\int_\Omega n_{1\varepsilon}^{\kappa-1}D_{1\varepsilon}(n_{1\varepsilon})\nabla n_{1\varepsilon}\cdot\nabla\varphi\Big|$$

$$\begin{aligned}
&\le c_5\cdot\Big(\int_\Omega n_{1\varepsilon}^{\kappa-1}|\nabla n_{1\varepsilon}|\Big)\cdot\|\nabla\varphi\|_{L^\infty(\Omega)}\\
&\le c_5\cdot\Big(\int_\Omega n_{1\varepsilon}^{\kappa-2}|\nabla n_{1\varepsilon}|^2+\int_\Omega n_{1\varepsilon}^{\kappa}\Big)\cdot\|\nabla\varphi\|_{L^\infty(\Omega)} \qquad (3.45)\\
&\le\Big(c_5\int_\Omega n_{1\varepsilon}^{\kappa-2}|\nabla n_{1\varepsilon}|^2+c_{24}c_{21}^{\kappa}|\Omega|\Big)\cdot\|\nabla\varphi\|_{L^\infty(\Omega)}
\end{aligned}$$

as well as

$$\begin{aligned}
&\Big|(\kappa-1)\int_\Omega n_{1\varepsilon}^{\kappa-1}\nabla n_{1\varepsilon}\cdot\Big(S_{1\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\varphi\Big|\\
&\le(\kappa-1)\cdot\Big(\int_\Omega n_{1\varepsilon}^{\kappa-1}|\nabla n_{1\varepsilon}|\Big)\cdot C_{S_1}c_3\|\varphi\|_{L^\infty(\Omega)}\\
&\le(\kappa-1)C_{S_1}c_3\cdot\Big(\int_\Omega n_{1\varepsilon}^{\kappa-2}|\nabla n_{1\varepsilon}|^2+c_2^{\kappa}|\Omega|\Big)\cdot\|\varphi\|_{L^\infty(\Omega)} \qquad (3.46)
\end{aligned}$$

and

$$\Big|\int_\Omega n_{1\varepsilon}^{\kappa}\Big(S_{1\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\cdot\nabla\varphi\Big|\le c_2^{\kappa}c_3C_{S_1}|\Omega|\|\nabla\varphi\|_{L^\infty(\Omega)} \qquad (3.47)$$

as well as

$$\Big|\frac{1}{\kappa}\int_\Omega n_{1\varepsilon}^{\kappa}u_\varepsilon\cdot\nabla\varphi\Big|\le\frac{1}{\kappa}c_2^{\kappa}c_4|\Omega|\|\nabla\varphi\|_{L^\infty(\Omega)} \qquad (3.48)$$

and

$$\Big|\mu_1\int_\Omega n_{1\varepsilon}^{\kappa}(1-n_{1\varepsilon}-a_1n_{2\varepsilon})\varphi\Big|\le\mu_1(c_2^{\kappa}+c_2^{\kappa+1})\|\varphi\|_{L^\infty(\Omega)}. \qquad (3.49)$$

Due to $W_0^{3,2}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$, combining with (3.44)-(3.49), there exists a constant $c_7>0$ such that

$$\|\frac{\partial}{\partial t}n_{1\varepsilon}^{\kappa}(\cdot,t)\|_{(W_0^{3,2}(\Omega))^*}\le c_7\cdot\Big(\int_\Omega n_{1\varepsilon}^{\kappa-2}|\nabla n_{1\varepsilon}|^2+1\Big)$$

for all $t\in(0,\infty)$ and any $\varepsilon\in(0,1)$. According to (3.43), for each $T>0$ we have

$$\int_0^T\|\frac{\partial}{\partial t}n_{1\varepsilon}^{\kappa}(\cdot,t)\|_{(W_0^{3,2}(\Omega))^*}dt\le c_6c_7+c_7T \qquad (3.50)$$

for all $\varepsilon\in(0,1)$. A similar argument we have

$$\int_0^T\|\frac{\partial}{\partial t}n_{2\varepsilon}^{\kappa}(\cdot,t)\|_{(W_0^{3,2}(\Omega))^*}dt\le c_8T+c_8 \qquad (3.51)$$

for all $\varepsilon\in(0,1)$, with some constant $c_8>0$. Multiplying both sides of the first equation in (2.10) by ψ and integrating by parts on Ω, we obtain

$$\begin{aligned}
&\int_\Omega\Big(\frac{\partial}{\partial t}n_{1\varepsilon}\Big)\psi \qquad (3.52)\\
&=\int_\Omega\Big[\nabla\cdot\Big(D_{1\varepsilon}(n_{1\varepsilon})\nabla n_{1\varepsilon}-n_{1\varepsilon}S_{1\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)\cdot\nabla v_\varepsilon\Big)-u_\varepsilon\nabla n_{1\varepsilon}\Big]\cdot\psi
\end{aligned}$$

