J. Korean Math. Soc. **57** (2020), No. 1, pp. 171–194
https://doi.org/10.4134/JKMS.j180880
pISSN: 0304-9914 / eISSN: 2234-3008

CHARACTERIZATION OF TEMPERED EXPONENTIAL DICHOTOMIES

Luis Barreira, João Rijo, and Claudia Valls

Abstract. For a nonautonomous dynamics defined by a sequence of bounded linear operators on a Banach space, we give a characterization of the existence of an exponential dichotomy with respect to a sequence of norms in terms of the invertibility of a certain linear operator between general admissible spaces. This notion of an exponential dichotomy contains as very special cases the notions of uniform, nonuniform and tempered exponential dichotomies. As applications, we detail the consequences of our results for the class of tempered exponential dichotomies, which are ubiquitous in the context of ergodic theory, and we show that the notion of an exponential dichotomy under sufficiently small parameterized perturbations persists and that their stable and unstable spaces are as regular as the perturbation.

1. Introduction

We give a characterization of the existence of an *exponential dichotomy with respect to a sequence of norms* for a nonautonomous dynamics defined by a sequence of bounded linear operators on a Banach space in terms of the invertibility of a certain linear operator. We note that this notion of an exponential dichotomy contains as very special cases the notions of uniform, nonuniform and tempered exponential dichotomies.

More precisely, let $(A_m)_{m\in\mathbb{Z}}$ be a two-sided sequence of bounded linear operators acting on a Banach space X. It induces the dynamics

$$x_m = A_{m-1}x_{m-1} \quad \text{for } m \in \mathbb{Z}, \tag{1}$$

on the space X. Our main aim is to give a characterization of the existence of an exponential dichotomy for this dynamics or, more precisely, of the existence of an exponential dichotomy with respect to a sequence of norms (see Section 2 for the definition), in terms of the invertibility of a linear operator T between appropriate Banach spaces of two-sided sequences in X. These Banach spaces,

Received December 30, 2018; Accepted June 12, 2019.
2010 *Mathematics Subject Classification.* Primary 37D99.
Key words and phrases. Tempered exponential dichotomies, admissible spaces.
L. Barreira and C. Valls were supported by FCT/Portugal through UID/MAT/04459/2013. J. Rijo was supported by FCT/Portugal through the grant PD/BD/128413/2017.

usually called *admissible spaces*, belong to a large class of spaces introduced by Coffman and Schäffer in [4] for discrete time. For simplicity of the exposition we refrain from introducing them here in detail. Nevertheless, they include for example all ℓ^p spaces with $p \in [1, +\infty]$ as well as many others (see Section 2 for details and examples).

Given two admissible spaces Y and Y' of two-sided sequences in X, we define the operator $T\colon \mathcal{D}(T) \subset Y' \to Y$ by $T\mathbf{x} = \mathbf{y}$, where

$$\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \quad \text{and} \quad \mathbf{y} = (x_m - A_{m-1}x_{m-1})_{m\in\mathbb{Z}},$$

on the domain formed by all $\mathbf{x} \in Y'$ such that $\mathbf{y} \in Y$. For $\mathbf{x} \in \mathcal{D}(T)$ we consider the graph norm

$$\|\mathbf{x}\|_T = \|\mathbf{x}\|_{Y'} + \|T\mathbf{x}\|_Y$$

and the linear operator

$$T\colon (\mathcal{D}(T), \|\cdot\|_T) \to (Y, \|\cdot\|_Y),$$

which from now on we denote simply by T. It is in terms of the invertibility of this operator that we shall characterize the existence of an exponential dichotomy for the dynamics in (1).

Here we formulate only a particular case of our results when $Y = Y'$.

Theorem 1.1. *When $Y = Y'$, a sequence $(A_m)_{m\in\mathbb{Z}}$ of bounded linear operators has an exponential dichotomy if and only if the operator T is invertible.*

Theorem 1.1 is an immediate consequence of Theorems 3.1 and 3.2 below for general spaces Y and Y'. The formulation of these results is left for the main text since it requires introducing much additional material. Theorem 1.1 is a version of work in [11] that considers a one-sided dynamics with discrete time and a constant sequences of norms (and thus uniform exponential dichotomies). A simple consequence is that the notion of an exponential dichotomy with respect to a sequence of norms persists under sufficiently small linear perturbations.

We emphasize that the notion of an exponential dichotomy with respect to a sequence of norms already occurs naturally in the study of a dynamics on a smooth manifold, in which case the derivative of the trajectory travels along tangent spaces, each with its own norm induced from the Riemannian metric. Another main motivation for the notion is given by ergodic theory. Namely, *almost all* linear variational equations with nonzero Lyapunov exponents obtained from an autonomous differential equation $x' = f(x)$ with a measure-preserving flow have a tempered exponential behavior (see Section 4 for details). In its turn, this exponential behavior can be expressed in terms of a sequence of Lyapunov norms (see Proposition 4.1 below). These are appropriate norms that transform the tempered exponential behavior into a uniform one and vice-versa, although possibly at the expense of having a ratio between these norms and the original norm that may diverge subexponentially when time goes to infinity. On the other hand, as noted above, a uniform exponential behavior corresponds to consider a constant sequence of norms. We refer the reader to the book [1] for details and references.

The characterization of the existence of an exponential behavior of the type considered in the present paper (such as in Theorem 1.1) goes back to work of Perron in [10] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + y(t)$$

on $\mathbb{R}^n$ for any bounded continuous perturbation $y\colon \mathbb{R}_0^+ \to \mathbb{R}^n$. A relatively simple modification of Perron's work yields the following statement for discrete time: given a sequence of $n\times n$ matrices $(A_m)_{m\in\mathbb{N}}$, if for each bounded sequence $(y_m)_{m\in\mathbb{N}}$ in $\mathbb{R}^n$ there exists $x_0 \in \mathbb{R}^n$ such that the sequence

$$x_m = A_{m-1}x_{m-1} + y_m \quad \text{for } m \in \mathbb{N}, \tag{2}$$

is bounded, then any bounded sequence $A_m \cdots A_1 x$ tends to zero as $m \to \infty$. In other words, under the former assumption stability leads to asymptotic stability. This is a first step towards showing that the dynamics has contracting and expanding directions, and ultimately an exponential dichotomy. From this point of view, our work can be seen as a far reaching generalization of Perron's work for a two-sided dynamics, also considering much more general spaces in which we take the perturbation $(y_m)_{m\in\mathbb{N}}$ and look for a solution $(x_m)_{m\in\mathbb{N}}$ of problem (2).

There exists an extensive literature on the relation between the stability or exponential stability of a dynamics and the invertibility properties of certain linear operators as the one described above (these properties are often called admissibility properties). For some of the most relevant early contributions in the direction initiated by Perron in [10], we refer to the books by Massera and Schäffer [9] (that culminates the development started in [8]) and by Dalec'kiĭ and Kreĭn [5]. Related results for discrete time were first obtained by Li in [7] and then by Coffman and Schäffer in [4]. We also refer to [6] for some early results on infinite-dimensional spaces.

2. Basic notions

In this section we introduce what is strictly necessary for the formulation of our main results. This includes the notions of an exponential dichotomy with respect to a sequence of norms and of an admissible space. The proofs require additional material that will be introduced only later on.

Let $X = (X, \|\cdot\|)$ be a Banach space and let $L(X)$ be the set of all bounded linear operators acting on X. Given a sequence $(A_m)_{m\in\mathbb{Z}}$ in $L(X)$, we define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n \end{cases}$$

for each $m, n \in \mathbb{Z}$ with $m \geq n$. Now let $\|\cdot\|_m$, for $m \in \mathbb{Z}$, be norms on X that are equivalent to the original norm $\|\cdot\|$. We say that $(A_m)_{m\in\mathbb{Z}}$ has an *exponential dichotomy with respect to the norms* $\|\cdot\|_m$ if:

1. there exist projections $P_m \colon X \to X$, for $m \in \mathbb{Z}$, satisfying

$$P_m \mathcal{A}(m,n) = \mathcal{A}(m,n) P_n \quad \text{for } m \geq n, \tag{3}$$

   such that, writing $Q_m = \mathrm{Id} - P_m$, the map

$$\mathcal{A}(m,n)_{|Q_n(X)} \colon Q_n(X) \to Q_m(X) \tag{4}$$

   is onto and invertible;
2. there exist constants $\lambda, D > 0$ such that for each $x \in X$ we have

$$\|\mathcal{A}(m,n) P_n x\|_m \leq D e^{-\lambda(m-n)} \|x\|_n, \tag{5}$$

   and

$$\|\mathcal{A}(n,m) Q_m x\|_n \leq D e^{-\lambda(m-n)} \|x\|_m \tag{6}$$

   for $m \geq n$, where $\mathcal{A}(n,m)$ denotes the inverse of $\mathcal{A}(m,n)_{|Q_n(X)}$.

We also introduce the class of admissible spaces. Let $\mathbb{R}^{\mathbb{Z}}$ be the set of all sequences $\mathbf{s} = (s_n)_{n\in\mathbb{Z}}$ of real numbers and denote by χ_A the characteristic function of a set $A \subset \mathbb{Z}$. A Banach space $B = (B, \|\cdot\|_B) \subset \mathbb{R}^{\mathbb{Z}}$ formed by all sequences $\mathbf{s} \in \mathbb{R}^{\mathbb{Z}}$ such that $\|\mathbf{s}\|_B < +\infty$ is called an *admissible space* if:

1. $\chi_{\{n\}} \in B$ for all $n \in \mathbb{Z}$;
2. if $\mathbf{s}' = (s'_n)_{n\in\mathbb{Z}} \in B$ and $|s_n| \leq |s'_n|$ for all $n \in \mathbb{Z}$, then $\mathbf{s} \in B$ and $\|\mathbf{s}\|_B \leq \|\mathbf{s}'\|_B$;
3. there exists $N > 0$ such that for each $\mathbf{s} = (s_n)_{n\in\mathbb{Z}} \in B$ and $m \in \mathbb{Z}$ the sequence $\mathbf{s}^m = (s_{n+m})_{n\in\mathbb{Z}}$ is in B and satisfies $\|\mathbf{s}^m\|_B \leq N\|\mathbf{s}\|_B$.

