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ON THE TANGENT SPACE OF A WEIGHTED HOMOGENEOUS PLANE CURVE SINGULARITY

Mario Morán Cañón and Julien Sebag

Abstract. Let k be a field of characteristic 0. Let $\mathscr{C} = \mathrm{Spec}(k[x,y]/\langle f\rangle)$ be a weighted homogeneous plane curve singularity with tangent space $\pi_{\mathscr{C}}\colon T_{\mathscr{C}/k} \to \mathscr{C}$. In this article, we study, from a computational point of view, the Zariski closure $\mathscr{G}(\mathscr{C})$ of the set of the 1-jets on $\mathscr{C}$ which define formal solutions (in $F[[t]]^2$ for field extensions F of k) of the equation $f = 0$. We produce Groebner bases of the ideal $\mathcal{N}_1(\mathscr{C})$ defining $\mathscr{G}(\mathscr{C})$ as a reduced closed subscheme of $T_{\mathscr{C}/k}$ and obtain applications in terms of logarithmic differential operators (in the plane) along $\mathscr{C}$.

1. Introduction

1.1. Let k be a field of characteristic zero. Let V be a *k-variety*, i.e., a k-scheme of finite type. One usually attaches to V its *tangent space* $\pi_V\colon T_{V/k} := \mathrm{Spec}(\mathrm{Sym}(\Omega^1_{V/k})) \to V$ and its arc scheme $\mathscr{L}_\infty(V)$ which is canonically endowed with a morphism of k-schemes $\pi_1^\infty : \mathscr{L}_\infty(V) \to T_{V/k}$. We set $\mathscr{G}(V) := \overline{\pi_1^\infty(\mathscr{L}_\infty(V))}$. (This closed subset is endowed with its reduced structure of closed subscheme of $T_{V/k}$.) If V is assumed to be integral, it is not hard to prove that $\mathscr{G}(V)$ is an irreducible component of $T_{V/k}$ that we call the *general component* of $T_{V/k}$ by analogy with the theory of differential equations (see Subsection 3.2). If $V \hookrightarrow \mathbf{A}^n_k$ is also assumed to be affine, we denote by $\mathcal{N}_1(V)$ the unique ideal of $\mathcal{O}(T_{V/k})$ such that $\mathcal{O}(\mathscr{G}(V)) = \mathcal{O}(T_{V/k})/\mathcal{N}_1(V)$. (*We do not denote differently the ideal $\mathcal{N}_1(V)$ and its preimage in the ring $\mathcal{O}(\mathbf{A}^n_k)$.*)

1.2. The general component of $T_{V/k}$ plays a role in various contexts. Assume, for simplicity, that the considered varieties are affine. First of all, we observe that the elements of the ideal $\mathcal{N}_1(V)$ define the nilpotent functions on the *arc scheme* $\mathscr{L}_\infty(V)$ associated with V which live on $\mathscr{L}_1(V)$ (see Subsection 3.1 for details and formula (3.3)). Furthermore, in [13], if $\mathscr{C}$ is a plane curve defined by the datum of an irreducible polynomial $f \in k[x,y]$, the second author has shown that every homogeneous element P of $\mathcal{N}_1(\mathscr{C})$, of degree d,

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defines a differential operator of the plane D_P such that $D_P(f^d) \in \langle f \rangle$, and conversely (see also [12] for a generalization to reduced polynomials). In this direction, by [12], one can also observe that the principal symbols of the Bernstein operators of f (i.e., the differential operators D of the plane such that $Df^{s+1} = b(s)f^s$ with $b \in \mathbf{Q}[s]$) belong to $\mathcal{N}_1(\mathscr{C})$. In the end, the existence of nontrivial nilpotent functions on $T_{V/k}$ has been used in the context of vertex algebras (see [1]).

1.3. Thus, obtaining a description and a complete understanding of $\mathcal{N}_1(V)$ for arbitrary varieties V, in particular from the computational point of view, appears as a challenging and tricky question. In the present article, we provide systems of generators and Groebner bases of the ideal $\mathcal{N}_1(\mathscr{C})$ in the case of an affine plane curve singularity defined by the datum of a homogeneous or weighted homogeneous polynomial (see Sections 6 and 7). Our key ingredient is differential algebra as developed by Ritt and Kolchin that we reinterpret in our context (see Sections 3 and 5). Let us stress that, as a by-product, our main results provide the following particular case:

Theorem. *Let k be a field of characteristic zero. Let $(r,s) \in \mathbf{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f = x^r - y^s \in k[x,y]$. Let $\mathscr{C}$ be the associated affine plane k-curve.*

(1) *The family of polynomials $\widetilde{D}_{-1} := sy_1x_0 - ry_0x_1$, $\widetilde{D}_i := s^i y_0^{s-i} y_1^i - r^i x_0^{r-i} x_1^i$, for every integer $i \in \{0,\dots,s\}$, is a Groebner basis of the ideal $\mathcal{N}_1(\mathscr{C})$.*

(2) *For every differential operator $D = \sum_{i+j\leq d} a_{i,j}(x,y)\partial_x^i\partial_y^j$ on $\mathbf{A}_k^2$, with order d, such that $D(f^d) \in \langle f \rangle$, its principal symbol*

$$\sigma(D) = \sum_{i+j=d} a_{i,j}(x,y)\partial_x^i\partial_y^j$$

is a combination (in the Weyl algebra) of the following differential operators

$$\begin{cases} f, \\ sx\partial_x - ry\partial_y, \\ s^i y^{s-i}\partial_x^i + (-1)^i r^i x^{r-i}\partial_y^i \quad \forall i \in \{1,\dots,s\}. \end{cases} \tag{1.1}$$

(3) *If $P_B = \sum_i P_i s^i$ is a Bernstein operator of f (i.e., there exists a polynomial $b \in \mathbf{Q}[s]$ such that $P_B(f^{s+1}) = b(s)f^s$), with order d, then each P_i of maximal order d in the expression of P_B is a combination (in the Weyl algebra) of the differential operators (1.1).*

Proof. Assertion (1) is a particular case of Theorem 7.6. Assertions (2) and (3) follow from assertion (1) and the main results of [13] and [12]. ☐

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2. Conventions, notation, recollection

2.1. In this article, the base field k is assumed to be of characteristic zero. Let $n \in \mathbf{N}$. For every $m \in \mathbf{N} \cup \{\infty\}$, we denote by A_m the polynomial ring $k[t_{i,j}; i \in \{1, \dots, n\}, j \in \{0, \dots, m\}]$ with the convention $\{0, \dots, \infty\} = \mathbf{N}$. For every $m \in \mathbf{N} \cup \{\infty\}$, a polynomial $f \in A_m$ (resp. an ideal I of A_m) is called *reduced* (resp. *radical* or *perfect*) if f has no multiple factor (resp. if $\mathrm{Rad}(I) := \sqrt{I} = I$). We denote by B_0 (resp. B_1) the polynomial ring $k[x_0, y_0]$ (resp. $k[x_0, y_0, x_1, y_1]$). If R is a ring, $H \in R$, and J an ideal of R, we denote by $(J : H^\infty)$ the *saturation* of H in J, i.e., the ideal of R formed by the elements y such that there exists an integer $M \in \mathbf{N}$ with $yH^M \in J$. We denote the *regular locus* (resp. *singular locus*) of V by $\mathrm{Reg}(V)$ (resp. $\mathrm{Sing}(V)$). If V is k-variety, we always assume that its singular locus $\mathrm{Sing}(V)$ is endowed with its *reduced* structure of subscheme. In the present article, with a slight abuse of notation, we will not make any difference between the ideal $\mathcal{N}_1(V)$ of the ring $\mathcal{O}(T_{V/k})$ and its unique preimage in the ring A_1.

2.2. On the polynomial ring A_1 (or B_1), we will use various graded structures associated with various *degree functions*.

(1) The *total degree* $\deg := \deg_{\mathrm{tot}}$ of the polynomial ring A_1; for this function, the monomial $M = t_{1,0}^{a_1} \cdots t_{n,0}^{a_n} t_{1,1}^{b_1} \cdots t_{n,1}^{b_n}$ of A_1 is of degree $\deg_{\mathrm{tot}}(M) = \sum_{i=1}^n a_i + b_i$;

(2) The *partial degree* $\deg_0$ of the polynomial ring A_1, where every polynomial is seen as a polynomial in the variables $t_{i,0}$ with coefficients in the ring $k[t_{i,1}; i \in \{1, \dots, n\}]$; for this function, the monomial M is of degree $\deg_0(M) = \sum_{i=1}^n a_i$

(3) The *partial degree* $\deg_1$ of the polynomial ring A_1, where every polynomial is seen as a polynomial in the variables $t_{i,1}$ with coefficients in the ring $k[t_{i,0}; i \in \{1, \dots, n\}]$; for this function, the monomial M is of degree $\deg_1(M) = \sum_{i=1}^n b_i$. For this grading, a homogeneous polynomial P is said to be 1-*homogeneous* of 1-*degree* $\deg_1(P)$.

We say that a polynomial $P \in A_1$ is *bi-homogeneous* if the polynomial P is *simultaneously* a homogeneous polynomial for the graded structure induced by (2) and that induced by (3). Equivalently, the polynomial P is bi-homogeneous if and only if there exist two integers e, d such that one has $\deg_0(T) = e$ and $\deg_1(T) = d$ for every nonzero term T of P. The pair (d, e) is the *bi-degree* of P. Let us stress that, in particular, the polynomial P then is homogeneous for the graded structure induced by (1) (but, obviously, the converse does not hold).

2.3. Let k be a field of characteristic zero. Let $f \in B_0$ be a polynomial. We say that f is *weighted homogeneous* of weight (w_1, w_2, w) if we have the formula

$$f(t^{w_1} x_0, t^{w_2} y_0) = t^w f(x_0, y_0)$$

in the polynomial ring $B_0[t]$. We recall the following usual characterizations of weighted homogeneous polynomials.

Proposition 2.4. *Let k be a field of characteristic zero. Let $f \in B_0 \setminus k$ be a reduced polynomial. The following assertions are equivalent*:

(1) *The polynomial f is weighted homogeneous of weight (w_1, w_2, w)*;

(2) *Every monomial $x_0^i y_0^j$ of f satisfies $iw_1 + jw_2 = w$*;

We assume that the field k is algebraically closed. The former assertions are equivalent to the following one:

(3) *There exist a k-automorphism σ of the ring B_0, an integer $n \geq 1$, a pair of coprime integers (r, s), with $r \geq s$, and $\lambda_1, \ldots, \lambda_n \in k^\times$ such that $\sigma(f) = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^n x_0^r - \lambda_i y_0^s$ with $\varepsilon, \varepsilon' \in \{0, 1\}$.*

Proof. Equivalence (1) $\Leftrightarrow$ (2) is clear. Equivalence (2) $\Leftrightarrow$ (3) is proved in [9, Lemmas 1,2,3]. ☐

3. Recollection of arc scheme and differential algebra

In this section, we provide some recollection of differential algebra in relation with arc scheme; basics on arc scheme are recalled. Direct consequences are also established (see Lemmas 3.4 and 3.7).