$$
\begin{aligned}
&+\mu_1\int_\Omega n_{1\varepsilon}(1-n_{1\varepsilon}-a_1n_{2\varepsilon})\psi\\
=&\int_\Omega G_1(n_{1\varepsilon})\triangle\psi+\int_\Omega\Big(n_{1\varepsilon}S_{1\varepsilon}(x,n_{1\varepsilon},v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\nabla\psi+\int_\Omega n_{1\varepsilon}u_\varepsilon\cdot\nabla\psi\\
&+\mu_1\int_\Omega n_{1\varepsilon}(1-n_{1\varepsilon}-a_1n_{2\varepsilon})\psi
\end{aligned}
$$

for all $t\in(0,\infty)$, where we have set $G_1(s):=\int_0^s D_1(\sigma)d\sigma$ for $s\geq 0$. Recalling that $D_{1\varepsilon}<D_1+2\varepsilon$ in $(0,\infty)$ we can estimate

$$G_1(n_{1\varepsilon})\leq c_9:=c_2\cdot\Big(\|D_1\|_{L^\infty(0,c_{21})}+2\Big) \text{ in } \Omega\times(0,\infty) \text{ for all } \varepsilon\in(0,1). \tag{3.53}$$

Similar to Lemma 3.23 in [43], there exist constants $c_{10},c_{11}>0$ such that

$$\int_0^T\|\frac{\partial}{\partial t}n_{1\varepsilon}(\cdot,t)\|_{(W_0^{2,2}(\Omega))^*}dt\leq c_{10} \tag{3.54}$$

and

$$\int_0^T\|\frac{\partial}{\partial t}n_{2\varepsilon}(\cdot,t)\|_{(W_0^{2,2}(\Omega))^*}dt\leq c_{11} \tag{3.55}$$

for all $\varepsilon\in(0,1)$. In accordance with Lemma 3.9, the Arzelá-Ascoli theorem along with a standard extraction procedure yields a sequence $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ with $\varepsilon_j\to 0$ as $j\to\infty$ such that

$$v_\varepsilon\to v \text{ in } C^0_{loc}(\overline{\Omega}\times[0,\infty)), \tag{3.56}$$

$$\nabla v_\varepsilon\to\nabla v \text{ in } C^0_{loc}(\overline{\Omega}\times(0,\infty)), \tag{3.57}$$

$$u_\varepsilon\to u \text{ in } C^0_{loc}(\overline{\Omega}\times[0,\infty)), \tag{3.58}$$

hold with some limit functions v and u belonging to the indicated spaces. Passing to a subsequence if necessary, by means of Proposition 3.1 we can achieve that for some $n_i\in L^\infty(\Omega\times(0,\infty))$ we moreover have

$$n_{i\varepsilon}\overset{*}{\rightharpoonup} n_i \text{ in } L^\infty(\Omega\times(0,\infty)),\ i=1,2, \tag{3.59}$$

$$\nabla v_\varepsilon\overset{*}{\rightharpoonup}\nabla v \text{ in } L^\infty(\Omega\times(0,\infty)), \tag{3.60}$$

$$Du_\varepsilon\overset{*}{\rightharpoonup}Du \text{ in } L^\infty(\Omega\times(0,\infty)). \tag{3.61}$$

We fix $\kappa>m_1$ satisfy $\kappa\geq 2(m_1-1)$ and combine (3.51) with (3.44) for $p:=2\kappa-m_1+1$ to see that for each $T>0$, $(\varepsilon^\kappa)_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T);W^{1,2}(\Omega))$ with $\left(\frac{\partial}{\partial t}n_{1\varepsilon}^\kappa\right)_{\varepsilon\in(0,1)}$ being bounded in $L^1((0,T);(W_0^{3,2}(\Omega))^\star)$. Therefore, an Aubin-Lions lemma applies to yield strong precompactness of $(n_{1\varepsilon}^\kappa)_{\varepsilon\in(0,1)}$ in $L^2(\Omega\times(0,T))$, whence along a suitable subsequence we have

$n_{1\varepsilon}^{\kappa} \to b^{\kappa}$ and hence $n_{1\varepsilon} \to b$ a.e. in $\Omega \times (0,\infty)$ for some nonnegative measurable $b : \Omega \times (0,\infty) \to \mathbb{R}$. By Egorov's theorem, we know that necessarily $n_1 = b$, thus, we have

$$n_{i\varepsilon} \to n_i \text{ a.e. in } \Omega \times (0,\infty),\ i = 1,2. \tag{3.62}$$