Examples of admissible spaces are the following:

1. the set

$$\ell^\infty = \left\{ \mathbf{s} \in \mathbb{R}^{\mathbb{Z}} : \sup_{m\in\mathbb{Z}} |s_m| < +\infty \right\}$$

   with the norm $\|\mathbf{s}\| = \sup_{m\in\mathbb{Z}} |s_m|$;
2. for each $p \in [1, +\infty)$, the set

$$\ell^p = \left\{ \mathbf{s} \in \mathbb{R}^{\mathbb{Z}} : \sum_{m\in\mathbb{Z}} |s_m|^p < +\infty \right\}$$

   with the norm $\|\mathbf{s}\| = (\sum_{m\in\mathbb{Z}} |s_m|^p)^{1/p}$;
3. taking $\psi(t) = \int_0^t \phi(s)\, ds$ for some nondecreasing left-continuous function $\phi \colon \mathbb{R}^+ \to (0, +\infty]$ and defining $M(\mathbf{s}) = \sum_{n\in\mathbb{Z}} \psi(|s_n|)$, let

$$B = \left\{ \mathbf{s} \in \mathbb{R}^{\mathbb{Z}} : M(c\mathbf{s}) < +\infty \text{ for some } c > 0 \right\}$$

   with the norm

$$\|\mathbf{s}\| = \inf\left\{ c > 0 : M(\mathbf{s}/c) \leq 1 \right\}.$$

Finally, we introduce a class of Banach subspaces of $X^{\mathbb{Z}}$. Namely, given an admissible space B, let

$$Y_B = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}} : (\|x_n\|_n)_{n\in\mathbb{Z}} \in B\right\}.$$

Moreover, for each $\mathbf{x} \in Y_B$ we define

$$\|\mathbf{x}\|_{Y_B} = \left\|(\|x_n\|_n)_{n\in\mathbb{Z}}\right\|_B.$$

Proposition 2.1. *$Y_B = (Y_B, \|\cdot\|_{Y_B})$ is a Banach space.*

Proof. Given a Cauchy sequence $(\mathbf{x}^k)_{k\in\mathbb{N}}$ in Y_B, there exists a subsequence $(\mathbf{x}^{\ell_k})_{k\in\mathbb{N}}$ such that

$$\|\mathbf{x}^{\ell_{k+1}} - \mathbf{x}^{\ell_k}\|_{Y_B} \le 2^{-k} \quad \text{for } k \in \mathbb{N}. \tag{7}$$

Writing $\mathbf{x}^k = (x_n^k)_{n\in\mathbb{Z}}$ we define

$$y_n = \sum_{k=1}^{+\infty} \|x_n^{\ell_{k+1}} - x_n^{\ell_k}\|_n \quad \text{and} \quad y_n^m = \sum_{k=1}^{m} \|x_n^{\ell_{k+1}} - x_n^{\ell_k}\|_n$$

for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Since B is a Banach space, it follows from (7) that $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ and $\mathbf{y}^m = (y_n^m)_{n\in\mathbb{Z}}$ belong to B for $m \in \mathbb{N}$. Moreover, $\mathbf{y}^m \to \mathbf{y}$ in B when $m \to +\infty$.

Now observe that

$$\|x_n^k - x_n^l\|_n \chi_{\{n\}}(m) \le \|x_m^k - x_m^l\|_m$$

for $k, l \in \mathbb{N}$ and $n, m \in \mathbb{Z}$, and so it follows from the properties in the notion of an admissible space that

$$\|x_n^k - x_n^l\|_n \le \frac{N}{\|\chi_{\{0\}}\|_B} \|\mathbf{x}^k - \mathbf{x}^l\|_{Y_B}.$$

Since the norms $\|\cdot\|$ and $\|\cdot\|_m$ are equivalent, we conclude that $(x_n^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in X for each $n \in \mathbb{Z}$. Let

$$x_n = \lim_{k\to\infty} x_n^k \quad \text{for } n \in \mathbb{Z},$$

and write $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$. It follows from the identity

$$x_n - x_n^{\ell_m} = \sum_{k=m}^{+\infty} (x_n^{\ell_{k+1}} - x_n^{\ell_k})$$

that

$$\|x_n - x_n^{\ell_m}\|_n \le y_n - y_n^{m-1}. \tag{8}$$

Since $\mathbf{y}, \mathbf{y}^{m-1} \in B$, we find that $\mathbf{x} - \mathbf{x}^{\ell_m} = (x_n - x_n^{\ell_m})_{n\in\mathbb{Z}} \in Y_B$ and so also

$$\mathbf{x} = (\mathbf{x} - \mathbf{x}^{\ell_m}) + \mathbf{x}^{\ell_m} \in Y_B.$$

Moreover, by (8) we conclude that

$$\|\mathbf{x} - \mathbf{x}^{\ell_m}\|_{Y_B} \le \|\mathbf{y} - \mathbf{y}^{m-1}\|_B \to 0$$

when $m \to +\infty$. This shows that the sequence $(\mathbf{x}^k)_{k\in\mathbb{N}}$ has a convergent subsequence in Y_B and since it is a Cauchy sequence in fact it converges. □

In the following section we shall consider an admissibility property with respect to a pair of spaces Y_B and $Y_{B'}$ obtained from some admissible spaces B and B'.

3. Main results

In this section we formulate our main results relating admissibility and hyperbolicity. More precisely, we shall give a characterization of an exponential dichotomy for a sequence $(A_m)_{m\in\mathbb{Z}}$ in terms of the invertibility of a certain linear operator. In its turn this invertibility can be expressed in terms of an appropriate admissibility property.

Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of linear operators in $L(X)$ and let $(B, \|\cdot\|_B)$ and $(B', \|\cdot\|_{B'})$ be two admissible spaces. We denote by N the maximum of the corresponding constants given by property 3 in the notion of an admissible space. For simplicity of the notation, we shall denote Y_B and $Y_{B'}$, respectively, by Y and Y'. Now we consider the linear operator $T\colon \mathcal{D}(T) \subset Y' \to Y$ given by

$$(T\mathbf{x})_{m+1} = x_{m+1} - A_m x_m \quad \text{for } m \in \mathbb{Z}, \tag{9}$$

on the domain $\mathcal{D}(T)$ formed by all sequences $\mathbf{x} \in Y'$ such that $T\mathbf{x} \in Y$. For $\mathbf{x} \in \mathcal{D}(T)$ we consider the graph norm

$$\|\mathbf{x}\|_T = \|\mathbf{x}\|_{Y'} + \|T\mathbf{x}\|_Y.$$

The linear operator T is closed (see Proposition 5.2) and so $(\mathcal{D}(T), \|\cdot\|_T)$ is a Banach space. Moreover, the operator

$$T\colon (\mathcal{D}(T), \|\cdot\|_T) \to (Y, \|\cdot\|_Y) \tag{10}$$

is bounded and we denote it from now on simply by T.

We can now formulate our main results. The first one shows that the existence of an exponential dichotomy with respect to a sequence of norms yields the invertibility of the operator T in (9) whenever $B \subset B'$.

Theorem 3.1. *Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of linear operators in $L(X)$ such that $(A_m)_{m\in\mathbb{Z}}$ has an exponential dichotomy with respect to the norms $\|\cdot\|_m$. If $B \subset B'$, then the operator T in* (10) *is invertible.*

Theorem 3.1 is proved in Section 6.

The next result can be seen as a partial converse of Theorem 3.1 (although not necessarily assuming that $B \subset B'$). Given an admissible space B, we define sequences $\alpha_B, \beta_B\colon \mathbb{N}_0 \to \mathbb{R}^+$ by

$$\alpha_B(n) = \|\chi_{\{0,\dots,n\}}\|_B \tag{11}$$

and

$$\beta_B(n) = \inf\left\{\beta > 0 : \sum_{k=0}^{n} |s_k| \le \beta \|\mathbf{s}\|_B \text{ for } \mathbf{s} \in B\right\}. \tag{12}$$

Theorem 3.2. *Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of linear operators in $L(X)$ such that the operator T in* (10) *is invertible. If*

$$\lim_{n\to\infty} \frac{\alpha_B(n)\beta_{B'}(n)}{n^2} = 0, \tag{13}$$

then $(A_m)_{m\in\mathbb{Z}}$ has an exponential dichotomy with respect to the norms $\|\cdot\|_m$.

Theorem 3.2 is proved in Section 7.

Condition (13) holds, in particular, when any of the following conditions is satisfied:

1. $B = B'$ (see Proposition 5.3);
2. $B = \ell^p$ and $B' = \ell^q$ with $p, q \in [1, +\infty]$ such that $1/p - 1/q < 1$;
3. the sequence $\alpha_B(n)\beta_{B'}(n)/n^2$ has 0 as a sublimit;
4. the sequence $\alpha_{B'}(n)\beta_B(n)$ diverges.

See the remarks at the end of Section 5 for details.

The following result is an immediate consequence of Theorems 3.1 and 3.2.

Corollary 3.3. *Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of linear operators in $L(X)$. If $B \subset B'$ and property* (13) *holds, then the following properties are equivalent*:

1. *the sequence $(A_m)_{m\in\mathbb{Z}}$ has an exponential dichotomy with respect to the norms $\|\cdot\|_m$;*
2. *the operator T is invertible.*

As noted above, when $B = B'$ condition (13) holds automatically and so a sequence $(A_m)_{m\in\mathbb{Z}}$ has an exponential dichotomy with respect to some norms $\|\cdot\|_m$ if and only if the operator T with $Y = Y'$ is invertible.