3.1. For every integer $m \in \mathbf{N}$, with every k-variety V, we may associate its *jet scheme* $\mathscr{L}_m(V)$ of level m defined by the existence of the natural bijection, for every k-scheme S, bi-functorial in S and V,

$$\mathrm{Hom}_{\mathrm{Sch}_k}(S, \mathscr{L}_m(V)) \cong \mathrm{Hom}_{\mathrm{Sch}_k}(S \otimes_k k[[T]]/\langle T^{m+1}\rangle, V). \tag{3.1}$$

One attaches to V its *arc scheme* $\mathscr{L}_\infty(V)$ by the bi-functorial property (in S, V):

$$\mathrm{Hom}_{\mathrm{Sch}_k}(S, \mathscr{L}_\infty(V)) \cong \varinjlim_m \mathrm{Hom}_{\mathrm{Sch}_k}(S \otimes_k k[[T]]/\langle T^{m+1}\rangle, V).$$

Thus, one has $\mathscr{L}_\infty(V) \cong \varprojlim_m(\mathscr{L}_m(V))$.

3.2. In this article, we focus on the case where $m = 1$. Because of formula (3.1) and the universal property of symmetric algebra, we easily conclude that $\mathscr{L}_1(V) \cong \mathrm{Spec}(\mathrm{Sym}(\Omega^1_{V/k}))$, i.e., it is the *tangent space* $T_{V/k}$ of V. One has the following decomposition:

$$(T_{V/k})_{\mathrm{red}} = \overline{\pi_V^{-1}(\mathrm{Reg}(V))} \bigcup \pi_V^{-1}(\mathrm{Sing}(V)). \tag{3.2}$$

By [6], one knows that the open subscheme $\mathscr{L}_\infty(V) \setminus \mathscr{L}_\infty(\mathrm{Sing}(V))$ is dense in $\mathscr{L}_\infty(V)$. In this way, if $\pi_1^\infty \colon \mathscr{L}_\infty(V) \to \mathscr{L}_1(V)$ is the canonical morphism and if V is assumed to be irreducible, we easily observe that

$$\mathscr{G}(V) := \overline{\pi_1^\infty(\mathscr{L}_\infty(V))} = \overline{\pi_V^{-1}(\mathrm{Reg}(V))} \tag{3.3}$$

since the closed subset $\overline{\pi_1^\infty(\mathscr{L}_\infty(V))}$ obviously contains the open subset $\pi_V^{-1}(\mathrm{Reg}(V)) = \mathscr{L}_1(\mathrm{Reg}(V))$ and $\overline{\pi_V^{-1}(\mathrm{Reg}(V))}$ also is an irreducible component of $T_{V/k}$. If $F \supset k$ is a field extension, the F-points of $\mathscr{G}(V)$ hence correspond to elements of $T_{V/k}(F)$, i.e., *F-jets of level* 1, which are Zariski closed to 1-jets with regular base-point.

3.3. Let R be a k-algebra. We denote by $\mathrm{Der}_k(R)$ the R-module formed by the *k-derivations* on R, i.e., the k-linear maps $R \to R$ which satisfy the Leibniz rule. We endow the k-algebra A_∞ with the k-derivation Δ defined by $\Delta(t_{i,j}) = t_{i,j+1}$, for every integer $i \in \{1, \dots, n\}$ and every integer $j \in \mathbf{N}$. The resulting differential k-algebra is denoted by $k\{t_1, \dots, t_n\}$ and called the *differential polynomial ring.* The injective morphism of k-algebras $k[t_1, \dots, t_n] \to k\{t_1, \dots, t_n\}$, defined by $t_i \mapsto t_{i,0}$, identifies the polynomial ring $k[t_1, \dots, t_n]$ in $k\{t_1, \dots, t_n\}$ with A_0 and gives rise to a structure of $k[t_1, \dots, t_n]$-algebra on $k\{t_1, \dots, t_n\}$. In particular, by a slight abuse of notation, we will not make any difference between the rings $k[t_1, \dots, t_n]$ and A_0. For every subset $S \subset k\{t_1, \dots, t_n\}$, we denote by $[S]$ the differential ideal generated by S in the differential ring $k\{t_1, \dots, t_n\}$ and by $\{S\}$ the radical of the ideal $[S]$. As usually, if S only contains the polynomial $P \in k\{t_1, \dots, t_n\}$, the notation $\{P\}$ refers to this radical differential ideal associated with S.

Let us mention the following useful and classical statement (which is, e.g., a direct consequence of [7, I/§9/Lemma 6] in the particular case of a single equation):

Lemma 3.4. *Let k be a field of characteristic zero. Let I be a prime ideal of A_0. Let $P \in A_0$. Then, the polynomial P belongs to the ideal $\{I\}$ if and only if it belongs to the ideal I.*

If I is an ideal of A_0, we denote by $\langle I, \Delta(I)\rangle$ the ideal of A_1 generated by the polynomials in I (seen in A_1) and the polynomials $\Delta(f)$ for all the polynomials $f \in I$.

3.5. If V is an affine k-variety with $\mathcal{O}(V) = A_0/I$, one verifies that

$$\mathscr{L}_m(V) \cong \mathrm{Spec}(A_m/\langle \Delta^{(i)}(f);\ f \in I, i \in \{0, \dots, m\}\rangle)$$

for every integer $m \in \mathbf{N}$, and that $\mathscr{L}_\infty(V) \cong \mathrm{Spec}(A_\infty/[I])$.

3.6. Let I be a reduced ideal of the ring A_0. Let $I = \cap_{j=1}^r I_j$ be a *prime decomposition* of I, i.e., the ideals I_j are prime for every integer $j \in \{1, \dots, r\}$ (and homogeneous if the ideal I is homogeneous). By the Kolchin irreducibility theorem (see [7, Ch. IV/S17/Proposition 10]), one knows that the reduced differential ideal $\{I_j\}$ is prime, for every integer $j \in \{1, \dots, r\}$ and

$$\{I\} = \{I_1\} \cap \dots \cap \{I_r\}. \tag{3.4}$$

Density statements in arc scheme (e.g., [10, Corollary 3.7]) provide presentations of the ideal $\{I\}$ (when the ideal I is assumed to be prime) which are very useful from a computational point of view. The following general formulation can be deduced from [7, Ch. IV/S17/Proposition 10] and the more general statement in [3, Proposition 3.3] (which is valid in arbitrary characteristic); we provide a direct proof for the convenience of the reader.

Lemma 3.7. *Let I be a prime ideal of the ring A_0. For every $H \in A_0$ such that* $\mathrm{Sing}(\mathrm{Spec}(A_0/I)) \subset V(H)$ *and $H \notin I$, then we have*

$$\{I\} = ([I] : H^\infty). \tag{3.5}$$

This formula, in the special case of systems of algebraic equations, can be linked to corresponding formulas of Lazard and to the Rosenfeld lemma (see [7, Ch. IV/§9/Lemma 2]) in the context of differential algebra. The proof of the Kolchin irreducibility theorem (see [7, Ch. IV/S17/Proposition 10]) in particular explains how to use these results in the differential setting for obtaining statements analogous to Lemma 3.7 in the algebraic framework. As a direct illustration, let us stress that, for every irreducible polynomial $f \in A_0$, [7, Ch. IV/§9/Lemma 2] directly implies that

$$\{f\} = ([f] : \partial(f)^\infty)$$

for every nonzero partial derivative $\partial(f)$ of f. This formula also is a particular form of Lemma 3.7; and the way we will use Lemma 3.7 in the present article.

Proof. Let $P \in ([I] : H^\infty)$. There exists an integer N such that $H^N P \in \{I\}$. By [7, Ch. IV/S17/Proposition 10] and Lemma 3.4, we conclude that $P \in \{I\}$. Let $D(H) = \mathrm{Spec}(A_0) \setminus V(H)$. Furthermore, we deduce from the very definitions, that $\mathscr{L}_\infty(D(H)) \subset V(([I] : H^\infty))$, which implies, thanks to the irreducibility of $\mathscr{L}_\infty(\mathrm{Spec}(A_0/I))$ (by [7, Ch. IV/S17/Proposition 10]), that $V(([I] : H^\infty)) = \mathscr{L}_\infty(\mathrm{Spec}(A_0/I))_{\mathrm{red}}$; hence, the radical of the ideal $([I] : H^\infty)$ coincides with the ideal $\{I\}$. In this end, we note that $\mathscr{L}_\infty(D(H))$ is an open subscheme of $\mathscr{L}_\infty(\mathrm{Reg}(\mathrm{Spec}(A_0/I))$ because of the choice of H. By [11, Lemma 3.4.2], we conclude that the localization $(A_\infty/[I])_H$ by H of the ring $A_\infty/[I]$ is a domain. The injective morphism

$$A_\infty/([I] : H^\infty) \hookrightarrow (A_\infty/[I])_H$$

then implies that the ideal $([I] : H^\infty)$ is prime, hence reduced, which concludes the proof. ☐

3.8. If V is assumed to be affine and integral with $\mathcal{O}(V) = A_0/I$, by formula 3.3 and Lemma 3.7, we have

$$\mathcal{N}_1(V) := (\{I\} \cap A_1)/\langle I, \Delta(I)\rangle = (([I] : H^\infty) \cap A_1)/\langle I, \Delta(I)\rangle. \tag{3.6}$$

Besides, following the previous ideas, we observe that $\langle I,\Delta(I)\rangle \subset (\langle I,\Delta(I)\rangle : H^\infty) \subset \mathcal{N}_1(V)$ and that the injective morphism

$$A_1/(\langle I,\Delta(I)\rangle : H^\infty) \hookrightarrow (A_1/\langle I,\Delta(I)\rangle)_H$$

implies that the ideal $(\langle I,\Delta(I)\rangle : H^\infty)$ is prime. We recall that we do not make any difference between the ideal $\mathcal{N}_1(V)$ of the ring $\mathcal{O}(T_{V/k})$ and its unique preimage in the ring A_1, which contains the ideal $\langle I,\Delta(I)\rangle$.Thus, we have $(\langle I,\Delta(I)\rangle : H^\infty) = \mathcal{N}_1(V)$.