Finally, as the embedding $L^{\infty}(\Omega) \hookrightarrow (W_0^{2,2}(\Omega))^{\star}$ is compact, the Arzelà-Ascoli once more applies to say that the equicontinuity property (3.54) together with the boundedness of $(n_{1\varepsilon})_{\varepsilon\in(0,1)}$ in $C^0([0,\infty); L^{\infty}(\Omega)$ ensures that

$$n_{i\varepsilon} \to n_i \text{ in } C^0_{loc}\Big([0,\infty); (W_0^{2,2}(\Omega))^*\Big),\ i = 1,2, \tag{3.63}$$

holds after a further extraction of an adequate subsequence. The additional regularity property

$$n_1 \in C^0_{w^{-*}}([0,\infty); L^{\infty}(\Omega)),\ n_2 \in C^0_{w^{-*}}([0,\infty); L^{\infty}(\Omega)). \tag{3.64}$$

thereafter is a consequence of (3.63) and the fact that $C_i := \|n_i\|_{L^{\infty}(\Omega\times(0,\infty))}$, $i = 1,2$ is finite: first, from the latter property it follows that there exists a null set $N \subset [0,\infty)$ such that for all $t \in [0,\infty)\backslash N$ we have $n_i(\cdot,t) \in L^{\infty}(\Omega)$ with $\|n_i(\cdot,t)\|_{L^{\infty}(\Omega)} \le C_i$, $i = 1,2$. As $[0,\infty)\backslash N$ is dense in $[0,\infty)$, for an arbitrary $t_0 \in [0,\infty)$ we can find $(t_j)_{j\in\mathbb{N}} \subset [0,\infty)\backslash N$ such that $t_j \to t_0$ as $j \to \infty$, and extracting a subsequence if necessary we can obtain that $n_i(\cdot,t) \overset{*}{\rightharpoonup} \bar{n}_i$ in $L^{\infty}(\Omega)$ as $j \to \infty$, with some $\bar{n}_i \in L^{\infty}(\Omega)$ satisfying $\|\bar{n}_i\|_{L^{\infty}(\Omega)} \le C_i$, $i = 1,2$. Since (3.61) assert that moreover $n_i(\cdot,t_j) \to n_i(\cdot,t_0)$ in $(W_0^{2,2}(\Omega))^{\star}$ as $j \to \infty$, this allows us to identify $\bar{n}_i = n_i(\cdot,t_0)$ and to conclude that thus actually $n_i(\cdot,t) \in L^{\infty}(\Omega)$ for all $t \in [0,\infty)$, with $\|n_i(\cdot,t)\|_{L^{\infty}(\Omega)} \le C_i$ for all $t \ge 0$. The property (3.62) can now be verified by partially repeating this argument: given any $t_0 \ge 0$ and $(t_j)_{j\in\mathbb{N}} \subset [0,\infty)\backslash N$ such that $t_j \to t_0$ as $j \to \infty$ we know that $(n_i(\cdot,t_j)))_{j\in\mathbb{N}}$ is bounded in $L^{\infty}(\Omega)$, and that for all $\varphi \in C_0^{\infty}(\Omega)$ we have $\int_{\Omega} n_i(\cdot,t_j))\varphi \to \int_{\Omega} n_i(\cdot,t_0)\varphi$ as $j \to \infty$ by (3.61). By density of $C_0^{\infty}(\Omega)$ in $L^1(\Omega)$, this proves that indeed $n_i(\cdot,t_j) \overset{*}{\rightharpoonup} n_i(\cdot,t_0)$ in $L^{\infty}(\Omega)$ as $j \to \infty$.

Now the verification of the claimed weak solution property of (n_1,n_2,v,u) is straightforward: whereas the nonnegativity of n_i, $i = 1,2$ and v and the integrability requirements in conditions of Definition 2.1 are immediate from (3.56)-(3.59) and (3.62), the integral identities in Definition 2.1 can be derived by standard arguments from the corresponding weak formulations in the approximate system (2.1) upon letting $\varepsilon = \varepsilon_j \to 0$ and using (3.59) and (3.62) as well as (3.56)-(3.58), (3.60) and (3.61). This completes the proof. □

4. Asymptotic behavior of the solution

In this section, we consider a special case with $D_1(n_1) = D_2(n_2) \equiv 1$, then the weak solution becomes classical solution, we begin with stating the main theorem, and construct the function to prove it. As a first step towards this, we show the main theorem.

Theorem 4.1. *Let $D_1(n_1) = D_2(n_2) \equiv 1$, assume that the condition of Theorem 3.1 holds. Then the solution of* (1.1) *has the following properties*:

(i) *Let $a_1, a_2 \in (0,1)$, under the condition that there exists γ_1 such that*

$$4\gamma_1 - (1+\gamma_1)^2 a_1 a_2 > 0$$

and

$$\frac{c_1 C_{S_1}^2 (1-a_1)}{4a_1\mu_1(1-a_1a_2)} + \frac{\gamma_1 c_2 C_{S_2}^2 (1-a_2)}{4a_2\mu_2(1-a_1a_2)} < \frac{4\gamma_1 - (1+\gamma_1)^2 a_1 a_2}{a_1\alpha^2\gamma_1 + a_2\beta^2 - a_1a_2\alpha\beta(1+\gamma_1)},$$

then

$$n_1(\cdot,t) \to N_1,\ n_2(\cdot,t) \to N_2,\ v(\cdot,t) \to V_1,\ u(\cdot,t) \to 0 \text{ in } L^\infty \text{ as } t \to \infty,$$

where

$$N_1 := \frac{1-a_1}{1-a_1a_2},\ N_2 := \frac{1-a_2}{1-a_1a_2},\ V_1 := \alpha N_1 + \beta N_2$$

as well as

$$c_1 = \max\{1, (1 + \|n_1\|_{L^\infty[0,\|n_{10}\|_{L^\infty(\Omega)}+1]})^{1-\alpha_1}\}$$

and

$$c_2 = \max\{1, (1 + \|n_2\|_{L^\infty[0,\|n_{20}\|_{L^\infty(\Omega)}+1]})^{1-\alpha_2}\}.$$