4. Tempered exponential dichotomies

In this section we apply Theorems 3.1 and 3.2 to the notion of a tempered exponential dichotomy. First we introduce the notion of an upper tempered sequence: a two-sided sequence $(c_m)_{m\in\mathbb{Z}}$ of positive real numbers is said to be *upper tempered* if

$$\limsup_{n\to\pm\infty} \frac{1}{|n|} \log c_n \le 0.$$

Now let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of linear operators in $L(X)$. We say that $(A_m)_{m\in\mathbb{Z}}$ has a *tempered exponential dichotomy* if:

1. there exist projections $P_m \colon X \to X$, for $m \in \mathbb{Z}$, satisfying (3) such that the map in (4) is onto and invertible;
2. there exist a constant $\lambda > 0$ and an upper tempered sequence $(D_n)_{n\in\mathbb{Z}}$ such that for $m \ge n$ we have
$$\|\mathcal{A}(m,n)P_n\| \le D_n e^{-\lambda(m-n)}$$
and
$$\|\mathcal{A}(n,m)Q_m\| \le D_m e^{-\lambda(m-n)},$$
where $Q_m = \mathrm{Id} - P_m$ and $\mathcal{A}(n,m) = (\mathcal{A}(m,n)_{|Q_n(X)})^{-1}$.

The following proposition establishes an equivalence between tempered exponential dichotomies and exponential dichotomies with respect to a certain sequence of norms. The proof follows along the lines of the proof analogous results in [2] for nonuniform exponential dichotomies. So, we outline only the differences.

Proposition 4.1. *The following properties are equivalent*:

1. $(A_m)_{m\in\mathbb{Z}}$ *has a tempered exponential dichotomy*;
2. $(A_m)_{m\in\mathbb{Z}}$ *has an exponential dichotomy with respect to some norms* $\|\cdot\|_m$ *satisfying*

$$\|x\| \le \|x\|_m \le C_m\|x\| \tag{14}$$

for $m \in \mathbb{Z}$, $x \in X$ *and some upper tempered sequence* $(C_m)_{m\in\mathbb{Z}}$.

Proof. Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ has a tempered exponential dichotomy. For each $n \in \mathbb{Z}$ we define a norm $\|\cdot\|_n$ on X by

$$\|x\|_n = \sup_{m\ge n}\big(\|\mathcal{A}(m,n)P_nx\|e^{\lambda(m-n)}\big) + \sup_{m\le n}\big(\|\mathcal{A}(m,n)Q_nx\|e^{\lambda(n-m)}\big).$$

One can easily verify that the norm is well defined and that (14) holds taking $C_m = 2D_m$. Since $(D_m)_{m\in\mathbb{Z}}$ is upper tempered, the same happens with the sequence $(C_m)_{m\in\mathbb{Z}}$. One can also show that for $m, n \in \mathbb{Z}$ with $m \ge n$ and $x \in X$, we have

$$\|\mathcal{A}(m,n)P_nx\|_m \le e^{-\lambda(m-n)}\|x\|_n, \quad \|\mathcal{A}(m,n)Q_nx\|_m \le e^{-\lambda(n-m)}\|x\|_n.$$

Hence, $(A_m)_{m\in\mathbb{Z}}$ has an exponential dichotomy with respect to norms $\|\cdot\|_m$.

Now we assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ has an exponential dichotomy with respect to some norms $\|\cdot\|_m$ satisfying (14) for some upper tempered sequence $(C_m)_{m\in\mathbb{Z}}$. Then

$$\|\mathcal{A}(m,n)P_nx\| \le DC_ne^{-\lambda(m-n)}\|x\|$$

and

$$\|\mathcal{A}(n,m)Q_mx\| \le DC_me^{-\lambda(m-n)}\|x\|$$

for $x \in X$ and $m \ge n$. Since $(C_n)_{n\in\mathbb{Z}}$ is upper tempered the same happens with $(DC_n)_{n\in\mathbb{Z}}$. Hence, the sequence $(A_m)_{m\in\mathbb{Z}}$ has a tempered exponential dichotomy taking $D_n = DC_n$. □

Combining Theorems 3.1 and 3.2 with Proposition 4.1 yields a corresponding result for tempered exponential dichotomies.

Theorem 4.2. *For a sequence of linear operators* $(A_m)_{m\in\mathbb{Z}}$ *in* $L(X)$ *and for some norms* $\|\cdot\|_m$ *on* X *satisfying* (14) *for* $m \in \mathbb{Z}$, $x \in X$ *and some upper tempered sequence* $(C_m)_{m\in\mathbb{Z}}$:

1. *if* $(A_m)_{m\in\mathbb{Z}}$ *has a tempered exponential dichotomy and* $B \subset B'$, *then the operator* T *in* (10) *is invertible*;
2. *if the operator* T *in* (10) *is invertible and property* (13) *holds, then* $(A_m)_{m\in\mathbb{Z}}$ *has a tempered exponential dichotomy.*

5. Admissible spaces: additional properties

In this section we describe some additional properties that are needed in the proofs of Theorems 3.1 and 3.2.

The following proposition collects some properties of admissible spaces.

Proposition 5.1. *Let B be an admissible space.*

1. *If $\mathbf{s} = (s_n)_{n\in\mathbb{Z}}$ and $\mathbf{s}' = (s'_n)_{n\in\mathbb{Z}}$ belong to $\mathbb{R}^{\mathbb{Z}}$ and $s_n = s'_n$ for all but finitely many integers $n \in \mathbb{Z}$, then $\mathbf{s} \in B$ if and only if $\mathbf{s}' \in B$.*
2. *If $\mathbf{s}^n \to \mathbf{s}$ in B when $n \to \infty$, then $s^n_k \to s_k$ when $n \to \infty$, for all $k \in \mathbb{Z}$.*
3. *Given $\mathbf{s} \in B$ and $\lambda \in (0,1)$, the sequences $\mathbf{v} = (p_n)_{n\in\mathbb{Z}}$ and $\mathbf{w} = (q_n)_{n\in\mathbb{Z}}$ defined by*
$$p_n = \sum_{m=0}^{+\infty} \lambda^m s_{n-m} \quad \text{and} \quad q_n = \sum_{m=1}^{+\infty} \lambda^m s_{n+m}$$
belong to B and satisfy
$$\|\mathbf{v}\|_B \le \frac{N}{1-\lambda}\|\mathbf{s}\|_B \quad \text{and} \quad \|\mathbf{w}\|_B \le \frac{N\lambda}{1-\lambda}\|\mathbf{s}\|_B.$$

One can use Proposition 5.1 to show that the operator T in (9) is closed.

Proposition 5.2. *The operator T is closed, that is, if $(\mathbf{x}^k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{D}(T)$ converging to $\mathbf{x} \in Y'$ such that $T\mathbf{x}^k$ converges to $\mathbf{y} \in Y$, then $\mathbf{x} \in \mathcal{D}(T)$ and $T\mathbf{x} = \mathbf{y}$.*

Proof. Let $(\mathbf{x}^k)_{k\in\mathbb{N}}$ be such a sequence. It follows from property 2 in Proposition 5.1 that
$$x_{m+1} - A_m x_m = \lim_{k\to+\infty}(x^k_{m+1} - A_m x^k_m) = \lim_{k\to+\infty}(T\mathbf{x}^k)_{m+1} = y_{m+1}$$
for each $m \in \mathbb{Z}$. Hence, $\mathbf{x} \in \mathcal{D}(T)$ and $T\mathbf{x} = \mathbf{y}$. □

Now we consider the sequences $\alpha_B(n)$ and $\beta_B(n)$ given by (11) and (12). One can show that both sequences are nondecreasing. Moreover,
$$\sum_{k=m}^{m+n} |s_k| \le N\beta_B(n)\|\mathbf{s}\|_B$$
for all $\mathbf{s} \in B$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The following proposition can essentially be found in [11] (see also [4]). For completeness we give a short proof.

Proposition 5.3. *If B is an admissible space, then*
$$n + 1 \le \alpha_B(n)\beta_B(n) \le N(2n+1) \quad \text{for all } n \in \mathbb{N}_0.$$

Proof. For the lower bound we note that taking $\mathbf{s} = \chi_{\{0,\dots,n\}}$ yields the inequality
$$\sum_{k=0}^{n} |s_k| \le \beta_B(n)\|\mathbf{s}\|_B \quad \text{and so} \quad n + 1 \le \beta_B(n)\alpha_B(n).$$

For the upper bound, take $n \in \mathbb{N}_0$, $\mathbf{s} \in B$ and consider the sequence $\bar{\mathbf{s}} = \sum_{k=0}^{n} \chi_{\{k\}} |s_k| \in B$. Then $\|\bar{\mathbf{s}}\|_B \le \|\mathbf{s}\|_B$. Using the notation in property 3 in the definition of an admissible space, we obtain

$$\left\| \sum_{k=-n}^{n} \bar{\mathbf{s}}^k \right\|_B \le \sum_{k=-n}^{n} \|\bar{\mathbf{s}}^k\|_B \le N \sum_{k=-n}^{n} \|\bar{\mathbf{s}}\|_B \le N(2n+1)\|\mathbf{s}\|_B.$$

On the other hand,

$$\left\| \sum_{k=-n}^{n} \bar{\mathbf{s}}^k \right\|_B \ge \left\| \sum_{k=0}^{n} |s_k| \chi_{\{0,\dots,n\}} \right\|_B = \sum_{k=0}^{n} |s_k| \alpha_B(n)$$

and so

$$\sum_{k=0}^{n} |s_k| \le \frac{N(2n+1)}{\alpha_B(n)} \|\mathbf{s}\|_B.$$

Hence,

$$\beta_B(n) \le \frac{N(2n+1)}{\alpha_B(n)},$$

which concludes the proof of the proposition. □

Now we show that condition (13) holds when any of the four conditions formulated after Theorem 3.2 is satisfied.