3.9. Let us summarize the previous remarks as the following observation:

Observation 3.10. *Let k be a field of characteristic zero. Let $m \in \mathbf{N}^*$. Let I be a reduced ideal of A_0 with $(I_j)_{j\in\{1,\ldots,m\}}$ as prime components. We set $V = \operatorname{Spec}(A_0/I)$ and $V_j = \operatorname{Spec}(A_0/I_j)$ for every integer $j \in \{1,\ldots,m\}$. Let $P \in A_1$. The following assertions are equivalent:*

1. *The polynomial P belongs to the ideal $\mathcal{N}_1(V)$;*
2. *The polynomial P belongs to the ideal $\cap_{j=1}^m(\{I_j\} \cap A_1)$;*
3. *For every integer $j \in \{1,\ldots,m\}$, the polynomial P belongs to the ideal $\mathcal{N}_1(V_j) = ([I_j] : H_j^\infty) = (\langle I_j,\Delta(I_j)\rangle : H_j^\infty)$ for every H_j satisfying the assumption of Lemma 3.7.11.*

In particular, if there exists an irreducible polynomial $f \in A_0$ (resp. B_0) such that $I = \langle f\rangle$, then we have $\mathcal{N}_1(V) = (\langle f,\Delta(f)\rangle : \partial(f)^\infty)$ for every nonzero partial derivative $\partial(f)$ of f.

Remark 3.11. By analogous arguments, Observation 3.10 can be extended for m-jet scheme of any level $m \geq 1$.

Remark 3.12. By [5], it is easy to deduce an algorithm to compute Groebner bases of the ideal $\mathcal{N}_1(V)$. (See [8] or [4].)

3.13. Let V be an affine k-variety with $\mathcal{O}(V) = A_0/I$. Since, for every generator g of I, the polynomial $\Delta(g)$ is homogeneous, with $\deg_1(\Delta(g)) = 1$, for the graded structure (3) in Subsection 2.2, we conclude from formula (3.6) that the ideal $\mathcal{N}_1(V)$ (in the ring A_1) is homogeneous. Besides, if the ideal I is assumed to be homogeneous (in the ring A_0), the same argument implies that the ideal $\mathcal{N}_1(V)$ (in the ring A_1) is bi-homogeneous.

3.14. Let k' be an algebraic closure of the field k. Let I be a prime ideal of A_0. We observe that the ideal $\{I\} \otimes_k k'$ of the ring $A_\infty \otimes_k k'$ coincides with the radical of the differential ideal generated by the ideal I in the differential ring $A'_\infty := A_\infty \otimes_k k'$. Besides, for every polynomial $P \in A'_1 := A_1 \otimes_k k'$, one can check directly from the very definition that, if $(e_i)_{i\in I}$ is a basis of the k-vector space k', then the polynomial $P = \sum_{i\in I} P_i e_i$ (with $P_i \in A_0$ for every $i \in I$) belongs to $(\{I\} \otimes_k k') \cap A'_1$ if and only if, for every $i \in I$, we have $P_i \in \{I\}$.

4. Technical results on polynomials

In this section, we establish technical results (see Propositions 4.4 and 4.6) which will be useful for our description of the general component attached to an affine plane curve defined by the datum of a homogeneous or weighted-homogeneous polynomial, but which are sufficiently general to be considered independently. In this section we fix the lexicographic order on B_1 associated with $y_1 > y_0 > x_1 > x_0$.

4.1. On the set $\mathbf{N}^4$, we introduce the following equivalence relation: for every pair of tuples $(a,b) \in \mathbf{N}^4 \times \mathbf{N}^4$, we say that a is equivalent to b if there exists an integer $s \in \mathbf{Z}$ such that

$$\begin{cases} b_1 = a_1 - s \\ b_2 = a_2 + s \\ b_3 = a_3 + s \\ b_4 = a_4 - s. \end{cases}$$

In this case, we write $a \sim b$.

Lemma 4.2. *Let $a, b \in \mathbf{N}^4$. The following assertions are equivalent:*

(1) *We have $a \sim b$;*
(2) *The 4-tuples a, b verify the following conditions:*

$$\begin{cases} a_1 + a_3 = b_1 + b_3 \\ a_2 + a_4 = b_2 + b_4 \\ a_1 + a_2 = b_1 + b_2 \\ a_3 + a_4 = b_3 + b_4. \end{cases}$$

Proof. We only have to prove (2) ⇒ (1). Let us set $s := a_1 - b_1$. We observe from equations in system (2) that

$$\begin{aligned} s &= b_2 - a_2 \\ &= b_3 - a_3 \\ &= a_4 - b_4. \end{aligned}$$

Thus, we deduce that $b_1 = a_1 - s$, $b_4 = a_4 - s$, $b_2 = a_2 + s$ and $b_3 = a_3 + s$. ☐

4.3. Let Γ be a system of representatives of $\sim$ in $\mathbf{N}^4$. For every polynomial $P \in B_1$, there exist bi-homogeneous polynomials $P_1, \ldots, P_m \in B_1$ with $P = \sum_{i=1}^m P_i$ and which satisfy the following property: for every integer $i \in \{1, \ldots, m\}$, one can find a unique $\alpha_i \in \Gamma$ such that

$$P_i = \sum_{a \in \mathbf{N}^4, a \sim \alpha_i \in \Gamma} \lambda_a y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}. \tag{4.1}$$

If we assume that the polynomial P is bi-homogeneous of bi-degree (d,e), we observe, thanks to Lemma 4.2, that, for every integer $i\in\{1,\dots,m\}$, there exist an integer $\ell_i\le d+e$ such that:

$$P_i=\sum_{\substack{(a_1,a_2)\in\mathbf{N}^2\\ a_1+a_2=\ell_i}}\lambda_{(a_1,a_2,d-a_1,e-a_2)}y_1^{a_1}y_0^{a_2}x_1^{d-a_1}x_0^{e-a_2}. \tag{4.2}$$

(Let us stress that, because of the assumption on P, we have $a_3=d-a_1$ and $a_4=e-a_2$ in formula (4.1).)

Proposition 4.4. *Let $P\in B_1$ be a polynomial. We set*

$$P=\sum_{a\in\mathbf{N}^4}\lambda_a y_1^{a_1}y_0^{a_2}x_1^{a_3}x_0^{a_4}.$$

Let $r,s\in\mathbf{N}\setminus\{0\}$. The following assertions are equivalent:

1. *The polynomial $sy_1x_0-ry_0x_1$ divides the polynomial P.*
2. *We have the formula* $\sum_{b\in\mathbf{N}^4,b\sim a}\lambda_b r^{b_1}s^{b_3}=0$ *for every tuple $a\in\mathbf{N}^4$.*

If we assume that the polynomial P is bi-homogeneous of bi-degree (d,e), then the former assertions are equivalent to the following one:

3. *For every integer ℓ, we have the formula*

$$\sum_{\substack{(a_1,a_2)\in\mathbf{N}^2\\ a_1+a_2=\ell}}\lambda_{(a_1,a_2,d-a_1,e-\ell+a_1)}r^{a_1}s^{d-a_1}=0.$$

Proof. Assertion (3) is equivalent to assertion (2) by Observation (4.2).

(1) ⇒ (2) We set $G=sy_1x_0-ry_0x_1$. Let $Q\in B_1$ be a polynomial. Each term $M=\mu_m y_1^{m_1}y_0^{m_2}x_1^{m_3}x_0^{m_4}$ of Q (with $\mu_m\in k$) provides two monomials in the expression of QG, whose degrees belong to the same equivalence class by $\sim$, namely $s\mu_m y_1^{m_1+1}y_0^{m_2}x_1^{m_3}x_0^{m_4+1}$ and $-r\mu_m y_1^{m_1}y_0^{m_2+1}x_1^{m_3+1}x_0^{m_4}$. One checks that their sum satisfies the required property since

$$\begin{aligned} s\mu_m r^{m_1+1}s^{m_3}-r\mu_m r^{m_1}s^{m_3+1}&=\mu_m r^{m_1}s^{m_3}(rs-rs)\\ &=0.\end{aligned}\tag{4.3}$$

(2) ⇒ (1) We may assume that

$$P=\sum_{a\in\mathbf{N}^4,a\sim\alpha\in\Gamma}\lambda_a y_1^{a_1}y_0^{a_2}x_1^{a_3}x_0^{a_4}$$

by Subsection 4.3. By assumption, we have

$$\sum_{a\in\mathbf{N}^4,a\sim\alpha\in\Gamma}\lambda_a r^{a_1}s^{a_3}=0.$$

We have to prove that G divides the polynomial P. Let us set

$$\mathrm{LM}(P)=\lambda_{\tilde a}y_1^{\tilde a_1}y_0^{\tilde a_2}x_1^{\tilde a_3}x_0^{\tilde a_4}$$

with $\tilde a\sim\alpha$. Various cases occur:

∘ If $\tilde{a}_1 = 0$, then $\lambda_a = 0$ whenever $a_1 > 0$ (otherwise it would contradict the fact that the tuple $\tilde{a}$ corresponds to $\mathrm{LM}(P)$). But, by the definition of the relation $\sim$, there is no tuple $a \sim \tilde{a}$ with $a_1 = \tilde{a}_1$ different from $\tilde{a}$ itself. Thus $P = \mathrm{LM}(P)$ and, by assumption, we have $\lambda_{\tilde{a}} = 0$; hence, $P = 0$.

∘ We assume that $\tilde{a}_4 = 0$. By the definition of relation $\sim$, every tuple a equivalent to $\tilde{a}$ must verify $a_1 \geq \tilde{a}_1$, since $a_1 = \tilde{a}_1 + a_4$. If $a_1 > \tilde{a}_1$, we deduce that $\lambda_a = 0$ because of the choice of $\tilde{a}$. Thus, we have $P = \mathrm{LM}(P)$, and we conclude as formerly.

∘ We assume that $\tilde{a}_1, \tilde{a}_4 > 0$. Then the polynomial

$$P^{(1)} := P - (\frac{\lambda_{\tilde{a}}}{s} y_1^{\tilde{a}_1 - 1} y_0^{\tilde{a}_2} x_1^{\tilde{a}_3} x_0^{\tilde{a}_4 - 1})G$$

still verifies $\sum_{a \in \mathbf{N}^4, a \sim \alpha \in \Gamma} \lambda_a r^{a_1} s^{a_3} = 0$ (by Observation (4.3) applied here), and we also have $\mathrm{LT}(P^{(1)}) < \mathrm{LT}(P)$. Using the previous cases, we observe that this construction can be iterated. In this way, we construct $P^{(2)}$ such that $\mathrm{LT}(P^{(2)}) < \mathrm{LT}(P^{(1)})$ and G divides $P^{(2)} - P^{(1)}$. After a finite number t of steps (at most $\min\{\tilde{a}_1, \tilde{a}_4\}$), we will obtain $P^{(t)} = 0$, which proves the property and concludes the proof. ☐

4.5. Let $r, s \in \mathbf{N}^*$. We introduce the morphism of B_0-algebras

$$\widetilde{\mathrm{ev}}_1 \colon B_1 \to B_0 \tag{4.4}$$

defined by $x_1 \mapsto s x_0$ and $y_1 \mapsto r y_0$.