(ii) *Let $a_1 \geq 1 > a_2$ under the condition that there exist γ_3 and $a_1' \in [1, a_1]$ such that*

$$4\gamma_3 - (1+\gamma_3)^2 a_1' a_2 > 0$$

and

$$\mu_2 > \frac{C_{S_2}^2 \gamma_3 c_2 (\alpha^2 a_1'\gamma_3 + \beta^2 a_2 - \alpha\beta a_1' a_2(1+\gamma_3))}{4a_2(4\gamma_3 - a_1'a_2(1+\gamma_3)^2)},$$

then

$$n_1(\cdot,t) \to 0,\ n_2(\cdot,t) \to 1,\ v(\cdot,t) \to \beta,\ u(\cdot,t) \to 0 \text{ in } L^\infty \text{ as } t \to \infty,$$

where

$$c_2 = \max\{1, (1 + \|n_2\|_{L^\infty[0,\|n_{20}\|_{L^\infty(\Omega)}+1]})^{1-\alpha_2}\}.$$

We will give the following lemma which will give stabilization in (1.1).

Lemma 4.1 ([9, Lemma 4.6])**.** *Let $n \in C^0(\overline{\Omega} \times [0,\infty))$ satisfying that there exist $C_1 > 0$ and $\theta \in (0,1)$ such that*

$$\|n\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \leq C \text{ for all } t \geq 1.$$

Assume that

$$\int_1^\infty \int_\Omega (n - N^*)^2 < \infty$$

with some constant N^*. Then

$$n(\cdot,t) \to N^* \text{ as } t \to \infty.$$

Lemma 4.2 ([23, Lemma 2.1]). *Let $a, b, c, d, e, f \in \mathbb{R}$. Suppose that*

$$a > 0,\ ad - \frac{b^2}{4} > 0,\ adf + \frac{bce}{4} - \frac{c^2 d}{4} - \frac{b^2 f}{4} - \frac{ae^2}{4} > 0.$$

Then there exists $\varepsilon > 0$ such that

$$ay_1^2 + by_1y_2 + cy_1y_3 + dy_2^2 + ey_2y_3 + fy_3^2 \geq \varepsilon(y_1^2 + y_2^2 + y_3^2)$$

holds for all $y_1, y_2, y_3 \in \mathbb{R}$.

4.1. Case 1: $a_1, a_2 \in (0, 1)$

In this subsection, we will analysis convergence of (n_1, n_2, v, u) as $t \to 0$. To get the desire results, we give the following key estimate for stabilization in (1.1) in the case $a_1, a_2 \in (0, 1)$.

Lemma 4.3. *Let $a_1, a_2 \in (0, 1)$. Under the assumption of Theorem 4.1*(i)*, the solution of* (1.1) *has the property that there exist $\gamma_1, \gamma_2 > 0$ and $\varepsilon_1 > 0$ such that the nonnegative function E_1 and F_1 defined by*

$$E_1 := \int_\Omega (n_1 - N_1 - N_1 \ln \frac{n_1}{N_1}) + \gamma_1 \frac{a_1\mu_1}{a_2\mu_2} \int_\Omega (n_2 - N_2 - N_2 \ln \frac{n_2}{N_2}) + \frac{\gamma_2}{2} \int_\Omega v^2$$

and

$$F_1 := \int_\Omega (n_1 - N_1)^2 + \int_\Omega (n_2 - N_2)^2 + \int_\Omega (v - V_1)^2$$

satisfy

$$E_1(t) \geq 0 \text{ for all } t > 0 \tag{4.1}$$

and

$$\frac{d}{dt} E_1(t) \leq -\varepsilon_1 F_1(t) \text{ for all } t > 0, \tag{4.2}$$

where $N_1 = \frac{1-a_1}{1-a_1a_2}$, $N_2 = \frac{1-a_2}{1-a_1a_2}$ and $V_1 = \alpha N_1 + \beta N_2$.