Remark 5.4. When $B = B'$ it follows from Proposition 5.3 that the sequence

$$\alpha_B(n)\beta_{B'}(n)/n = \alpha_B(n)\beta_B(n)/n$$

is bounded and so condition (13) holds.

Remark 5.5. It follows by direct computation that

$$\alpha_{\ell^p}(n) = \begin{cases} (n+1)^{1/p} & \text{if } p < \infty, \\ 1 & \text{if } p = \infty \end{cases}$$

and

$$\beta_{\ell^p}(n) = \begin{cases} (n+1)^{1-1/p} & \text{if } p < \infty, \\ n+1 & \text{if } p = \infty. \end{cases}$$

In fact, writing $1/\infty = 0$ one could simply write

$$\alpha_{\ell^p}(n) = (n+1)^{1/p} \quad \text{and} \quad \beta_{\ell^p}(n) = (n+1)^{1-1/p}$$

for $p \in [1, +\infty]$. Using these formulas, one can easily verify that condition (13) holds for $B = \ell^p$ and $B' = \ell^q$ with $p, q \in [1, +\infty]$ such that $1/p - 1/q < 1$.

Remark 5.6. It follows from Proposition 5.3 for B and B' that

$$n + 1 \le \alpha_B(n)\beta_B(n) \le N(2n+1)$$

and

$$n + 1 \le \alpha_{B'}(n)\beta_{B'}(n) \le N(2n+1).$$

Hence,

$$\frac{(n+1)^2}{n^2\alpha_{B'}(n)\beta_B(n)} \le \frac{\alpha_B(n)\beta_{B'}(n)}{n^2} \le \frac{N^2(2n+1)^2}{n^2\alpha_{B'}(n)\beta_B(n)}$$

and so

$$\frac{1}{\alpha_{B'}(n)\beta_B(n)} \le \frac{\alpha_B(n)\beta_{B'}(n)}{n^2} \le \frac{9N^2}{\alpha_{B'}(n)\beta_B(n)} \tag{15}$$

for all n. It follows from (15) that if the sequence $\alpha_B(n)\beta_{B'}(n)/n^2$ has 0 as a sublimit, then $\alpha_{B'}(n)\beta_B(n)$ has ∞ as a sublimit. But since both sequences $\alpha_{B'}(n)$ and $\beta_B(n)$ are nondecreasing, in fact their product converges to ∞. Hence, it follows again from (15) that condition (13) holds.

Remark 5.7. Alternatively, if we assume from the beginning that the sequence $\alpha_{B'}(n)\beta_B(n)$ diverges, then it follows from (15) that condition (13) holds.

6. Proof of Theorem 3.1

We need to establish the injectivity and the surjectivity of the operator T.

Step 1. Injectivity of T. We first show that the operator T in (9) is one-to-one. Take $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y'$ with $T\mathbf{x} = 0$. Then $x_{n+1} = A_n x_n$ for $n \in \mathbb{Z}$. Moreover, it follows from (3) that

$$P_{n+1}x_{n+1} = A_n P_n x_n \quad \text{and} \quad Q_{n+1}x_{n+1} = A_n Q_n x_n$$

for all $n \in \mathbb{Z}$. For $k \ge 0$, we have

$$P_m x_m = \mathcal{A}(m, m-k)P_{m-k}x_{m-k}$$

and so

$$\begin{aligned}\|P_m x_m\|_m &= \|\mathcal{A}(m, m-k)P_{m-k}x_{m-k}\|_m \\ &\le De^{-\lambda k}\|x_{m-k}\|_{m-k} \\ &\le \frac{DN}{\alpha_{B'}(0)}e^{-\lambda k}\|\mathbf{x}\|_{Y'}.\end{aligned}$$

Letting $k \to +\infty$ we obtain $P_m x_m = 0$ for $m \in \mathbb{Z}$. Similarly, since

$$Q_m x_m = \mathcal{A}(m, m+k)Q_{m+k}x_{m+k}$$

for $k \ge 0$, we have

$$\begin{aligned}\|Q_m x_m\|_m &= \|\mathcal{A}(m, m+k)Q_{m+k}x_{m+k}\|_m \\ &\le De^{-\lambda k}\|x_{m+k}\|_{m+k} \\ &\le \frac{DN}{\alpha_{B'}(0)}e^{-\lambda k}\|\mathbf{x}\|_{Y'}.\end{aligned}$$

Letting $k \to +\infty$ we obtain $Q_m x_m = 0$ for $m \in \mathbb{Z}$. Therefore, $x_m = 0$ for $m \in \mathbb{Z}$ and so $\mathbf{x} = 0$.

Step 2. Surjectivity of T. Now we show that the operator T is onto. Take $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y$. Since $B \subset B'$, we have $Y \subset Y'$ and so $\mathbf{y} \in Y'$. For each $n \in \mathbb{Z}$, let

$$v_n = \sum_{m=0}^{+\infty} \mathcal{A}(n, n-m) P_{n-m} y_{n-m}, \quad w_n = \sum_{m=1}^{+\infty} \mathcal{A}(n, n+m) Q_{n+m} y_{n+m}.$$

It follows from (5) and (6) that

$$\begin{aligned} \sum_{m=0}^{+\infty} \|\mathcal{A}(n, n-m) P_{n-m} y_{n-m}\|_n &\le \sum_{m=0}^{+\infty} D e^{-\lambda m} \|y_{n-m}\|_{n-m} \\ &\le \frac{DN}{\alpha_{B'}(0)} \|\mathbf{y}\|_{Y'} \sum_{m=0}^{+\infty} e^{-\lambda m} \end{aligned} \tag{16}$$

and

$$\begin{aligned} \sum_{m=1}^{+\infty} \|\mathcal{A}(n, n+m) Q_{n+m} y_{n+m}\|_n &\le \sum_{m=1}^{+\infty} D e^{-\lambda m} \|y_{n+m}\|_{n+m} \\ &\le \frac{DN}{\alpha_{B'}(0)} \|\mathbf{y}\|_{Y'} \sum_{m=1}^{+\infty} e^{-\lambda m}. \end{aligned} \tag{17}$$

This shows that v_n and w_n are well defined. Moreover, it follows from (16) and (17) together with Proposition 5.1 and property 2 in the definition of an admissible space that $\mathbf{v}$ and $\mathbf{w}$ belong to Y'. Now let $\mathbf{x} = \mathbf{v} - \mathbf{w} \in Y'$. One can easily verify that

$$x_{n+1} - A_n x_n = y_{n+1} \quad \text{for all } n \in \mathbb{Z},$$

and so $T\mathbf{x} = \mathbf{y}$. This completes the proof of Theorem 3.1.

7. Proof of Theorem 3.2

Again we divide the proof into steps.

Step 1. Invariant subspaces. For each $n \in \mathbb{Z}$, let Z_n be the set of all $x \in X$ for which there exists a sequence $\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \in Y'$ with $x_n = x$ and

$$x_{m+1} = A_m x_m \quad \text{for } m \ge n. \tag{18}$$

Moreover, let W_n be the set of all $x \in X$ for which there exists a sequence $\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \in Y'$ with $x_n = x$ and

$$x_m = A_{m-1} x_{m-1} \quad \text{for } m \le n. \tag{19}$$

Note that Z_n and W_n are subspaces of X.

Lemma 7.1. *For each $n \in \mathbb{Z}$ we have*

$$X = Z_n \oplus W_n. \tag{20}$$

Proof. Take $v \in X$ and define a sequence $\mathbf{y} = (y_m)_{m\in\mathbb{Z}}$ by

$$y_m = \begin{cases} v & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Clearly, $\mathbf{y} \in Y$ and since T is invertible, there exists $\mathbf{x} \in Y'$ such that $T\mathbf{x} = \mathbf{y}$, that is,

$$x_n - A_{n-1}x_{n-1} = v \tag{21}$$

and

$$x_m = A_{m-1}x_{m-1} \quad \text{for } m \neq n. \tag{22}$$

It follows from (22) that

$$x_{m+1} = A_m x_m \quad \text{for } m \geq n,$$

and

$$x_m = A_{m-1}x_{m-1} \quad \text{for } m \leq n-1.$$

Hence, $x_n \in Z_n$ and $x_{n-1} \in W_{n-1}$. Then $A_{n-1}x_{n-1} \in W_n$ and it follows from (21) that $v \in Z_n + W_n$.

Now we show that $Z_n \cap W_n = \{0\}$. Take $v \in Z_n \cap W_n$ and let $\mathbf{x} = (x_m)_{m\in\mathbb{Z}}$ and $\mathbf{x}' = (x'_m)_{m\in\mathbb{Z}}$ be sequences in Y' with $x_n = x'_n = v$ satisfying (18) and (19), respectively. We define a sequence $\mathbf{y} = (y_m)_{m\in\mathbb{Z}}$ by

$$y_m = \begin{cases} x_m & \text{if } m \geq n, \\ x'_m & \text{if } m < n. \end{cases}$$

Then $\mathbf{y} \in Y'$ and $T\mathbf{y} = 0$. Since T is invertible, $\mathbf{y} = 0$ and so $y_n = v = 0$. □

We denote by $P_n \colon X \to X$ and $Q_n \colon X \to X$ the projections associated with the splitting in (20).