Proposition 4.6. *Let $P \in B_1$ be a 1-homogeneous polynomial of 1-degree d. Let $r, s \in \mathbf{N} \setminus \{0\}$. The following assertions are equivalent:*

1. *The polynomial P is divisible by $s y_1 x_0 - r y_0 x_1$.*
2. *We have $\widetilde{\mathrm{ev}}_1(P) = 0$.*

Proof. We only have to prove (2) ⇒ (1). We set

$$P = \sum_{\substack{a \in \mathbf{N}^4 \\ a_1 + a_3 = d}} \lambda_a y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}.$$

By assumption, we have

$$\begin{aligned} 0 &= \widetilde{\mathrm{ev}}_1(P) \\ &= \sum_{\substack{a \in \mathbf{N}^4 \\ a_1 + a_3 = d}} \lambda_a r^{a_1} s^{a_3} y_0^{a_1 + a_2} x_0^{a_3 + a_4} \end{aligned} \tag{4.5}$$

Let $(\ell, m) \in \mathbf{N}^2$. If $a_3 + a_4 = m$ and $a_1 + a_2 = \ell$, we conclude that $a_2 + a_4 = \ell - a_1 + m + a_1 - d = \ell + m - d$. Thus, the sum $P_{(\ell,m)}$ of the terms $T = \lambda_a y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$ of P with $a_3 + a_4 = m$ and $a_1 + a_2 = \ell$ is a bi-homogeneous polynomial of bi-degree $(d, \ell + m - d)$. Formula (4.5) implies that, for every pair of integers $(\ell, m) \in \mathbf{N}^2$, we have

$$\sum_{\substack{(a_1,a_2)\in\mathbf{N}^2\\ a_1+a_2=\ell\\ d+a_4=m+a_1}} \lambda_{(a_1,a_2,d-a_1,\ell+m-d-a_2)} r^{a_1} s^{d-a_1} = 0.$$

We deduce from Proposition 4.4 that each polynomial $P_{(\ell,m)}$ is divisible by $sy_1x_0 - ry_0x_1$, which concludes the proof. □

5. Differential properties of homogeneous polynomials

Let k be a field of characteristic zero. In this section, we establish various technical results which will be used in the next sections. We exhibit in particular an additional differential structure on the ring A_1 (see Theorem 5.4).

5.1. Let us introduce the following k-derivations of the k-algebra A_1. We denote by $D \in \mathrm{Der}_k(A_1)$ (resp. $E \in \mathrm{Der}_k(A_1)$) the derivation defined by $\sum_{i=1}^n t_{i,1}\partial_{t_{i,0}}$ (resp. $\sum_{i=1}^n t_{i,0}\partial_{t_{i,1}}$). The derivations D, E have the following first properties.

Proposition 5.2. *Let k be a field of characteristic zero.*

1. *For every polynomial $P \in A_0$, we have $D(P) = \Delta(P)$ and $E(P) = 0$.*
2. *For every pair $(i, j) \in \{1, \dots, n\}^2$ of integers, we have*
$$D(t_{i,1}t_{j,0} - t_{j,1}t_{i,0}) = E(t_{i,1}t_{j,0} - t_{j,1}t_{i,0}) = 0.$$
3. *For every homogeneous ideal I of the ring A_0, we have $E(\langle I, \Delta(I)\rangle) \subset \langle I, \Delta(I)\rangle$.*
4. *Let $\mathrm{ev}_1 \colon A_1 \to A_0$ be the (surjective) morphism of A_0-algebras which sends the variable $t_{i,1}$ to $t_{i,0}$ for every integer $i \in \{1, \dots, n\}$. For every reduced homogeneous ideal I of the ring A_0, for every polynomial $P \in \mathcal{N}_1(\mathrm{Spec}(A_0/I))$, we have $\mathrm{ev}_1(P) \in I$.*

Proof. Assertions (1) and (2) are obvious and follow from a direct computation. Let us prove assertion (3). Let $g \in A_0$ be a nonzero homogeneous generator of I of degree $\deg_0(g)$. Then, thanks to the Euler identity, we have

$$\begin{aligned} E(\Delta(g)) &= \sum_{i=1}^n \partial_{t_{i,0}}(g)E(t_{i,1}) \\ &= \sum_{i=1}^n \partial_{t_{i,0}}(g)t_{i,0} \\ &= \deg_0(g)g. \end{aligned}$$

Let us prove assertion (4). Up to replacing the ideal I by each of its prime components, we may assume that the ideal I is prime by Observation 3.10. Then, by Subsection 3.8, there exist an integer N and a polynomial $H \notin I$

such that $H^N P \in \langle I, \Delta(I)\rangle$. Then, the polynomial $H^N \mathrm{ev}_1(P)$ belongs to the ideal $\langle I, \mathrm{ev}_1(\Delta(I))\rangle$. And we conclude the proof by observing that, for every homogeneous polynomial $g \in A_0$, we have $\mathrm{ev}_1(\Delta(g))$ equals $\deg_0(g)g$; hence, we have $\langle I, \mathrm{ev}_1(\Delta(I))\rangle = I$. ☐

5.3. The following theorem can be interpreted as a formal "almost" integration of homogeneous polynomials in the ring A_1. Precisely, we show in Theorem 5.4 that the action of the derivation D (resp. E) on the image of E (resp. D) is near from "the" reverse action.

Theorem 5.4. *Let k be a field of characteristic zero. For every bi-homogeneous polynomial $P \in A_1$ of bi-degree (d, e), with $d, e \geq 1$, there exists a positive integer α such that*

$$D(E(P)) - \alpha P \in \langle t_{i,1}t_{j,0} - t_{j,1}t_{i,0}; i, j \in \{1, \dots, n\}\rangle.$$

The same formula holds for the polynomial $E(D(P))$.

Proof. We only prove the formula for the polynomial $D(E(P))$. The proof is based on a direct computation. We have

$$\begin{aligned} D(E(P)) &= \left(\sum_{i=1}^{n} t_{i,1}\partial_{t_{i,0}}\right) \circ \left(\sum_{j=1}^{n} t_{j,0}\partial_{t_{j,1}}\right)(P) \\ &= \left(\sum_{i=1}^{n} t_{i,1}\partial_{t_{i,1}}(P)\right) + T. \end{aligned} \tag{5.1}$$

The first parenthesis in formula (5.1) equals, by the Euler identity, the polynomial dP. Besides, we have

$$\begin{aligned} T &:= \sum_{i=1}^{n} t_{i,1}\left(\sum_{j=1}^{n} t_{j,0}\partial_{t_{i,0}}\partial_{t_{j,1}}(P)\right) \\ &=: \sum_{i=1}^{n} T_i, \end{aligned}$$

where we set, for every integer $i \in \{1, \dots, n\}$,

$$T_i := t_{i,1}\sum_{j=1}^{n} t_{j,0}\partial_{t_{i,0}}\partial_{t_{j,1}}(P).$$

Let us fix an integer $i \in \{1, \dots, n\}$. We write

$$\begin{aligned} T_i = & \left(t_{i,0} \sum_{j=1}^{n} t_{j,1} \partial_{t_{j,1}} \partial_{t_{i,0}} (P) \right) + \left(t_{i,1} \sum_{j=1, j \neq i}^{n} t_{j,0} \partial_{t_{i,0}} \partial_{t_{j,1}} (P) \right) \\ & - \left(t_{i,0} \sum_{j=1, j \neq i}^{n} t_{j,1} \partial_{t_{j,1}} \partial_{t_{i,0}} (P) \right) . \end{aligned} \tag{5.2}$$

For every integer $j \in \{1, \dots, n\}$, with $j \neq i$, we set

$$T_{i,j} = \left(t_{i,1} t_{j,0} - t_{i,0} t_{j,1} \right) \partial_{t_{i,0}} \partial_{t_{j,1}} (P) \in \langle t_{\alpha,1} t_{\beta,0} - t_{\beta,1} t_{\alpha,0} ; \alpha, \beta \in \{1, \dots, n\} \rangle$$

and observe, thanks to the Euler identity, that formula (5.2) can be rewritten under the following form:

$$T_i = d t_{i,0} \partial_{t_{i,0}} (P) + \textstyle\sum_{j=1, j \neq i}^{n} T_{i,j} . \tag{5.3}$$

Thus, formula (5.1) can be rewritten as

$$\begin{aligned} D(E(P)) &= dP + T \\ &= dP + \sum_{i=1}^{n} T_i \\ &= dP + d \left(\sum_{i=1}^{n} t_{i,0} \partial_{t_{i,0}} (P) \right) + \left(\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} T_{i,j} \right) . \end{aligned} \tag{5.4}$$

By the Euler identity applied to the second term in formula (5.4) we conclude that

$$\begin{aligned} D(E(P)) &= dP + T \\ &= dP + deP + \left(\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} T_{i,j} \right) \\ &= d(e+1)P + \left(\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} T_{i,j} \right) \end{aligned}$$

which concludes the proof. □

5.5. From now on and up to the end of the section, we restrict ourselves to the case of affine plane curves. The polynomial $y_1 x_0 - y_0 x_1$ plays an important role in our study as the following lemma underlines it:

Lemma 5.6. *Let k be a field of characteristic zero. Let $\mathscr{C}$ be an affine plane curve defined by the datum of a homogeneous polynomial $g \in B_0$. Then the polynomial $y_1 x_0 - y_0 x_1$ belongs to the ideal $\mathcal{N}_1(\mathscr{C})$.*

Example 5.7. Lemma 5.6 does not hold in higher dimension. Let us consider the hypersurface $\mathscr{S}$ of $\mathbf{A}_k^3$ defined by the datum of the polynomial $f = x_0^2 + y_0^2 + z_0^2 \in k[x, y, z]$. Then $x_1y_0 - x_0y_1, x_1z_0 - x_0z_1, y_1z_0 - z_1y_0 \notin \mathcal{N}_1(\mathscr{S})$.