Proof. Let $A_1(t), B_1(t)$ and $C_1(t)$ defined as

$$A_1(t) := \int_\Omega (n_1 - N_1 - N_1 \ln \frac{n_1}{N_1}),$$

$$B_1(t) := \int_\Omega (n_2 - N_2 - N_2 \ln \frac{n_2}{N_2})$$

and

$$C_1(t) := \frac{1}{2} \int_\Omega v^2,$$

and we write

$$E_1(t) = A_1(t) + \gamma_1 \frac{a_1\mu_1}{a_2\mu_2} B_1(t) + \gamma_2 C_1(t).$$

Taking the similar procedure in [9, Lemma 4.1], we know (4.1) holds. By the straightforward calculation we have

$$\frac{d}{dt} E_1(t) = \frac{d}{dt} A_1(t) + \gamma_1 \frac{a_1\mu_1}{a_2\mu_2} \frac{d}{dt} B_1(t) + \gamma_2 \frac{d}{dt} C_1(t)$$

$$
\begin{aligned}
(4.3) \qquad \le & -\mu_1 \int_\Omega (n_1 - N_1)^2 - a_1\mu_1(1+\gamma_1)\int_\Omega (n_1-N_1)(n_2-N_2) \\
& - N_1 \int_\Omega \frac{|\nabla n_1|^2}{n_1^2} + C_{S_1} N_1 \int_\Omega \frac{\nabla n_1 \cdot \nabla v}{(1+n_1)^{\alpha_1+1}} \\
& - \frac{\gamma_1 a_1 \mu_1}{a_2}\int_\Omega (n_2-N_2)^2 - \frac{\gamma_1 a_1\mu_1 N_2}{a_2\mu_2}\int_\Omega \frac{|\nabla n_2|^2}{n_2^2} \\
& + \frac{\gamma_1 a_1 \mu_1 N_2 C_{S_2}}{a_2\mu_2}\int_\Omega \frac{\nabla n_2\cdot\nabla v}{(1+n_2)^{\alpha_2+1}} - \gamma_2\int_\Omega |\nabla v|^2 - \gamma_2 \int_\Omega (v-V_1)^2 \\
& + \gamma_2\alpha\int_\Omega (n_1-N_1)(v-V_1) + \gamma_2\beta\int_\Omega (n_2-N_2)(v-V_1).
\end{aligned}
$$

Using the (2.5) and Young's inequality we have

$$
(4.4) \qquad N_1 C_{S_1}\int_\Omega \frac{\nabla n_1\cdot\nabla v}{(1+n_1)^{\alpha_1+1}} \le N_1\int_\Omega \frac{|\nabla n_1|^2}{n_1^3} + \frac{N_1 C_{S_1}^2}{4}\int_\Omega \frac{|\nabla v|^2}{(1+n_1)^{\alpha_1-1}},
$$

due to

(4.5)

$$
\int_\Omega \frac{|\nabla v|^2}{(1+n_1)^{\alpha_1-2}} \le \begin{cases} (1+\|n_1\|_{L^\infty[0,\|n_{01}\|_{L^\infty(\Omega)}]})^{1-\alpha_1}\int_\Omega |\nabla v|^2, \ \text{if } \alpha_1<1, \\ \int_\Omega |\nabla v|^2, \ \text{if } \alpha_1 \ge 1, \end{cases}
$$

combining with (3.30) and (4.5), there exists a positive constant c_1 such that

$$
(4.6) \qquad \int_\Omega \frac{|\nabla v|^2}{(1+n_1)^{\alpha_1-1}} \le c_1 \int_\Omega |\nabla v|^2,
$$

substituting (4.6) into (4.4), we have

$$
(4.7) \qquad N_1 C_{S_1}\int_\Omega \frac{\nabla n_1\cdot\nabla v}{(1+n_1)^{\alpha_1+1}} \le N_1 \int_\Omega \frac{|\nabla n_1|^2}{n_1^2} + \frac{c_1 N_1 C_{S_1}^2}{4}\int_\Omega |\nabla v|^2.
$$

Similar to (4.7), there exists a constant $c_2 > 0$ such that

$$
\begin{aligned}
& \frac{\gamma_1 a_1\mu_1 N_2 C_{S_2}}{a_2\mu_2}\int_\Omega \frac{\nabla n_2\cdot\nabla v}{(1+n_2)^{\alpha_2+1}} \\
(4.8) \qquad \le & \frac{\gamma_1 a_1 \mu_1 N_2 C_{S_2}^2}{4a_2\mu_2}\int_\Omega \frac{|\nabla v|^2}{(1+n_2)^{\alpha_2-1}} + \frac{\gamma_1 a_1\mu_1 N_2}{a_2\mu_2}\int_\Omega \frac{|\nabla n_2|^2}{n_2^2} \\
\le & \frac{\gamma_1 a_1\mu_1 N_2}{a_2\mu_2}\int_\Omega \frac{|\nabla n_2|^2}{n_2^2} + \frac{c_2\gamma_1 a_1\mu_1 N_2 C_{S_2}^2}{4a_2\mu_2}\int_\Omega |\nabla v|^2.
\end{aligned}
$$

Combining with (4.3)-(4.8), we obtain

$$
\begin{aligned}
\frac{d}{dt}E_1(t) \le & -\mu_1\int_\Omega (n_1-N_1)^2 - a_1\mu_1(1+\gamma_1)\int_\Omega (n_1-N_1)(n_2-N_2) \\
& + \gamma_2\alpha\int_\Omega (n_1-N_1)(v-V_1) - \frac{\gamma_1 a_1\mu_1}{a_2}\int_\Omega (n_2-N_2)^2 \\
& + \gamma_2\beta\int_\Omega (n_2-N_2)(v-V_1) - \gamma_2\int_\Omega (v-V_1)^2
\end{aligned}
$$

$$-\Big(\gamma_2-\frac{c_1N_1C_{S_1}^2}{4}-\frac{c_2\gamma_1a_1\mu_1N_2C_{S_2}^2}{4a_2\mu_2}\Big)\int_\Omega|\nabla v|^2.$$