Lemma 7.2. *Property* (3) *holds.*

Proof. Let $n \in \mathbb{Z}$ and note that it suffices to show that

$$\mathcal{A}(m,n)Z_n \subset Z_m \quad \text{and} \quad \mathcal{A}(m,n)W_n \subset W_m$$

for $m \geq n$. It is clear from the definition that if $x \in Z_n$ then $\mathcal{A}(m,n)x \in Z_m$ for $m \geq n$. Now given $x \in W_n$ and $m \in \mathbb{Z}$ with $m \geq n$, let $\mathbf{x} = (x_m)_{m\in\mathbb{Z}}$ be a sequence in Y' with $x_n = x$ satisfying (19). Consider a new sequence $\mathbf{y} = (y_m)_{m\in\mathbb{Z}}$ given by

$$y_k = \begin{cases} \mathcal{A}(k,n)x & \text{if } n < k \leq m, \\ x_k & \text{otherwise.} \end{cases}$$

Then $\mathbf{y} \in Y'$ and $y_k = A_{k-1}y_{k-1}$ for $k \leq m$. So $y_m = \mathcal{A}(m,n)x \in W_m$. □

Step 2. Invertibility along the spaces $Q_n(X)$. Now we establish the invertibility of the dynamics along the spaces $Q_n(X)$.

Lemma 7.3. *The map $A_{n|Q_n(X)}\colon Q_n(X) \to Q_{n+1}(X)$ is invertible for $n \in \mathbb{Z}$.*

Proof. Take $x \in Q_n(X) = W_n$ such that $A_n x = 0$. Then $\mathcal{A}(m,n)x = 0$ for all $m \ge n$ and so $x \in Z_n$. Hence, $x \in Z_n \cap W_n$ and so $x = 0$. This shows that the map in the lemma is one-to-one.

To show that the map is onto, take $x \in Q_{n+1}(X)$ and let $\mathbf{x} = (x_m)_{m\in\mathbb{Z}}$ be a sequence in Y' with $x_{n+1} = x$ satisfying (19) for $m \le n+1$. Clearly, $x_n \in Q_n(X)$ and $A_n x_n = x_{n+1} = x$, which shows that the map is onto. □

Step 3. Bound for the projections.

Lemma 7.4. *There exists $L > 0$ such that $\|P_n x\|_n \le L\|x\|_n$ for all $n \in \mathbb{Z}$ and $x \in X$.*

Proof. Take $n \in \mathbb{Z}$ and $v \in X$. Using the same notation as in the proof of Lemma 7.1 we have $P_n v = x_n = (T^{-1}\mathbf{y})_n$. Now we observe that since the operator T in (9) is invertible, its inverse

$$T^{-1}\colon (Y, \|\cdot\|_Y) \to (\mathcal{D}(T), \|\cdot\|_T)$$

is a bounded linear operator. Therefore,

$$\begin{aligned}\|P_n v\|_n = \|(T^{-1}\mathbf{y})_n\|_n &\le \frac{N}{\alpha_{B'}(0)}\|T^{-1}\mathbf{y}\|_{Y'}\\ &\le \frac{N}{\alpha_{B'}(0)}\|T^{-1}\| \cdot \|\mathbf{y}\|_{Y'} \le \frac{N^2\alpha_B(0)}{\alpha_{B'}(0)}\|T^{-1}\| \cdot \|v\|_n.\end{aligned}$$

This establishes the desired inequality. □

Finally, we establish the exponential bounds in (5) and (6).

Step 4. Bounds along the spaces $P_n(X)$. We first obtain two auxiliary results.

Lemma 7.5. *There exists $M > 0$ such that for each $n \in \mathbb{Z}$ and $x \in P_n(X)$ we have*

$$\|\mathcal{A}(m,n)x\|_m \le M\|x\|_n \quad \text{for } m \ge n. \tag{23}$$

Proof. Given $n \in \mathbb{Z}$ and $v \in P_n(X)$, let

$$y_m = \begin{cases} v & \text{if } m = n, \\ 0 & \text{if } m \ne n \end{cases} \quad \text{and} \quad x_m = \begin{cases} \mathcal{A}(m,n)v & \text{if } m \ge n, \\ 0 & \text{if } m < n. \end{cases}$$

Clearly, $\mathbf{y} = (y_m)_{m\in\mathbb{Z}} \in Y$ and since $v \in P_n(X)$, the sequence $\mathbf{x} = (x_m)_{m\in\mathbb{Z}}$ belongs to Y'. Moreover, $\mathbf{x} = T^{-1}\mathbf{y}$ and

$$\|\mathbf{x}\|_{Y'} \le \|\mathbf{x}\|_T \le \|T^{-1}\| \cdot \|\mathbf{y}\|_Y. \tag{24}$$

Using the properties of an admissible space we obtain

$$\|\mathcal{A}(m,n)v\|_m = \|x_m\|_m \le \frac{N}{\alpha_{B'}(0)}\|\mathbf{x}\|_{Y'}$$

for $m \geq n$ and

$$\|\mathbf{y}\|_Y \leq N\alpha_B(0)\|v\|_n.$$

Combining these inequalities with (24) we find that

$$\|\mathcal{A}(m,n)v\|_m \leq \frac{N^2\alpha_B(0)}{\alpha_{B'}(0)}\|T^{-1}\| \cdot \|v\|_n$$

for $m \geq n$. This establishes the bound in (23). □

Lemma 7.6. *There exists $p \in \mathbb{N}$ such that for $n \in \mathbb{Z}$, $m \geq p$ and $x \in P_n(X)$ we have*

$$\|\mathcal{A}(n+m,n)x\|_{n+m} \leq \frac{1}{2}\|x\|_n. \tag{25}$$

Proof. Given $n \in \mathbb{Z}$ and $x \in P_n(X)$, let $x_m = \mathcal{A}(m,n)x$ for $m \geq n$ and assume that there exists $b > n$ such that $\|x_b\|_b > \|x_n\|_n/2$. It follows from (23) that

$$\frac{\|x_n\|_n}{2} < \|x_b\|_b = \|\mathcal{A}(b,m)x_m\|_b \leq M\|x_m\|_m$$

for $n \leq m \leq b$. Therefore,

$$\frac{1}{2M}\|x_n\|_n < \|x_m\|_m \leq M\|x_n\|_n \tag{26}$$

for $n \leq m \leq b$. Now let

$$w_m = \begin{cases} 0 & \text{if } m < n, \\ x_m/\|x_m\|_m & \text{if } n \leq m < b, \\ 0 & \text{if } m \geq b. \end{cases}$$

Clearly, $\mathbf{w} = (w_m)_{m\in\mathbb{Z}} \in Y$. We define a sequence $\mathbf{v} = (v_m)_{m\in\mathbb{Z}}$ by

$$v_m = \begin{cases} 0 & \text{if } m < n, \\ x_m \sum_{k=n}^{m} \frac{1}{\|x_k\|_k} & \text{if } n \leq m < b, \\ x_m \sum_{k=n}^{b-1} \frac{1}{\|x_k\|_k} & \text{if } m \geq b. \end{cases}$$

Since $x \in P_n(X)$, we have $\mathbf{v} \in Y'$ and so $\mathbf{v} = T^{-1}\mathbf{w}$. We obtain

$$\begin{aligned}\sum_{k=n}^{b-1}\|v_k\|_k &\leq N\beta_{B'}(b-n-1)\|\mathbf{v}\|_{Y'} \\ &\leq N\beta_{B'}(b-n-1)\|T^{-1}\| \cdot \|\mathbf{w}\|_Y\end{aligned}$$

and since $\|\mathbf{w}\|_Y \leq N\alpha_B(b-n-1)$,

$$\sum_{k=n}^{b-1}\|v_k\|_k \leq N^2\|T^{-1}\|\alpha_B(b-n-1)\beta_{B'}(b-n-1).$$

On the other hand, by (26), we have

$$\sum_{k=n}^{b-1}\|v_k\|_k = \sum_{k=n}^{b-1}\sum_{j=n}^{k}\frac{\|x_k\|_k}{\|x_j\|_j} > \frac{1}{2M^2}\sum_{k=n}^{b-1}\sum_{j=n}^{k}1$$
$$= \frac{1}{2M^2}\cdot\frac{(b-n)(b-n+1)}{2} > \frac{(b-n)^2}{4M^2}.$$

Therefore,

$$\frac{\alpha_B(b-n-1)\beta_{B'}(b-n-1)}{(b-n)^2} > \frac{1}{4M^2N^2\|T^{-1}\|}.$$

Since the right hand side of this inequality is positive, it follows from (13) that there exists $p \in \mathbb{N}$ such that $b-n<p$. This shows that inequality (25) holds for $m \ge p$. ☐

Lemma 7.7. *There exist $\lambda, D > 0$ such that for each $n \in \mathbb{Z}$ and $x \in P_n(X)$ we have*

$$\|\mathcal{A}(m,n)x\|_m \le De^{-\lambda(m-n)}\|x\|_n \quad \text{for } m \ge n. \tag{27}$$

Proof. Take $m, n \in \mathbb{Z}$ with $m \ge n$ and write $m-n = kp+r$, with $k \in \mathbb{N}_0$ and $0 \le r < p$. By (23) and (25), for $x \in P_n(X)$ we obtain

$$\begin{aligned}\|\mathcal{A}(m,n)x\|_m &= \|\mathcal{A}(n+r+kp,n)x\|_{n+r+kp}\\ &= \|\mathcal{A}(n+r+kp,n+r)\mathcal{A}(n+r,n)x\|_{n+r+kp}\\ &\le \frac{1}{2^k}\|\mathcal{A}(n+r,n)x\|_{n+r} \le \frac{M}{2^k}\|x\|_n.\end{aligned}$$

Since $0 \le r < p$, we have $k \ge (m-n)/p - 1$ and so $1/2^k \le 2/2^{(m-n)/p}$. Therefore,

$$\|\mathcal{A}(m,n)x\|_m \le 2Me^{-(m-n)\log 2/p}\|x\|_n.$$

Hence, property (27) holds taking $D = 2M$ and $\lambda = \log 2/p$. This completes the proof of the lemma. ☐

Step 5. Bounds along the spaces $Q_n(X)$. Now we consider the spaces $Q_n(X)$. Again, we first obtain two auxiliary results.