Proof. By Observation 3.10, up to replacing g by each of its irreducible factors, we may assume that the polynomial g is irreducible; then, we have to prove that the polynomial $y_1x_0 - y_0x_1$ belongs to the ideal $([g] : \partial(g)^\infty)$ (for some nonzero partial derivative). Let us assume that $\partial_y(g) \neq 0$ (a symmetrical argument works if $\partial_x(g) \neq 0$). We write

$$\begin{aligned}\partial_y(g)(y_1x_0 - y_0x_1) &\equiv -x_0\partial_x(g)x_1 - y_0\partial_y(g)x_1 \pmod{\Delta(g)}\\ &\equiv -\deg_0(g)x_1g \pmod{\Delta(g)},\end{aligned}$$

which concludes the proof. □

5.8. We consider the morphism of B_0-algebras $\mathrm{ev}_1\colon B_1 \to B_0$ defined by $x_1 \mapsto x_0$ and $y_1 \mapsto y_0$.

Lemma 5.9. *Let k be a field of characteristic zero. Let $g \in B_0$ be a reduced homogeneous polynomial with $\mathscr{C} = \mathrm{Spec}(B_0/\langle g\rangle)$. Let $P \in B_1$ be a homogeneous polynomial (in x_1, y_1) of degree $\deg_1(P) = d$. The following assertions are equivalent*:

1. *The polynomial P belongs to $\mathcal{N}_1(\mathscr{C})$;*
2. *The polynomial g divides $\mathrm{ev}_1(P)$ in the ring B_0.*

Example 5.10. The analog of Lemma 5.9 does not hold in higher dimensions. Let us consider the hypersurface of $\mathbf{A}_k^3$ defined by the datum of the polynomial $f = x_0^2+y_0^2+z_0^2 \in k[x, y, z]$. Then the polynomial $x_1y_0-x_0y_1$ satisfies condition (2) but does not belong to $\mathcal{N}_1(\mathscr{C})$.

Proof. By (4) in Proposition 5.2, we only have to prove (2) ⇒ (1). By Observation 3.10, we may assume that the polynomial g is irreducible up to replacing it by each of its irreducible factors. Two cases occur.

∘ Let us assume that there exists $u \in k^*$ such that $g = ux$. (By symmetrical arguments, we could prove the case $g = uy$.) In this case, we have $\mathcal{N}_1(\mathscr{C}) = \langle g, \Delta(g)\rangle = \langle x_0, x_1\rangle$. Hence, the polynomial P belongs to $\mathcal{N}_1(\mathscr{C})$ if and only if it belongs to the kernel of the morphism of k-algebras $\mathrm{ev}\colon B_1 \to k$ sending the variables x_0, x_1 to zero. Let us assume that the polynomial g divides $\mathrm{ev}_1(P)$. Since P is 1-homogeneous, there exists $q \in k[y_0]$ such that $\mathrm{ev}(P) = P(0, y_0, 0, y_1) = q(y_0)y_1^d$. Since $\mathrm{ev}(\mathrm{ev}_1(P)) = \mathrm{ev}_1(\mathrm{ev}(P))$, we conclude that $q = 0$. In other words, we have $P \in \mathcal{N}_1(\mathscr{C})$.

∘ Let us assume that g is not divisible by x_0 or y_0, cases for which we have proved the property. We have the formula

$$\begin{aligned}x_0^dP(x_0, y_0, x_1, y_1) &= P(x_0, y_0, x_0x_1, x_0y_1)\\ &\equiv P(x_0, y_0, x_0x_1, x_1y_0) \pmod{y_1x_0 - x_1y_0}\\ &\equiv x_1^dP(x_0, y_0, x_0, y_0) \pmod{y_1x_0 - x_1y_0}.\end{aligned}\tag{5.5}$$

By assumption, the polynomial g divides $x_1^d P(x_0, y_0, x_0, y_0)$. By formula (5.5) and Lemma 5.6, we conclude that $x_0^d P(x_0, y_0, x_1, y_1)$ belongs to $\mathcal{N}_1(\mathscr{C})$. By Lemma 3.4, we conclude that $P(x_0, y_0, x_1, y_1)$ belongs to $\mathcal{N}_1(\mathscr{C})$ (which is prime). ☐

Proposition 5.11. *Let k be a field of characteristic zero. For every affine plane curve $\mathscr{C}$ defined by the datum of a homogeneous polynomial $g \in B_0$, the ideal $\mathcal{N}_1(\mathscr{C})$ is stable under the actions of D, E.*

In general, this assertion does not hold true. See Example 5.12.

Example 5.12. Let us consider the hypersurface $\mathscr{S}$ of $\mathbf{A}_k^3$ defined by the datum of the polynomial $f = z^3 + y^2 x + y^3 \in k[x, y, z]$. One can check that $D(\Delta(f)) \notin \mathcal{N}_1(\mathscr{S})$. The polynomial $P = 3y_0x_0z_1 + 3y_0^2z_1 - y_0x_1z_0 - 2y_1x_0z_0 - 3y_0y_1z_0$ belongs to $\mathcal{N}_1(\mathscr{S}) \setminus \langle f, \Delta(f)\rangle$ but $D(P) \notin \mathcal{N}_1(\mathscr{S})$.

Proof. By Observation 3.10, we may assume that the polynomial g is irreducible. Let $P \in \mathcal{N}_1(\mathscr{C})$. By Observation 3.10, there exist an integer $M \in \mathbf{N}$, a nonzero partial derivative $\partial(g)$ of g and polynomials $\alpha, \beta \in B_0$ such that

$$\partial(g)^M P = \alpha g + \beta \Delta(g). \tag{5.6}$$

By Proposition 5.2, we know that the derivation E stabilizes the ideal $\langle g, \Delta(g)\rangle$. Then, by applying E to equation (5.6), we conclude that $\partial(g)^M E(P)$ belongs to the ideal $\langle g, \Delta(g)\rangle$ which concludes the proof by Observation 3.10. Let us prove the assertion for the derivation D. By a direct computation, we obtain $D(\Delta(g)) = \Delta(\partial_x(g))x_1 + \Delta(\partial_y(g))y_1$. Then, we observe, thanks to the Euler identity, that

$$D(\Delta(g))(x_0, y_0, x_0, y_0) = \deg(g)(\deg(g) - 1)g,$$

and we conclude by Lemma 5.9. ☐

6. The general component of an affine plane curve defined by a homogeneous polynomial

Let k be a field of characteristic zero. The aim of this section is to describe presentations for the ideal $\mathcal{N}_1(\mathscr{C})$ when $\mathscr{C}$ is an affine plane curve defined by the datum of a homogeneous polynomial in B_0.

6.1. We introduce the following notation. Let $m \in \mathbf{N}$. Let $g \in B_0$ be a homogeneous polynomial with $\deg_0(g) = m$ and $\mathscr{C} = \mathrm{Spec}(B_0/\langle g\rangle)$. For every integer $i \in \mathbf{N}$, for every polynomial $g \in B_0$, we denote by $D_i(g)$ the element $D^{(i)}(g)/i$, if $i \geq 1$, and $D_0(g) = g$, which belongs to the ideal $\mathcal{N}_1(\mathscr{C})$. In particular, for every integer $i \geq m + 1$, we have $D_i(g) = 0$.

Proposition 6.2. *Let k be a field of characteristic zero. Let $m \geq 1$ be an integer. Let $\mathscr{C}$ be an affine plane curve defined by the datum of a reduced homogeneous polynomial $g \in B_0$ with $\deg_0(g) = m$. The ideal $\mathcal{N}_1(\mathscr{C})$ is generated by the family $D_{-1} := y_1x_0 - y_0x_1$ and the $D_i(g)$ for every integer $i \in \{0, \ldots, m\}$.*

Proof. By Proposition 5.11, for every integer $i \in \{0, \dots, m\}$ we have $D_i(g) \in \mathscr{N}_1(\mathscr{C})$. Thanks to this observation and Lemma 5.6, we deduce that $\langle y_1x_0 - y_0x_1\rangle + \langle D_i(g); i \in \{0, \dots, m\}\rangle \subset \mathscr{N}_1(\mathscr{C})$. Conversely, we have now to prove that $\mathscr{N}_1(\mathscr{C}) \subset \langle y_1x_0 - y_0x_1\rangle + \langle D_i(g); i \in \{0, \dots, m\}\rangle$. We show the result by an induction on the degree d (in x_1, y_1) of the polynomials in $\mathscr{N}_1(\mathscr{C})$ considered as polynomials in the ring $B_0[x_1, y_1]$. By Observation 3.10, we may assume that g is irreducible. If $P \in \mathscr{N}_1(\mathscr{C})$ is a polynomial with degree $d = 0$, then the polynomial g divides P by Lemma 3.4. Let $d \geq 1$ and $P \in \mathscr{N}_1(\mathscr{C})$ with $\deg_1(P) = d$. By Subsection 3.13, we may assume that P is bi-homogeneous. We observe that the degree of the polynomial $E(P)$ equals $d-1$ and belongs to the ideal $\mathscr{N}_1(\mathscr{C})$ by Proposition 5.11. By the induction hypothesis, we deduce that $E(P) \in \langle y_1x_0 - y_0x_1\rangle + \langle D_i(g); i \in \{0, \dots, m\}\rangle$. We conclude the proof by applying the operator D to $E(P)$ thanks to Theorem 5.4. ☐

Example 6.3. Let k be a field of characteristic zero. Let $f = x_0y_0 \in B_0$. In this case, a direct argument on polynomials provides the formula

$$\mathscr{N}_1(\mathscr{C}) = \{x_0\} \cap \{y_0\} \cap B_1 = ([x_0] \cdot [y_0]) \cap B_1 = \langle f, x_0y_1, x_1y_0, x_1y_1\rangle.$$

By Proposition 6.2, we deduce another presentation of the ideal $\mathscr{N}_1(\mathscr{C})$ given by

$$\langle f, y_1x_0 - y_0x_1, x_0y_1 + y_0x_1, x_1y_1\rangle.$$

6.4. Let $t \in \mathbf{N}^*$. For every integer $i \in \{1, \dots, t\}$, let $\gamma_i \in k^\times$ be mutually distinct elements. For every integer $i \in \{1, \dots, t\}$, we set $f_i = y_0 - \gamma_i x_0 \in B_0$ and $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^t f_i$ with $\varepsilon, \varepsilon' \in \{0, 1\}$. Let us denote by J the ideal of the ring B_1 defined by

$$J := \langle f_1, \Delta(f_1)\rangle \cdot \langle f_2, \Delta(f_2)\rangle \cdot \ldots \cdot \langle f_t, \Delta(f_t)\rangle.$$

The following bi-homogeneous polynomials of the ring B_1 belong to this ideal

$$\delta_i := \left(\prod_{\ell=1}^{i} \Delta(f_\ell)\right) \times \left(\prod_{\ell=i+1}^{t} f_\ell\right) \tag{6.1}$$

for every integer $i \in \{0, \dots, t\}$. We set $\mathscr{C} = \mathrm{Spec}(B_0/\langle f\rangle)$.