By the condition of Theorem 4.1(i), taking $\gamma_2>0$ satisfies

$$\frac{c_1C_{S_1}^2N_1}{4}+\frac{\gamma_1a_1\mu_1c_2C_{S_2}^2N_2}{4a_2\mu_2}<\gamma_2<\frac{a_1\mu_1[4\gamma_1-(1+\gamma_1)^2a_1a_2]}{a_1\alpha^2\gamma_1+a_2\beta^2-a_1a_2\alpha\beta(1+\gamma_1)},$$

it is easy to know that

$$\gamma_2-\frac{c_1N_1C_{S_1}^2}{4}-\frac{c_2\gamma_1a_1\mu_1N_2C_{S_2}^2}{4a_2\mu_2}>0.$$

Letting

$$a=\mu_1,\ b=a_1\mu_1(1+\gamma_1),\ c=-\gamma_2\alpha,\ d=\frac{\gamma_1a_1\mu_1}{a_2},\ e=-\gamma_2\beta,\ f=\gamma_2.$$

By the straightforward calculation we have

$$a=\mu_1>0,\ ad-\frac{b^2}{4}=\frac{a_1\mu_1^2(4\gamma_1-(1+\gamma_1)^2a_1a_2)}{4a_2}>0$$

and

$$\begin{aligned}&adf+\frac{bce}{4}-\frac{c^2d}{4}-\frac{b^2f}{4}-\frac{ae^2}{4}\\=&\frac{\mu_1\gamma_2}{4a_2}[a_1\mu_1(4\gamma_1-(1+\gamma_1)^2a_1a_2)-(a_1\alpha^2\gamma_1+a_2\beta^2-a_1a_2\alpha\beta(1+\gamma_1))\gamma_2]>0.\end{aligned}$$

From Lemma 4.2, there exists a constant $\varepsilon_1>0$ such that

$$\frac{d}{dt}E_1(t)\le-\varepsilon_1\Big(\int_\Omega(n_1-N_1)^2+\int_\Omega(n_2-N_2)^2+\int_\Omega(v-V_1)^2\Big)\ \text{for all } t>0.$$

This completes the proof. □

Lemma 4.4. *Let $a_1,a_2\in(0,1)$. Under the assumption of Theorem 4.1*(i)*, the solution of* (1.1) *satisfies that there exists a constant $C_2>0$ such that*

$$\int_1^\infty\int_\Omega(n_1-N_1)^2+\int_1^\infty\int_\Omega(n_2-N_2)^2+\int_1^\infty\int_\Omega(v-V_1)^2\le C_2. \tag{4.9}$$

Proof. Integrating (4.2) over $(1,t)$, we infer

$$E_1(t)+\varepsilon_1\int_1^tF_1(s)ds\le E_1(0). \tag{4.10}$$

Therefore, combination of (4.10) with (4.2) implies (4.9). This completes the proof. □

4.2. Case 2: $a_1 \geq 1 > a_2$

In this subsection, we will analysis convergence of (n_1, n_2, v, u) as $t \to 0$. To get the desire results, we give the following key estimate for stabilization in (1.1) in the case $a_1 \geq 1 > a_2$.

Lemma 4.5. *Let $a_1 \geq 1 > a_2$. Under the assumption of Theorem 4.1*(ii)*, the solution of* (1.1) *has the property that there exist $\gamma_3, \gamma_4 > 0$ and $\varepsilon_2 > 0$ such that the nonnegative function E_2 and F_2 defined by*

$$E_2 := \int_\Omega n_1 + \gamma_3 \frac{a_1' \mu_1}{a_2 \mu_2} \int_\Omega (n_2 - 1 - \ln n_2) + \frac{\gamma_4}{2} \int_\Omega (v - \beta)^2$$

and

$$F_2 := \int_\Omega n_1^2 + \int_\Omega (n_2 - 1)^2 + \int_\Omega (v - \beta)^2$$

satisfy

$$E_2(t) \geq 0 \text{ for all } t > 0 \tag{4.11}$$

and

$$\frac{d}{dt} E_2(t) \leq -\varepsilon_2 F_2(t) \text{ for all } t > 0. \tag{4.12}$$

Proof. Letting

$$E_2(t) = \int_\Omega n_1 + \gamma_3 \frac{a_1' \mu_1}{a_2 \mu_2} \int_\Omega (n_2 - 1 - \ln n_2) + \frac{\gamma_4}{2} \int_\Omega (v - \beta)^2.$$