Lemma 7.8. *There exists $M > 0$ such that for each $n \in \mathbb{Z}$ and $x \in Q_n(X)$ we have*

$$\|\mathcal{A}(m,n)\|_m \le M\|x\|_n \quad \text{for } m \le n. \tag{28}$$

Proof. Given $n \in \mathbb{Z}$ and $v \in Q_n(X)$, let

$$y_m = \begin{cases} -v & \text{if } m = n, \\ 0 & \text{if } m \ne n \end{cases} \quad \text{and} \quad x_m = \begin{cases} \mathcal{A}(m,n)v & \text{if } m < n, \\ 0 & \text{if } m \ge n. \end{cases}$$

Clearly, $\mathbf{y} = (y_m)_{m\in\mathbb{Z}} \in Y$ and since $v \in Q_n(X)$, the sequence $\mathbf{x} = (x_m)_{m\in\mathbb{Z}}$ belongs to Y'. Moreover, $\mathbf{x} = T\mathbf{y}$ and

$$\|\mathbf{x}\|_{Y'} \le \|\mathbf{x}\|_T \le \|T^{-1}\|\cdot\|\mathbf{y}\|_Y. \tag{29}$$

Using the properties of an admissible space we obtain

$$\|\mathcal{A}(m,n)v\|_m = \|x_m\|_m \le \frac{N}{\alpha_{B'}(0)}\|\mathbf{x}\|_{Y'}$$

for $m < n$ and

$$\|\mathbf{y}\|_Y \le N\alpha_B(0)\|v\|_n.$$

Combining these inequalities with (29) we find that

$$\|\mathcal{A}(m,n)v\|_m \le \frac{N^2\alpha_B(0)}{\alpha_{B'}(0)}\|T^{-1}\| \cdot \|v\|_n$$

for $m < n$. For $m = n$ we obtain $\mathcal{A}(m,n)v = v$ and so taking

$$M = \max\left\{\frac{N^2\alpha_B(0)}{\alpha_{B'}(0)}\|T^{-1}\|, 1\right\}$$

we obtain inequality (28). □

Lemma 7.9. *There exists $p \in \mathbb{N}$ such that for $n \in \mathbb{Z}$, $m \ge p$ and $x \in Q_n(X)$ we have*

$$\|\mathcal{A}(n-m,n)x\|_{n-m} \le \frac{1}{2}\|x\|_n. \tag{30}$$

Proof. Given $n \in \mathbb{Z}$ and $x \in Q_n(X)$, let $x_m = \mathcal{A}(m,n)x$ for $m \le n$ and assume that there exists $b < n$ such that $\|x_b\|_b > \|x_n\|_n/2$. It follows from (28) that

$$\frac{\|x_n\|_n}{2} < \|x_b\|_b = \|\mathcal{A}(b,m)x_m\|_b \le M\|x_m\|_m$$

for $b \le m \le n$. Therefore,

$$\frac{1}{2M}\|x_n\|_n < \|x_m\|_m \le M\|x_n\|_n \tag{31}$$

for $b \le m \le n$. Now let

$$w_m = \begin{cases} 0 & \text{if } m \le b, \\ -x_n/\|x_n\|_n & \text{if } b < m \le n, \\ 0 & \text{if } m > n. \end{cases}$$

Clearly, $\mathbf{w} = (w_m)_{m\in\mathbb{Z}} \in Y$. We define a sequence $\mathbf{v} = (v_m)_{m\in\mathbb{Z}}$ by

$$v_m = \begin{cases} x_m \sum_{k=b+1}^{n} \frac{1}{\|x_k\|_k} & \text{if } m < b, \\ x_m \sum_{k=m+1}^{n} \frac{1}{\|x_k\|_k} & \text{if } b \le m < n, \\ 0 & \text{if } n \ge m. \end{cases}$$

Since $x \in Q_n(X)$, we have $\mathbf{v} \in Y'$ and so $\mathbf{v} = T^{-1}\mathbf{w}$. We have

$$\begin{aligned}\sum_{k=b}^{n-1}\|v_k\|_k &\le N\beta_{B'}(n-b-1)\|\mathbf{v}\|_{Y'} \\ &\le N\beta_{B'}(n-b-1)\|T^{-1}\| \cdot \|\mathbf{w}\|_Y\end{aligned}$$

and since $\|\mathbf{w}\|_Y \le N\alpha_B(n-b-1)$,

$$\sum_{k=b}^{n-1} \|v_k\|_k \le N^2 \|T^{-1}\| \alpha_B(n-b-1)\beta_{B'}(n-b-1).$$

On the other hand, by (31), we have

$$\begin{aligned}\sum_{k=b}^{n-1} \|v_k\|_k &= \sum_{k=b}^{n-1} \sum_{j=k+1}^{n} \frac{\|x_k\|_k}{\|x_j\|_j} > \frac{1}{2M^2} \sum_{k=b}^{n-1} \sum_{j=k+1}^{n} 1 \\ &= \frac{1}{2M^2} \cdot \frac{(n-b)(n-b+1)}{2} > \frac{(n-b)^2}{4M^2}.\end{aligned}$$

Therefore,

$$\frac{\alpha_B(n-b-1)\beta_{B'}(n-b-1)}{(n-b)^2} > \frac{1}{4M^2N^2\|T^{-1}\|}.$$

Since the right hand side is positive, it follows from (13) that there exists $p \in \mathbb{N}$ such that $n-b<p$. Hence, inequality (30) holds for $m \ge p$. □

Lemma 7.10. *There exist $\lambda, D > 0$ such that for each $n \in \mathbb{Z}$ and $x \in Q_n(X)$ we have*

$$\|\mathcal{A}(m,n)x\|_m \le De^{-\lambda(n-m)}\|x\|_n \quad \text{for } m \le n. \tag{32}$$

Proof. Take $m, n \in \mathbb{Z}$ with $m \le n$ and write $n-m = kp+r$ with $k \in \mathbb{N}_0$ and $0 \le r < p$. By (28) and (30), for $x \in Q_m(X)$ we obtain

$$\begin{aligned}\|\mathcal{A}(m,n)x\|_m &= \|\mathcal{A}(n-r-kp,n)x\|_{n-r-kp} \\ &= \|\mathcal{A}(n-r-kp,n-r)\mathcal{A}(n-r,n)x\|_{n-r-kp} \\ &\le \frac{1}{2^k}\|\mathcal{A}(n-r,n)x\|_{n-r} \le \frac{M}{2^k}\|x\|_n.\end{aligned}$$

Since $0 \le r < p$, we have $k \ge (n-m)/p - 1$ and so $1/2^k \le 2/2^{(n-m)/p}$. Therefore,

$$\|\mathcal{A}(m,n)x\|_m \le 2Me^{-(n-m)\log 2/p}\|x\|_n.$$

Hence property (32) holds taking $D = 2M$ and $\lambda = \log 2/p$. This completes the proof of Lemma 7.10. □

8. Robustness under perturbations

In this section we describe an application of the characterization of an exponential dichotomy given by Theorems 3.1 and 3.2 to robustness. More precisely, we show that the notion of an exponential dichotomy under sufficiently small parameterized $C^{\kappa,\alpha}$ perturbations (more precisely, by C^κ maps with Hölder continuous κth derivative with Hölder exponent α) persists and that their stable and unstable spaces are as regular as the perturbation. To the best of our knowledge the case of C^1 perturbations for the discrete time was first considered in [3], although with an unrelated approach using fixed points problems.

Let $(A_m)_{m\in\mathbb{Z}}$ be a two-sided sequence of bounded linear operators acting on a Banach space X. It induces the dynamics in (1). Now we consider a perturbation $(B_m(\lambda))_{m\in\mathbb{Z}}$ given by continuous functions $B_m\colon I \to L(X)$, for $m \in \mathbb{Z}$, on a Banach space I. Thus we consider the perturbed dynamics

$$x_m = (A_{m-1} + B_{m-1}(\lambda))x_{m-1} \quad \text{for } m \in \mathbb{Z},$$

on the space X. For simplicity of the exposition we introduce the following notations. For each $m \in \mathbb{Z}$ we write

$$\|B_m(\lambda)\|' = \sup_{x\neq 0} \frac{\|B_m(\lambda)x\|_{m+1}}{\|x\|_m}.$$

Moreover, when the maps $\lambda \mapsto B_m(\lambda)$, for $m \in \mathbb{Z}$, have derivatives up to order $\kappa \in \mathbb{N}$, for each $m \in \mathbb{Z}$ and $i = 1, \ldots, \kappa$ we write

$$\|B_m^{(i)}(\lambda)\|' = \sup_{x\neq 0} \sup_{\mu_1\neq 0} \cdots \sup_{\mu_i\neq 0} \frac{\|B_m^{(i)}(\lambda)(\mu_1, \ldots, \mu_i)x\|_{m+1}}{\|\mu_1\| \cdots \|\mu_i\| \cdot \|x\|_m},$$

where the multilinear maps

$$B_m^{(i)}(\lambda)\colon I^i \to L(X)$$

are the derivatives of order i. We shall also write $B_m(\lambda) = B_m^{(0)}(\lambda)$.