Theorem 6.5. *Keep the assumptions and notation of Subsection 6.4. The family*

$$\mathfrak{B} := \{y_1x_0 - y_0x_1, x_{h_1}^{\varepsilon} y_{h_2}^{\varepsilon'} \delta_i(f),\ i \in \{0, \dots, t\}, h_1, h_2 \in \{0, 1\}\}$$

is a Groebner basis of $\mathscr{N}_1(\mathscr{C})$ *for the monomial order* $y_1 >_{\mathrm{lex}} y_0 >_{\mathrm{lex}} x_1 >_{\mathrm{lex}} x_0$ *in* B_1.

Proof. Let us prove Theorem 6.5. By Lemma 5.6 and the very definition of the ideal J, we conclude that the family is contained in the ideal $\mathscr{N}_1(\mathscr{C})$. By [2, Proposition 5.38], in order to prove that the family $\mathfrak{B}$ is a Groebner basis of the ideal $\mathscr{N}_1(\mathscr{C})$, it suffices to show that every element in $\mathscr{N}_1(\mathscr{C})$ has some term in $\langle \mathrm{LT}(\mathfrak{B})\rangle$. Let us denote the polynomial $\delta_{-1} := y_1x_0 - y_0x_1$. We observe that

$$\langle \mathrm{LT}(\mathfrak{B})\rangle = \left\langle y_1x_0, \{x_{h_1}^{\varepsilon} y_{h_2}^{\varepsilon'} y_1^{\ell} y_0^{t-\ell}\}_{\substack{\ell\in\{0,\dots,t\}\\ h_1,h_2\in\{0,1\}}} \right\rangle.$$

Let $P \in \mathcal{N}_1(\mathscr{C})$ that we may assume to be bi-homogeneous. We apply Lemma 5.9 and deduce that f divides $\mathrm{ev}_1(P)$ in the ring B_0. Two cases occur:

(i) Assume that $\mathrm{ev}_1(P) \neq 0$. Then, the polynomial P has some term of the form $y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$ such that $\mathrm{LT}(f) = x_0^{\varepsilon} y_0^{\varepsilon'} y_0^{t}$ divides $y_0^{a_1+a_2} x_0^{a_3+a_4}$; hence, we have $a_1 + a_2 \geq t + \varepsilon'$ and $a_3 + a_4 \geq \varepsilon$. The second inequality shows that either x_0^{ε} or x_1^{ε} divide the term $x_1^{a_3} x_0^{a_4}$.

On the other hand, for $\ell \in \{0, \dots, t\}$, the pairs $(\ell + \varepsilon', t - \ell)$, $(\ell, t - \ell + \varepsilon')$ range over all possible pairs of nonnegative integers whose sum equals $t + \varepsilon'$, and thus some monomial in $\{y_{h_2}^{\varepsilon'} y_1^{\ell} y_0^{t-\ell}\}_{\substack{\ell\in\{0,\dots,t\}\\ h_2\in\{0,1\}}}$ divides the term $y_1^{a_1} y_0^{a_2}$. We deduce that $y_1^{a_1} y_0^{a_2} x_1^{a_3} x_0^{a_4}$ is divisible by some monomial in $\{x_{h_1}^{\varepsilon} y_{h_2}^{\varepsilon'} y_1^{\ell} y_0^{t-\ell}\}_{\substack{\ell\in\{0,\dots,t\}\\ h_1,h_2\in\{0,1\}}}$ and we have proved the property.

(ii) Assume that $\mathrm{ev}_1(P) = 0$. Then, by Proposition 4.6, the polynomial $y_1x_0 - y_0x_1$ divides P; hence, the monomial y_1x_0 divides $\mathrm{LT}(P)$, and the property holds. ☐

Remark 6.6. Along the whole article we have chosen the monomial order in B_1 to be the lexicographic one with $y_1 >_{\mathrm{lex}} y_0 >_{\mathrm{lex}} x_1 >_{\mathrm{lex}} x_0$. In fact the proof would work with slight modifications for the lexicographic order for every ordering of the variables. In the homogeneous case the graded lexicographic order also works because it coincides with the lexicographic order, which is not true for the weighted homogeneous setting in Section 7. Computer tests have shown that we do not obtain the preceding Groebner basis for every monomial order.

6.7. Let us mention the following consequence of 6.5, which improves Proposition 6.2.

Corollary 6.8. *Let k be a field of characteristic zero. Let $f \in B_0$ be a reduced homogeneous polynomial which is not divisible by neither x_0 nor y_0. We set $\mathscr{C} = \mathrm{Spec}(B_0/\langle f\rangle)$. The family formed by the polynomial $y_1x_0 - y_0x_1$ and the $D_i(f)$ for every integer $i \in \{0, \dots, \deg_0(f)\}$ is a Groebner basis of the ideal $\mathcal{N}_1(\mathscr{C})$ for the monomial order $y_1 >_{\mathrm{lex}} y_0 >_{\mathrm{lex}} x_1 >_{\mathrm{lex}} x_0$ in B_1.*

Proof. Let k' be an algebraic closure of k. Let us consider the differential ideal $\{f\}$ in $k\{x, y\}$ and let P be a polynomial in $\{f\} \cap B_1$. By Subsection 3.14, the ideal $\{f\} \otimes_k k'$ equals the radical of the differential ideal generated by f in the differential ring $k'\{x, y\}$. By Theorem 6.5, the leading term $\mathrm{LT}(P)$ of the polynomial P is divisible (in $k'\{x, y\}$) by the leading term of some of the

δ_i, for $i \in \{-1, \ldots, \deg_0(f)\}$. We observe that these leading terms are the same as those of the family $\{D_i\}_{i\in\{-1,\ldots,\deg_0(f)\}}$, in the notation of Proposition 6.2. But the polynomials in this family belong to $k\{x, y\}$, which concludes the proof. $\square$

Example 6.9. Let us fix the field $k = \mathbf{Q}$. Let us consider the polynomial $f = x_0^4 + x_0^3y_0 + y_0^4$, which is irreducible in B_0 and homogeneous of degree 4. From a direct computation we obtain a Groebner basis of $\{f\} \cap B_1$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$:

$$\mathfrak{B} = \left\{ \begin{array}{c} y_1x_0 - y_0x_1, f, y_0^3y_1 + x_0^2x_1y_0 + x_0^3x_1, y_0^2y_1^2 + x_0x_1^2y_0 + x_0^2x_1^2, \\ y_0y_1^3 + x_1^3y_0 + x_0x_1^3, y_1^4 + x_1^3y_1 + x_1^4 \end{array} \right\}.$$

The family given in Proposition 6.2 is the following:

$$\begin{aligned} \mathfrak{C} = \{&D_{-1} := y_1x_0 - y_0x_1, D_0 := f, D_1 := 4y_0^3y_1 + x_0^3y_1 + 3x_0^2x_1y_0 + 4x_0^3x_1, \\ &D_2 := 12y_0^2y_1^2 + 6x_0^2x_1y_1 + 6x_0x_1^2y_0 + 12x_0^2x_1^2, \\ &D_3 := 24y_0y_1^3 + 18x_0x_1^2y_1 + 6x_1^3y_0 + 24x_0x_1^3, D_4 := 24y_1^4 + 24x_1^3y_1 + 24x_1^4\}. \end{aligned}$$

We observe that the leading terms of the elements in $\mathfrak{B}$ and $\mathfrak{C}$ are the same. Hence $\mathfrak{C}$ is also a Groebner basis of $\{f\} \cap B_1$.

7. The general component of an affine plane curve defined by a weighted homogeneous polynomial

In this section we compute a system of generators of the ideal $B_1 \cap \{f\}$ when the polynomial f is weighted homogeneous. Let $(r, s) \in \mathbf{N}^2$ be a pair of coprime integers with $r > s \geq 2$. The techniques we will use are partly similar to those of the homogeneous case (see Section 6).

7.1. We begin by giving an analogue to Lemma 5.6 in the weighted homogeneous case.

Lemma 7.2. *Let k be a field of characteristic zero. Let $\lambda \in k^\times$. Let $(r, s) \in \mathbf{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f = x_0^r - \lambda y_0^s \in B_0$. We set $\mathscr{C} = \operatorname{Spec}(B_0/\langle f\rangle)$. Then the polynomial $sy_1x_0 - ry_0x_1$ belongs to the ideal $\mathcal{N}_1(\mathscr{C})$.*

Proof. The polynomial f being irreducible, we have to prove that $sy_1x_0 - ry_0x_1$ belongs to the ideal $([f] : \partial(f)^\infty)$ (for some nonzero partial derivative). Let us reason with $\partial_y(f) \neq 0$ (a symmetrical argument works for $\partial_x(f) \neq 0$). We write

$$\begin{aligned} \partial_y(f)(sy_1x_0 - ry_0x_1) &\equiv -sx_0\partial_x(f)x_1 - ry_0\partial_y(f)x_1 \pmod{\Delta(f)} \\ &\equiv -rsx_1f \pmod{\Delta(f)}, \end{aligned}$$

which concludes the proof. $\square$

7.3. Now we give an analogue to Lemma 5.9. We consider the morphism of B_0-algebras $\widetilde{\mathrm{ev}}_1 : B_1 \longrightarrow B_0$ defined by $x_1 \longmapsto sx_0$ and $y_1 \longmapsto ry_0$.