Taking the similar procedure in [9, Lemma 4.1], we know (4.11) holds. By the straightforward calculation we have

$$\begin{aligned} \frac{d}{dt} E_2(t) \leq & -\mu_1 \int_\Omega n_1^2 - a_1' \mu_1 (1 + \gamma_3) \int_\Omega n_1 (n_2 - 1) - \mu_1 (a_1' - 1) \int_\Omega n_1 \\ & - \frac{\gamma_3 a_1' \mu_1}{a_2} \int_\Omega (n_2 - N_2)^2 - \frac{\gamma_3 a_1' \mu_1}{a_2 \mu_2} \int_\Omega \frac{|\nabla n_2|^2}{n_2^2} \\ & + \frac{\gamma_3 a_1' \mu_1 C_{S_2}}{a_2 \mu_2} \int_\Omega \frac{\nabla n_2 \cdot \nabla v}{(1 + n_2)^{\alpha_2 + 1}} - \gamma_4 \int_\Omega |\nabla v|^2 \\ & - \gamma_4 \int_\Omega (v - \beta)^2 + \gamma_4 \alpha \int_\Omega n_1 (v - \beta) + \gamma_4 \beta \int_\Omega (n_2 - 1)(v - \beta). \end{aligned} \tag{4.13}$$

Similar to (4.8), there exists a constant $c_2 > 0$ such that

$$\frac{\gamma_3 a_1' \mu_1 C_{S_2}}{a_2 \mu_2} \int_\Omega \frac{\nabla n_2 \cdot \nabla v}{(1 + n_2)^{\alpha_2 + 1}} \leq \frac{\gamma_3 a_1' \mu_1}{a_2 \mu_2} \int_\Omega \frac{|\nabla n_2|^2}{n_2^2} + \frac{c_2 \gamma_3 a_1' \mu_1 C_{S_2}^2}{4 a_2 \mu_2} \int_\Omega |\nabla v|^2. \tag{4.14}$$

Combining (4.13) with (4.14), we obtain

$$\frac{d}{dt} E_2(t) \leq -\mu_1 \int_\Omega n_1^2 - a_1' \mu_1 (1 + \gamma_3) \int_\Omega n_1 (n_2 - 1) + \gamma_4 \alpha \int_\Omega n_1 (v - \beta)$$

$$
\begin{aligned}
(4.15)\qquad & -\frac{\gamma_3 a_1'\mu_1}{a_2}\int_\Omega (n_2-1)^2+\gamma_4\beta\int_\Omega (n_2-1)(v-\beta)\\
& -\gamma_4\int_\Omega (v-\beta)^2-\Big(\gamma_4-\frac{c_2\gamma_3 a_1'\mu_1 C_{S_2}^2}{4a_2\mu_2}\Big)\int_\Omega |\nabla v|^2.
\end{aligned}
$$

By the condition of Theorem 4.1(ii), taking $\gamma_4>0$ satisfies

$$
\frac{\gamma_3 a_1'\mu_1 c_2 C_{S_2}^2}{4a_2\mu_2}<\gamma_2<\frac{a_1'\mu_1[4\gamma_3-(1+\gamma_3)^2a_1'a_2]}{a_1'\alpha^2\gamma_3+a_2\beta^2-a_1'a_2\alpha\beta(1+\gamma_3)},
$$

it is easy to know that

$$
\gamma_4-\frac{c_2\gamma_3 a_1'\mu_1 C_{S_2}^2}{4a_2\mu_2}>0.
$$

Letting

$$
a=\mu_1,\ b=a_1'\mu_1(1+\gamma_3),\ c=-\gamma_4\alpha,\ d=\frac{\gamma_3 a_1'\mu_1}{a_2},\ e=-\gamma_4\beta,\ f=\gamma_4.
$$

By the straightforward calculation we have

$$
a=\mu_1>0,\ ad-\frac{b^2}{4}=\frac{a_1'\mu_1^2(4\gamma_3-(1+\gamma_3)^2a_1'a_2)}{4a_2}>0
$$

and

$$
\begin{aligned}
& adf+\frac{bce}{4}-\frac{c^2d}{4}-\frac{b^2f}{4}-\frac{ae^2}{4}\\
=\ & \frac{\mu_1\gamma_4}{4a_2}[a_1'\mu_1(4\gamma_3-(1+\gamma_3)^2a_1'a_2)-(a_1'\alpha^2\gamma_3+a_2\beta^2-a_1'a_2\alpha\beta(1+\gamma_3))\gamma_4]>0.
\end{aligned}
$$

From Lemma 4.2, there exists a constant $\varepsilon_2>0$ such that

$$
\frac{d}{dt}E_2(t)\le-\varepsilon_2\Big(\int_\Omega (n_1-N_1)^2+\int_\Omega (n_2-N_2)^2+\int_\Omega (v-V_1)^2\Big)\ \text{for all } t>0.
$$

This completes the proof. □

Lemma 4.6. *Let $a_1\ge 1>a_2$. Under the assumption of Theorem 4.1*(ii)*, the solution of* (1.1) *satisfies that there exists a constant $C_3>0$ such that*

$$
\int_1^\infty\int_\Omega n_1^2+\int_1^\infty\int_\Omega (n_2-1)^2+\int_1^\infty\int_\Omega (v-\beta)^2\le C_3. \tag{4.16}
$$