Given an integer $\kappa \in \mathbb{N} \cup \{0\}$, we say that the perturbation $(B_m(\lambda))_{m\in\mathbb{Z}}$ is of class C^κ if:

1. all maps $\lambda \mapsto B_m(\lambda)$, for $m \in \mathbb{Z}$, have derivatives up to order κ (when $\kappa = 0$ this means that they are continuous, which is already assumed from the beginning);
2. the derivatives up to order κ are continuous uniformly on $m \in \mathbb{Z}$, that is, given $\lambda \in I$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|B_m^{(i)}(\lambda) - B_m^{(i)}(\lambda')\|' \leq \varepsilon \tag{33}$$

   for each $i = 0, \ldots, \kappa$ and all $m \in \mathbb{Z}$ and $\lambda' \in I$ with $\|\lambda - \lambda'\| \leq \delta$.

Moreover, given $\kappa \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0, 1]$, we say that the perturbation $(B_m(\lambda))_{m\in\mathbb{Z}}$ is of class $C^{\kappa,\alpha}$ if it is of class C^κ and the derivatives of order κ are locally Hölder continuous with Hölder exponent α uniformly on $m \in \mathbb{Z}$, that is, given $\lambda \in I$, there exist $\delta, L > 0$ such that

$$\|B_m^{(\kappa)}(\lambda) - B_m^{(\kappa)}(\lambda')\|' \leq L\|\lambda - \lambda'\|^\alpha \tag{34}$$

for all $m \in \mathbb{Z}$ and $\lambda' \in I$ with $\|\lambda - \lambda'\| \leq \delta$. Of course, the perturbation is of class $C^{\kappa,1}$ if it is of class C^κ and the derivatives of order κ are locally Lipschitz continuous uniformly on $m \in \mathbb{Z}$. We shall also write $C^\kappa = C^{\kappa,0}$.

The following theorem is our robustness result.

Theorem 8.1. *Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ has an exponential dichotomy with respect to the norms $\|\cdot\|_m$ and that $(B_m(\lambda))_{m\in\mathbb{Z}}$ is of class $C^{\kappa,\alpha}$*

for some $\kappa \in \mathbb{N} \cup \{0\}$ *and* $\alpha \in [0,1]$. *If*

$$c := \sup_{m\in\mathbb{Z}} \sup_{\lambda\in I} \|B_m(\lambda)\|' \tag{35}$$

is sufficiently small, then:

1. *for each* $\lambda \in I$ *the sequence* $(A_m + B_m(\lambda))_{m\in\mathbb{Z}}$ *has an exponential dichotomy with respect to the norms* $\|\cdot\|_m$;
2. *for the corresponding projections* $P_{m,\lambda}$ *onto the stable spaces, if*

$$d_i := \sup_{m\in\mathbb{Z}} \sup_{\lambda\in I} \|B_m^{(i)}(\lambda)\|' < +\infty \quad \textit{for } i = 1, \dots, \kappa,$$

   then each map $\lambda \mapsto P_{m,\lambda}$ *is of class* $C^{\kappa,\alpha}$.

Proof. We divide the proof into steps.

Step 1. Existence of an exponential dichotomy. Given an admissible space $B = (B, \|\cdot\|_B)$, let

$$Y = \big\{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}} : (\|x_n\|_n)_{n\in\mathbb{Z}} \in B\big\}$$

be the associated Banach space equipped with the norm

$$\|\mathbf{x}\|_Y = \big\|(\|x_n\|_n)_{n\in\mathbb{Z}}\big\|_B.$$

Moreover, we consider the linear operator $T\colon \mathcal{D}(T) \subset Y \to Y$ given by

$$(T\mathbf{x})_{m+1} = x_{m+1} - A_m x_m \quad \text{for } m \in \mathbb{Z},$$

on the domain $\mathcal{D}(T)$ formed by all sequences $\mathbf{x} \in Y$ such that $T\mathbf{x} \in Y$. For $\mathbf{x} \in \mathcal{D}(T)$ we consider the graph norm

$$\|\mathbf{x}\|_T = \|\mathbf{x}\|_Y + \|T\mathbf{x}\|_Y.$$

Since $(A_m)_{m\in\mathbb{Z}}$ has an exponential dichotomy, by Theorem 3.1 the operator T is invertible. For each $\lambda \in I$, we consider the sequence $(A_m + B_m(\lambda))_{m\in\mathbb{Z}}$ and the associated operator T_λ given by

$$(T_\lambda\mathbf{x})_{m+1} = x_{m+1} - (A_m + B_m(\lambda))x_m \quad \text{for } m \in \mathbb{Z},$$

on the domain $\mathcal{D}(T_\lambda)$ formed by all sequences $\mathbf{x} \in Y$ such that $T\mathbf{x} \in Y$. By (35) we have

$$\|(T\mathbf{x} - T_\lambda\mathbf{x})_{m+1}\|_{m+1} = \|B_m(\lambda)x_m\|_{m+1} \le c\|x_m\|_m$$

for each $\mathbf{x} \in Y$, $m \in \mathbb{Z}$ and $\lambda \in I$. Therefore, $T\mathbf{x} - T_\lambda\mathbf{x} \in Y$ and thus $\mathcal{D}(T) = \mathcal{D}(T_\lambda)$, for each $\lambda \in I$. Moreover,

$$\|(T - T_\lambda)\mathbf{x}\|_Y \le cN\|\mathbf{x}\|_Y \le cN\|\mathbf{x}\|_T$$

for $\mathbf{x} \in \mathcal{D}(T)$ and so $T_\lambda\colon (\mathcal{D}(T), \|\cdot\|_T) \to Y$ is bounded for each $\lambda \in I$. When c is sufficiently small, the operator T_λ is also invertible and it follows from Theorem 3.2 that the sequence $(A_m+B_m(\lambda))_{m\in\mathbb{Z}}$ has an exponential dichotomy with respect to the norms $\|\cdot\|_m$.

Moreover, for each $m \in \mathbb{Z}$, $\lambda \in I$ and $v \in X$ it follows from the proof of Theorem 3.2 (see the proof of Lemma 7.1) that the associated projections $P_{m,\lambda}$ are defined by

$$P_{m,\lambda} v = (T_\lambda^{-1} \mathbf{y})_m, \tag{36}$$

where $\mathbf{y} = (y_n)_{n \in \mathbb{Z}}$ is given by

$$y_n = \begin{cases} v & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \tag{37}$$

We want to show that each map $\lambda \mapsto P_{m,\lambda}$ is of class $C^{\kappa,\alpha}$. We start by showing that the map $\lambda \mapsto T_\lambda$ is of class C^κ.

Step 2. Continuity. First take $\kappa = 0$. By (33), given $\lambda \in I$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|B_m(\lambda) - B_m(\lambda')\|' \leq \varepsilon$$

for all $m \in \mathbb{Z}$ and $\lambda' \in I$ with $\|\lambda - \lambda'\| \leq \delta$. Since

$$(T_\lambda \mathbf{x} - T_{\lambda'} \mathbf{x})_{m+1} = -(B_m(\lambda) - B_m(\lambda')) x_m \quad \text{for } m \in \mathbb{Z},$$

we obtain

$$\|(T_\lambda - T_{\lambda'})\mathbf{x}\|_Y \leq \varepsilon N \|\mathbf{x}\|_Y \leq \varepsilon N \|\mathbf{x}\|_T$$

and so

$$\|T_\lambda - T_{\lambda'}\| \leq \varepsilon N \quad \text{whenever } \|\lambda - \lambda'\| \leq \delta.$$

In other words, the map $\lambda \mapsto T_\lambda$ is continuous.

Step 3. Construction of linear maps. Before proceeding we define maps

$$G_i \colon I \to L(I^i, L(\mathcal{D}(T), Y))$$

for $i = 1, \ldots, \kappa$ by

$$([G_i(\lambda)\nu_i]\mathbf{x})_{m+1} = -[B_m^{(i)}(\lambda)\nu_i] x_m$$

for $m \in \mathbb{Z}$, where $\nu_i = (\mu_1, \ldots, \mu_i) \in I^i$. We have

$$\begin{aligned} \|([G_i(\lambda)\nu_i]\mathbf{x})_{m+1}\|_{m+1} &= \|[B_m^{(i)}(\lambda)\nu_i] x_m\|_{m+1} \\ &\leq d_i \|\mu_1\| \cdots \|\mu_i\| \cdot \|x_m\|_m \end{aligned}$$

and so indeed $[G_i(\lambda)\nu_i]\mathbf{x} \in Y$ for $\mathbf{x} \in \mathcal{D}(T)$. Moreover,

$$\|[G_i(\lambda)\nu_i]\mathbf{x}\|_Y \leq d_i N \|\mu_1\| \cdots \|\mu_i\| \cdot \|\mathbf{x}\|_T$$

and so

$$\|G_i(\lambda)\nu_i\| \leq d_i N \|\mu_1\| \cdots \|\mu_i\| \quad \text{and} \quad \|G_i(\lambda)\| \leq d_i N.$$

Therefore,

$$G_i(\lambda)\nu_i \in L(\mathcal{D}(T), Y) \quad \text{and} \quad G_i(\lambda) \in L(I^i, L(\mathcal{D}(T), Y)).$$