Lemma 7.4. *Let k be a field of characteristic zero. Let $\lambda \in k^\times$. Let $(r,s) \in \mathbf{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f = x_0^r - \lambda y_0^s \in B_0$. We set $\mathscr{C} = \operatorname{Spec}(B_0/\langle f\rangle)$. Let $P \in B_1$ be a 1-homogeneous polynomial of degree $\deg_1(P) =: d$. The following assertions are equivalent*:

(1) *The polynomial P belongs to $\mathcal{N}_1(\mathscr{C})$.*
(2) *The polynomial $\widetilde{\mathrm{ev}}_1(P)$ belongs to the ideal $\langle f\rangle$ in the ring B_0.*

Proof. Let $P \in \mathcal{N}_1(\mathscr{C})$. By Observation 3.10, there exist an integer $N \in \mathbf{N}$ and a polynomial $H \notin \langle f\rangle$ in B_0 such that $H^N P \in \langle f, \Delta(f)\rangle$. Then, taking the image via $\widetilde{\mathrm{ev}}_1$, the polynomial $H^N \widetilde{\mathrm{ev}}_1(P)$ belongs to the prime ideal $\langle f, \widetilde{\mathrm{ev}}_1(\Delta(f))\rangle = \langle f\rangle$, because $\widetilde{\mathrm{ev}}_1(\Delta(f)) = rsf$. We conclude by Lemma 3.4. Conversely, let $P \in B_1$ such that $\widetilde{\mathrm{ev}}_1(P) = P(x_0, y_0, sx_0, ry_0) \in \langle f\rangle \subset B_0$. We have the formula

$$\begin{aligned} s^d x_0^d P(x_0,y_0,x_1,y_1) &= P(x_0,y_0,sx_0x_1,sx_0y_1) \\ &\equiv P(x_0,y_0,sx_0x_1,rx_1y_0) \pmod{sy_1x_0 - ry_0x_1} \\ &\equiv x_1^d P(x_0,y_0,sx_0,ry_0) \pmod{sy_1x_0 - ry_0x_1}. \end{aligned} \tag{7.1}$$

By assumption, $x_1^d P(x_0,y_0,sx_0,ry_0) \in \langle f\rangle$, then by formula 7.1 the polynomial $s^d x_0^d P(x_0,y_0,x_1,y_1)$ belongs to $\mathcal{N}_1(\mathscr{C})$, which is a prime ideal. By Lemma 3.4 the polynomial $P(x_0,y_0,x_1,y_1)$ belongs to $\mathcal{N}_1(\mathscr{C})$. ☐

7.5. Let us state the main result of this section. Let $\lambda_1, \ldots, \lambda_t \in k$ be nonzero elements. Let $(r,s) \in \mathbf{N}^2$ be a pair of coprime integers with $r > s \geq 2$. For every integer $i \in \{1,\ldots,t\}$, we set $\tilde{D}_{-1} := \widetilde{D}_{\lambda_i,-1} := sy_1x_0 - ry_0x_1$ and $\widetilde{D}_{\lambda_i,j_i} := \lambda_i s^{j_i} y_0^{s-j_i} y_1^{j_i} - r^{j_i} x_0^{r-j_i} x_1^{j_i}$, where $j_i \in \{0,\ldots,s\}$. For every $i \in \{1,\ldots,t\}$, if $j_i \in \{-1,\ldots,s\}$, we denote

$$\widetilde{D}_{j_1,\ldots,j_t} = \widetilde{D}_{\lambda_1,j_1} \cdots \widetilde{D}_{\lambda_t,j_t}. \tag{7.2}$$

Theorem 7.6. *Let k be a field of characteristic zero. Let $\lambda_1,\ldots,\lambda_t \in k$ be nonzero elements. Let $(r,s) \in \mathbf{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f \in B_0$ be the polynomial $f = x_0^\varepsilon y_0^{\varepsilon'} \prod_{i=1}^t (x_0^r - \lambda_i y_0^s)$ with $\varepsilon, \varepsilon' \in \{0,1\}$. We set $\mathscr{C} = \operatorname{Spec}(B_0/\langle f\rangle)$. Then the family*

$$\mathfrak{B} = \{\tilde{D}_{-1}, x_{h_1}^\varepsilon y_{h_2}^{\varepsilon'} \widetilde{D}_{j_1,\ldots,j_t}, j_i \in \{-1,\ldots,s\}, i \in \{1,\ldots,t\}, h_1, h_2 \in \{0,1\}\}$$

is a Groebner basis of $\mathcal{N}_1(\mathscr{C})$ for the monomial order $y_1 >_{\mathrm{lex}} y_0 >_{\mathrm{lex}} x_1 >_{\mathrm{lex}} x_0$ in B_1.

Let us stress that, if the field k is assumed to be algebraically closed, Theorem 7.6 provides a complete answer for weighted homogeneous polynomials by Proposition 2.4. Subsection 3.14 explains how Theorem 7.6 also gives an explicit Groebner basis in case the field k is not assumed to be algebraically closed.

Example 7.7. Let us fix the field $k = \mathbf{R}$ and let $k' := \mathbf{C}$. We keep the notation in Subsection 3.14. Let us consider the polynomial $f = x_0^7 + x_0y_0^4$, which is weighted homogeneous of weight $(2, 3, 14)$. In $k'[x_0, y_0]$, we have $f = x_0(x_0^3 - iy_0^2)(x_0^3 + iy_0^2)$. By Remark 7.16, which follows from the proof of Theorem 7.6, the following family is a Groebner basis of $(\{f\} \otimes_k k') \cap B_1'$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$:

$$\mathfrak{B}' = \{2y_1x_0 - 3y_0x_1\} \cup \{(x_\ell\widetilde{D}_{0,0}, x_\ell\widetilde{D}_{1,0}, x_\ell\widetilde{D}_{2,0}, x_\ell\widetilde{D}_{2,1}, x_\ell\widetilde{D}_{2,2})_{\ell\in\{0,1\}}\}.$$

From the preceding family we obtain the following one by computing the components (on the basis $\{1, i\}$) of its elements:

$$\begin{aligned}\mathfrak{C} = \{&2y_1x_0 - 3y_0x_1, f, x_1x_0^6 + x_1y_0^4, 2y_0^3y_1x_0 + 3x_0^6x_1, 2y_0y_1x_0^4 - 3y_0^2x_0^3x_1,\\ &2y_0^3y_1x_1 + 3x_0^5x_1^2, 2y_0y_1x_0^3x_1 - 3y_0^2x_0^2x_1^2, 4y_1^2y_0^2x_0 + 9x_0^5x_1^2, 4y_1^2x_0^4 - 9y_0^2x_0^2x_1^2,\\ &4y_1^2y_0^2x_1 + 9x_0^4x_1^3, 4y_1^2x_0^3x_1 - 9y_0^2x_0x_1^3, 8y_1^3y_0x_0 + 27x_0^4x_1^3,\\ &12y_1^2x_0^3x_1 - 18y_0y_1x_0^2x_1^2, 8y_1^3y_0x_1 + 27x_0^3x_1^4, 12y_1^2x_0^2x_1^2 - 18y_0y_1x_0x_1^3,\\ &16y_1^4x_0 + 81x_0^3x_1^4, 16y_1^4x_1 + 81x_0^2x_1^5\}.\end{aligned}$$

Now, from a direct computation we obtain a Groebner basis of $\{f\} \cap B_1$ for the monomial order $y_1 >_{\text{lex}} y_0 >_{\text{lex}} x_1 >_{\text{lex}} x_0$:

$$\begin{aligned}\mathfrak{B} = \{&2y_1x_0 - 3y_0x_1, f, x_1x_0^6 + x_1y_0^4, 2y_0^3y_1x_1 + 3x_0^5x_1^2, 4y_1^2y_0^2x_1 + 9x_0^4x_1^3,\\ &8y_1^3y_0x_1 + 27x_0^3x_1^4, 16y_1^4x_1 + 81x_0^2x_1^5\}.\end{aligned}$$

We observe that $\langle\mathfrak{B}\rangle = \langle\mathfrak{C}\rangle$.

The proof of Theorem 7.6 is presented in Subsection 7.15 and is based on results in Subsections 7.8 and 7.11. A key ingredient in our proof is to pass from the weighted homogeneous setting to the homogeneous one. For this, let us call $C_0 = k[u_0, v_0]$ and $C_1 = k[u_0, v_0, u_1, v_1]$. We consider the morphism of k-modules $\rho : B_1 \longrightarrow C_1$ given by $x_0 \mapsto u_0^s$, $y_0 \mapsto v_0^r$, $x_1 \mapsto su_0^{s-1}u_1$, $y_1 \mapsto rv_0^{r-1}v_1$. Let us stress that the morphism ρ is *injective* and satisfies the formula

$$\rho(\widetilde{\text{ev}}_1(P)) = \text{ev}_1(\rho(P)) \tag{7.3}$$

for every polynomial $P \in B_1$.

7.8. Let $\lambda \in k^\times$. Let us begin by important remarks in case $f = x_0^r - \lambda y_0^s \in B_0$. We set $g := \rho(f) = u_0^{rs} - \lambda v_0^{rs} \in C_0$; it is a homogeneous polynomial. We set $\mathscr{C} = \text{Spec}(B_0/\langle f\rangle)$ and $\mathscr{D} = \text{Spec}(C_0/\langle g\rangle)$. We use this notation in Subsection 7.8.

Lemma 7.9. *Let $M = v_1^{a_1}v_0^{a_2}u_1^{a_3}u_0^{a_4} \in C_1$. Then, the following assertions are equivalent:*

1. *The monomial M belongs to the image* $\text{Im}(\rho)$ *of the morphism ρ.*

(2) *The following conditions hold true*:

$$\left\{\begin{array}{l} r|a_1+a_2; \\ \dfrac{a_1+a_2}{r} \geq a_1; \\ s|a_3+a_4; \\ \dfrac{a_3+a_4}{s} \geq a_3. \end{array}\right. \tag{7.4}$$

If these conditions hold, we have $\rho^{-1}(M) = \dfrac{1}{r^{a_1}s^{a_3}} y_1^{a_1} y_0^{\frac{a_1+a_2}{r}-a_1} x_1^{a_3} x_0^{\frac{a_3+a_4}{s}-a_3}$.

Proof. Since the morphism ρ is injective, this assertion is straightforward. ☐

Lemma 7.10. *We have the formula* $\rho(\mathcal{N}_1(\mathscr{C})) = \mathcal{N}_1(\mathscr{D}) \cap \operatorname{Im}(\rho)$.

Proof. Let $P \in \mathcal{N}_1(\mathscr{C})$, then by Lemma 7.4 we know that $\widetilde{\mathrm{ev}}_1(P) \in \langle f \rangle$; hence, we have $\rho(\widetilde{\mathrm{ev}}_1(P)) \in \rho(\langle f \rangle) \subset \langle g \rangle$. By formula (7.3), it means that $\mathrm{ev}_1(\rho(P)) \in \langle g \rangle$. Since the polynomial g is reduced and homogeneous, by Lemma 5.9, we deduce that this condition is equivalent to $\rho(P) \in \mathcal{N}_1(\mathscr{D})$. Conversely, let $Q \in \mathcal{N}_1(\mathscr{D}) \cap \operatorname{Im}(\rho)$. Since the morphism ρ is injective, there exists a unique $P \in B_1$ such that $\rho(P) = Q$. By formula (7.3) and Lemma 5.9, $\rho(\widetilde{\mathrm{ev}}_1(P)) = \mathrm{ev}_1(\rho(P)) = \mathrm{ev}_1(Q) \in \langle g \rangle = \langle \rho(f) \rangle$.