Proof. The proof is similar to Lemma 4.4, so we omitted it. □

Lemma 4.7. *Under the assumptions of Theorem 2.1 and Theorem 2.2, the solution of* (1.1) *has the following property*:

$$
\|u(\cdot,t)\|_{L^\infty(\Omega)}\to 0 \ \text{as}\ t\to\infty. \tag{4.17}
$$

Proof. Noting from Lemma 4.3 and Lemma 4.5 that exists a constant $T>0$ such that $\int_T^\infty\int_\Omega(\alpha n_1+\beta n_2)^2v^2<\infty$ and using Lemma 4.5 in [5], we can obtain that

$$
\int_T^\infty\int_\Omega v^2<\infty,
$$

which entails $\|v\|_{L^\infty(\Omega)} \to 0$. Next we will show that $\|u\|_{L^\infty(\Omega)} \to 0$, similar to [5, Lemma 4.6], there exist some positive constants c_1, c_2, c_3 such that

$$\|u\|_{L^\infty(\Omega)} \le c_1\|\mathcal{A}^\theta u\|_{L^2(\Omega)} \le c_2\|\mathcal{A}^\epsilon u\|^a_{L^2(\Omega)}\|\mathcal{A}^\theta u\|^{1-a}_{L^2(\Omega)} \le c_3\|u\|^{1-a}_{L^2(\Omega)},$$

where $\theta \in (\frac{3}{4}, \epsilon)$ and $a = \frac{\theta}{\epsilon} \in (0,1)$, which means that it is sufficient to show that

$$\|u(\cdot,t)\|_{L^2(\Omega)} \to 0 \text{ as } t \to \infty.$$

From the Poincaré inequality that there exists a constant $c_4 > 0$ such that

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \le c_4\|\nabla u(\cdot,t)\|^2_{L^2(\Omega)}$$

for all $t \in (0,\infty)$. Put $(\bar{n}_1, \bar{n}_2) := (N_1, N_2)$ if $a_1, a_2 \in (0,1)$ or $(\bar{n}_1, \bar{n}_2) := (0,1)$ if $a_1 \ge 1 > 0$. We infer from the fourth equation in (2.10) and Young's inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 \\
=& \int_\Omega (n_1 - \bar{n}_1 + n_2 - \bar{n}_2)\nabla\phi \cdot u + (\bar{n}_1 + \bar{n}_2)\int_\Omega \nabla\phi \cdot u \\
\le& \frac{1}{4c_4}\int_\Omega u^2 + c_5\Big(\int_\Omega (n_1 - \bar{n}_1)^2 + \int_\Omega (n_2 - \bar{n}_2)^2\Big) + (\bar{n}_1 + \bar{n}_2)\int_\Omega \nabla\phi \cdot u
\end{aligned}$$

for all $t \in (0,\infty)$ with some constant $c_5 > 0$. Letting

$$y(t) := \int_\Omega u^2 \text{ and } h(t) := 2c_5\Big(\int_\Omega (n_1 - \bar{n}_1)^2 + \int_\Omega (n_2 - \bar{n}_2)^2\Big)$$

satisfy

$$y'(t) + c_6 y(t) \le h(t)$$

with some $c_6 > 0$. Hence it holds that

$$\begin{aligned}
y(t) &\le y(0)e^{-c_6 t} + \int_0^t e^{-c_6(t-s)}h(s)ds \\
&\le y(0)e^{-c_6 t} + \int_0^{\frac{t}{2}} e^{-c_6(t-s)}h(s)ds + \int_{\frac{t}{2}}^t e^{-c_6(t-s)}h(s)ds. \qquad (4.18)
\end{aligned}$$

From Proposition 3.1 that there exists a constant $c_7 > 0$ such that $h(s) \le c_7$ for all $s > 0$ and hence we have

$$\int_0^{\frac{t}{2}} e^{-c_6(t-s)}h(s)ds \le c_7 e^{-c_6 t}\int_0^{\frac{t}{2}} e^{c_6 s}ds \le c_8 e^{-\frac{c_6}{2}t} \qquad (4.19)$$

with some $c_7 > 0$. On the other hand, noting from (4.9) and (4.13) that $\int_0^\infty h(s)ds < \infty$, we have

$$0 \le \int_{\frac{t}{2}}^t e^{-c_6(t-s)}h(s)ds \le \int_{\frac{t}{2}}^t h(s)ds \to 0 \ as \ t \to \infty. \qquad (4.20)$$

Thus, combination of (4.18) with (4.19) and (4.20) leads to

$$\|u(\cdot,t)\|_{L^2(\Omega)} = y(t) \to 0 \text{ as } t \to \infty.$$

(4.17) follows from the dominated convergence theorem. This completes the proof. □

Finally, we prove the main theorem.

The proof of Theorem 4.1. A combination of Lemmata 4.1-4.7 directly leads to Theorem 4.1. □

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