Step 4. C^κ regularity. Now take $\kappa > 0$. We proceed by induction on i. Namely, assume that the map $R(\lambda) = T_\lambda$ has derivatives up to order $i < \kappa$ given by $d^j_\lambda R = G_i(\lambda)$ for $j = 1, \ldots, i$. We have

$$\begin{aligned}
&\big([(d^i_{\lambda'} R - d^i_\lambda R)\nu_i - G_{i+1}(\lambda)(\nu_i, \lambda' - \lambda)]\mathbf{x}\big)_{m+1} \\
&= \Big[\big(B^{(i)}_m(\lambda) - B^{(i)}_m(\lambda')\big)\nu_i + B^{(i+1)}_m(\lambda)(\nu_i, \lambda' - \lambda)\Big] x_m \\
&= -\left[\left(\int_0^1 [B^{(i+1)}_m(\lambda + t(\lambda' - \lambda)) - B^{(i+1)}_m(\lambda)]\, dt\right)(\nu_i, \lambda' - \lambda)\right] x_m
\end{aligned}$$

for $m \in \mathbb{Z}$. By (33), given $\lambda \in I$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}
&\big\|\big([(d^i_{\lambda'} R - d^i_\lambda R)\nu_i - G_{i+1}(\lambda)(\nu_i, \lambda' - \lambda)]\mathbf{x}\big)_{m+1}\big\|_{m+1} \\
&\le \varepsilon \|\mu_1\| \cdots \|\mu_i\| \cdot \|\lambda' - \lambda\| \cdot \|x_m\|_m
\end{aligned}$$

for each $m \in \mathbb{Z}$ and $\lambda' \in I$ with $\|\lambda - \lambda'\| \le \delta$. Therefore,

$$\begin{aligned}
\|(d^i_{\lambda'} R - d^i_\lambda R)\nu_i - G_{i+1}(\lambda)(\nu_i, \lambda' - \lambda)\| &\le \varepsilon N \|\mu_1\| \cdots \|\mu_i\| \cdot \|\lambda' - \lambda\| \cdot \|\mathbf{x}\|_Y \\
&\le \varepsilon N \|\mu_1\| \cdots \|\mu_i\| \cdot \|\lambda' - \lambda\| \cdot \|\mathbf{x}\|_T
\end{aligned}$$

for $\mathbf{x} \in \mathcal{D}(T)$ and $\lambda' \in I$ satisfying $\|\lambda - \lambda'\| < \delta$. Hence, the map $d^i_\lambda R$ is differentiable, with derivative $d^{i+1}_\lambda R = G_{i+1}(\lambda)$.

This shows that R has derivatives up to order κ. In order to show that it is of class C^κ it remains to show that $d^\kappa_\lambda R$ is continuous. By (33), given $\lambda \in I$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|B^{(\kappa)}_m(\lambda) - B^{(\kappa)}_m(\lambda')\|' \le \varepsilon$$

for all $m \in \mathbb{Z}$ and $\lambda' \in I$ with $\|\lambda - \lambda'\| \le \delta$. Since

$$((d^\kappa_\lambda R - d^\kappa_{\lambda'} R)\nu_\kappa \mathbf{x})_{m+1} = -(B^{(\kappa)}_m(\lambda) - B^{(\kappa)}_m(\lambda'))\nu_\kappa x_m \quad \text{for } m \in \mathbb{Z},$$

we obtain

$$\|d^\kappa_\lambda R - d^\kappa_{\lambda'} R\| \le \varepsilon N \quad \text{whenever } \|\lambda - \lambda'\| \le \delta$$

and so $d^\kappa_\lambda R$ is continuous. This readily implies that $\lambda \mapsto T_\lambda^{-1}$ is of class C^κ.

Step 5. $C^{\kappa,\alpha}$ regularity. Finally, we show that when $\alpha > 0$ the κth derivative of the map $S(\lambda) = T_\lambda^{-1}$ is locally Hölder continuous with Hölder exponent α. By (34), given $\lambda \in I$, there exist $\delta, L > 0$ such that

$$\|B^{(\kappa)}_m(\lambda) - B^{(\kappa)}_m(\lambda')\|' \le L\|\lambda - \lambda'\|^\alpha$$

for all $m \in \mathbb{Z}$ and $\lambda' \in I$ with $\|\lambda - \lambda'\| \le \delta$. Since

$$((d^\kappa_\lambda R - d^\kappa_{\lambda'} R)\nu_\kappa \mathbf{x})_{m+1} = -\big(B^{(\kappa)}_m(\lambda) - B^{(\kappa)}_m(\lambda')\big)\nu_\kappa x_m \quad \text{for } m \in \mathbb{Z},$$

we obtain

$$\|d^\kappa_\lambda R - d^\kappa_{\lambda'} R\| \le LN\|\lambda - \lambda'\|^\alpha \quad \text{whenever } \|\lambda - \lambda'\| \le \delta. \tag{38}$$

On the other hand, since the derivatives $d^i_\lambda R$ are continuous for $i = 1, \ldots, \kappa$, each map $d^{i-1}_\lambda R$ is locally Lipschitz, that is, given $\lambda \in I$, there exist $\delta, M > 0$ such that

$$\|d^i_\lambda R - d^i_{\lambda'} R\| \le M\|\lambda - \lambda'\| \quad \text{whenever } \|\lambda - \lambda'\| \le \delta, \tag{39}$$

for $i = 1, \ldots, \kappa$.

Now observe that since $R(\lambda)S(\lambda) = \mathrm{Id}$, we have

$$S(\lambda) - S(\lambda') = S(\lambda)(R(\lambda') - R(\lambda))S(\lambda')$$

and so it follows readily from (38) that the map S is locally Hölder continuous with Hölder exponent α. That is, given $\lambda \in I$, there exist $\delta, M' > 0$ such that

$$\|S(\lambda) - S(\lambda')\| \le M'\|\lambda - \lambda'\|^\alpha \quad \text{whenever } \|\lambda - \lambda'\| \le \delta. \tag{40}$$

For the derivatives we first observe that it follows from $R(\lambda)S(\lambda) = \mathrm{Id}$ that

$$\sum_{k=0}^{i} \binom{i}{k} d^k_\lambda R d^{i-k}_\lambda S = 0 \quad \text{and so} \quad d^i_\lambda S = -S(\lambda) \sum_{k=1}^{i} \binom{i}{k} d^k_\lambda R d^{i-k}_\lambda S$$

for $i = 1, \ldots, \kappa$. Therefore,

$$\begin{aligned} d^i_\lambda S - d^i_{\lambda'} S = & -(S(\lambda) - S(\lambda')) \sum_{k=1}^{i} \binom{i}{k} d^k_\lambda R d^{i-k}_\lambda S \\ & - S(\lambda') \sum_{k=1}^{i} \binom{i}{k} (d^k_\lambda R - d^k_{\lambda'} R) d^{i-k}_\lambda S \\ & - S(\lambda') \sum_{k=1}^{i} \binom{i}{k} d^k_{\lambda'} R (d^{i-k}_\lambda S - d^{i-k}_{\lambda'} S) \end{aligned}$$

and so it follows readily from (39) and (40) by induction on i that the map $d^i_\lambda S$ is locally Lipschitz continuous, for $i = 1, \ldots, \kappa - 1$. Finally, it follows from (38) and (40) by induction on i that the map $d^\kappa_\lambda S$ is locally Hölder continuous with Hölder exponent α.

Step 6. Regularity of the projections. Note that the projections $P_{m,\lambda}$ in (36) can be written in the form

$$P_{m,\lambda} = C_m T_\lambda^{-1} D_m,$$

where $D_m \colon X \to Y$ is the linear map $v \mapsto \mathbf{y}$ (see (37)) and $C_m \colon \mathcal{D}(T) \to X$ is the projection $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \mapsto x_m$. Since both C_m and D_m are bounded, the map $\lambda \mapsto P_{m,\lambda}$ is of class $C^{\kappa,\alpha}$. This completes the proof of the theorem. □

References

[1] L. Barreira and Y. B. Pesin, *Lyapunov Exponents and Smooth Ergodic Theory*, University Lecture Series, **23**, American Mathematical Society, Providence, RI, 2002.
[2] L. Barreira and C. Valls, *Lyapunov functions for trichotomies with growth rates*, J. Differential Equations **248** (2010), no. 1, 151–183. https://doi.org/10.1016/j.jde.2009.07.001
[3] ———, *Smooth robustness of exponential dichotomies*, Proc. Amer. Math. Soc. **139** (2011), no. 3, 999–1012. https://doi.org/10.1090/S0002-9939-2010-10531-2
[4] C. V. Coffman and J. J. Schäffer, *Dichotomies for linear difference equations*, Math. Ann. **172** (1967), 139–166. https://doi.org/10.1007/BF01350095
[5] Ju. Dalec'kiĭ and M. Kreĭn, *Stability of solutions of differential equations in Banach space*, translated from the Russian by S. Smith, American Mathematical Society, Providence, RI, 1974.
[6] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, translated from the Russian by L. W. Longdon, Cambridge University Press, Cambridge, 1982.
[7] T. Li, *Die Stabilitätsfrage bei Differenzengleichungen*, Acta Math. **63** (1934), no. 1, 99–141. https://doi.org/10.1007/BF02547352
[8] J. L. Massera and J. J. Schäffer, *Linear differential equations and functional analysis. I*, Ann. of Math. (2) **67** (1958), 517–573. https://doi.org/10.2307/1969871
[9] ———, *Linear Differential Equations and Function Spaces*, Pure and Applied Mathematics, Vol. 21, Academic Press, New York, 1966.
[10] O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), no. 1, 703–728. https://doi.org/10.1007/BF01194662
[11] P. Preda, A. Pogan, and C. Preda, *Discrete admissibility and exponential dichotomy for evolution families*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **12** (2005), no. 5, 621–631.

Luis Barreira
Departamento de Matemática
Instituto Superior Técnico
Universidade de Lisboa
1049-001 Lisboa, Portugal
Email address: barreira@math.tecnico.ulisboa.pt

João Rijo
Departamento de Matemática
Instituto Superior Técnico
Universidade de Lisboa
1049-001 Lisboa, Portugal
Email address: joaorijo@ist.utl.pt

Claudia Valls
Departamento de Matemática
Instituto Superior Técnico
Universidade de Lisboa
1049-001 Lisboa, Portugal
Email address: cvalls@math.tecnico.ulisboa.pt