∘ We assume that $\rho_{|B_0}(\langle f \rangle) = \langle \rho(f) \rangle \cap \operatorname{Im}(\rho_{|B_0})$ (where we see $\langle f \rangle$ and $\langle \rho(f) \rangle$ respectively as ideals in B_0 and C_0). Then, by the injectivity of the morphism ρ, we deduce that $\widetilde{\mathrm{ev}}_1(P) \in \langle f \rangle$ and conclude the proof by Lemma 7.4.

∘ Let us prove $\rho_{|B_0}(\langle f \rangle) = \langle \rho(f) \rangle \cap \operatorname{Im}(\rho_{|B_0})$. We only have to prove that $\rho_{|B_0}(\langle f \rangle) \supset \langle \rho(f) \rangle \cap \operatorname{Im}(\rho_{|B_0})$. Let $R \in B_0$ such that $\rho(R) \in \langle \rho(f) \rangle$ (seen as an ideal in C_0). Then there exists a polynomial $S \in C_0$ such that $\rho(R) = S\rho(f)$. Let us show that $S \in \operatorname{Im}(\rho)$. Each monomial $v_0^{a_2}u_0^{a_4}$ of $\rho(R)$ is, by our assumption, in the form $v_0^{a_2}u_0^{a_4} = v_0^{b_2+c_2}u_0^{b_4+c_4}$, where $v_0^{b_2}u_0^{b_4}$ (respectively $v_0^{c_2}u_0^{c_4}$) is a term of S (respectively $\rho(f)$). But $\rho(R)$ and $\rho(f)$ being in the image of ρ, by Lemma 7.9, r divides a_2 and c_2; from $a_2 = b_2 + c_2$ we then deduce that r also divides b_2. Analogously we have that $s|b_4$, then $v_0^{b_2}u_0^{b_4}$ belongs to $\operatorname{Im}(\rho)$; hence, the polynomial S also does. ☐

7.11. Let $\lambda_1, \dots, \lambda_t \in k$ be nonzero elements. We set $f_i = x_0^r - \lambda_i y_0^s$, for every integer $i \in \{1, \dots, t\}$, and $\mathscr{C}_i = \operatorname{Spec}(B_0/\langle f_i \rangle)$. We begin by proving that Lemma 7.10 can be extended to this setting.

Proposition 7.12. *Let k be a field of characteristic zero. Let $\lambda_1, \dots, \lambda_t \in k$ be nonzero elements. Let $(r, s) \in \mathbf{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f \in B_0$ be the polynomial $f = \prod_{i=1}^{t}(x_0^r - \lambda_i y_0^s)$ and $g := \rho(f) \in C_0$ its image by the morphism ρ. We set $\mathscr{C} = \operatorname{Spec}(B_0/\langle f \rangle)$ and $\mathscr{D} = \operatorname{Spec}(C_0/\langle g \rangle)$. Then $\rho(\mathcal{N}_1(\mathscr{C})) = \mathcal{N}_1(\mathscr{D}) \cap \operatorname{Im}(\rho)$.*

Proof. As the f_i are the irreducible factors of f, from the Kolchin irreducibility theorem (see formula 3.4), we deduce the formulas:

$$\mathcal{N}_1(\mathscr{C}) := \{f\} \cap B_1 = \bigcap_{i=1}^{t} (\{f_i\} \cap B_1) =: \bigcap_{i=1}^{t} \mathcal{N}_1(\mathscr{C}_i). \tag{7.5}$$

For every integer $i \in \{1, \ldots, t\}$, we set $g_i := \rho(f_i) = u_0^{rs} - \lambda_i v_0^{rs} \in C_0$ and $\mathscr{D}_i = \operatorname{Spec}(C_0/\langle g_i \rangle)$. From formulas 7.5, the injectivity of the morphism ρ and Lemma 7.10, we deduce the following equalities:

$$\rho\left(\mathcal{N}_1(\mathscr{C})\right) = \bigcap_{i=1}^{t} \rho\left(\mathcal{N}_1(\mathscr{C}_i)\right) = \bigcap_{i=1}^{t} \left(\mathcal{N}_1(\mathscr{D}_i)\right) \cap \operatorname{Im}(\rho). \tag{7.6}$$

On the other hand, by Subsection 3.14, we may assume that the field k is algebraically closed. For every integer $i \in \{1, \ldots, t\}$, let $g_i^{(j)}$ $(j \in J_i)$ be the irreducible factors of the polynomial g_i; hence the decomposition $g = \prod_{i=1}^{t} \prod_{j \in J_i} g_i^{(j)}$ is the decomposition of g into irreducible factors. By applying the Kolchin irreducibility theorem, we obtain

$$\mathcal{N}_1(\mathscr{D}_i) = \{g_i\} \cap C_1 = \bigcap_{j \in J_i} \{g_i^{(j)}\} \cap C_1,$$

and

$$\mathcal{N}_1(\mathscr{D}) = \{g\} \cap C_1 = \bigcap_{i=1}^{t} \bigcap_{j \in J_i} \{g_i^{(j)}\} \cap C_1 = \bigcap_{i=1}^{t} \mathcal{N}_1(\mathscr{D}_i). \tag{7.7}$$

We conclude the proof directly by formulas (7.6) and (7.7). □

Remark 7.13. Let us observe that Proposition 7.12 and Lemma 5.9 yield the following characterization. Let $P \in B_1$. Then, we have $P \in \mathcal{N}_1(\mathscr{C})$ if and only if $\rho(P) \in \rho(\mathcal{N}_1(\mathscr{C}))$ (because of the injectivity of the morphism ρ) if and only if $\rho(P) \in \mathcal{N}_1(\mathscr{D}) \cap \operatorname{Im}(\rho)$ if and only if $\operatorname{ev}_1(\rho(P)) \in \langle g \rangle = \langle g_1 \cdots g_t \rangle$.

Proposition 7.14. *Let k be a field of characteristic zero. Let $\lambda_1, \ldots, \lambda_t \in k$ be nonzero elements. Let $(r, s) \in \mathbf{N}^2$ be a pair of coprime integers with $r > s \geq 2$. Let $f \in B_0$ be the polynomial $f = \prod_{i=1}^{t}(x_0^r - \lambda_i y_0^s)$. We set $\mathscr{C} = \operatorname{Spec}(B_0/\langle f \rangle)$. Then the family*

$$\mathfrak{B} = \{\tilde{D}_{-1}, \widetilde{D}_{j_1,\ldots,j_t} : j_i \in \{-1, \ldots, s\}, i \in \{1, \ldots, t\}\}$$

is a Groebner basis of $\mathcal{N}_1(\mathscr{C})$ for the monomial order $y_1 >_{\mathrm{lex}} y_0 >_{\mathrm{lex}} x_1 >_{\mathrm{lex}} x_0$ in B_1.

Proof. By applying Lemma 7.4 to every element in $\mathfrak{B}$ for each of the f_i, $i \in \{1, \ldots, t\}$, and equality (7.5), we deduce that $\mathfrak{B} \subset \mathcal{N}_1(\mathscr{C})$. By [2, Proposition 5.38], in order to show that the family $\mathfrak{B}$ is a Groebner basis of $\mathcal{N}_1(\mathscr{C})$ it suffices to prove that every element in $\mathcal{N}_1(\mathscr{C})$ has some term in $\langle \mathrm{LT}(\mathfrak{B}) \rangle$.

Let us compute the leading terms of the elements of $\mathfrak{B}$ for the considered monomial order.

∘ We have $\textsc{lt}(sy_1x_0 - ry_0x_1) = y_1x_0$.

∘ For $\widetilde{D}_{j_1,\dots,j_t}$, let us denote $\ell := \sharp\{j_i : j_i \neq -1\}$. Then, for every $j_i \in \{-1,\dots,s\}$, $i \in \{1,\dots,t\}$, we have

$$\begin{aligned}\widetilde{D}_{j_1,\dots,j_t} &= \widetilde{D}_{\lambda_1,j_1}\cdots\widetilde{D}_{\lambda_t,j_t}\\ &= \prod_{j_i\neq -1}(\lambda_i s^{j_i}y_0^{s-j_i}y_1^{j_i} - r^{j_i}x_0^{r-j_i}x_1^{j_i})(sy_1x_0 - ry_0x_1)^{t-\ell}\\ &= \lambda_{i_1}\cdots\lambda_{i_\ell}s^{t-\ell+j_{i_1}+\cdots+j_{i_\ell}}y_1^{t-\ell+(j_{i_1}+\cdots+j_{i_\ell})}y_0^{\ell s-(j_{i_1}+\cdots+j_{i_\ell})}x_0^{t-\ell}+\cdots.\end{aligned}$$

Hence, we deduce that $\textsc{lt}(\widetilde{D}_{j_1,\dots,j_t}) = y_1^{t-\ell+j_{i_1}+\cdots+j_{i_\ell}}y_0^{\ell s-(j_{i_1}+\cdots+j_{i_\ell})}x_0^{t-\ell}$.

We conclude

$$\langle\textsc{lt}(\mathfrak{B})\rangle = \langle y_1x_0, \{y_1^{t-\ell+(j_{i_1}+\cdots+j_{i_\ell})}y_0^{\ell s-(j_{i_1}+\cdots+j_{i_\ell})}x_0^{t-\ell}\}_{\substack{0\leq\ell\leq t\\ 0\leq j_{i_1},\dots,j_{i_\ell}\leq s}}\rangle.$$

Let $P \in \mathcal{N}_1(\mathscr{C})$. We aim to prove that some of its terms belongs to $\langle\textsc{lt}(\mathfrak{B})\rangle$. By Remark 7.13 we know that $\mathrm{ev}_1(\rho(P)) \in \langle g\rangle = \langle g_1\cdots g_t\rangle$. Two cases occur:

∘ If $\mathrm{ev}_1(\rho(P)) = 0$, then, by formula 7.3, we have $\rho(\widetilde{\mathrm{ev}}_1(P)) = 0$. By the injectivity of the morphism ρ, we deduce that $\widetilde{\mathrm{ev}}_1(P) = 0$. But, by Proposition 4.6, this means that $P \in \langle sy_1x_0 - ry_0x_1\rangle$; hence, we conclude that the monomial y_1x_0 divides $\textsc{lt}(P)$.

∘ If $\mathrm{ev}_1(\rho(P)) \neq 0$, then P has some term $y_1^{a_1}y_0^{a_2}x_1^{a_3}x_0^{a_4}$ such that $\textsc{lt}(g) = v_0^{trs}$ (we are considering the monomial order $v_1 >_{\mathrm{lex}} v_0 >_{\mathrm{lex}} u_1 >_{\mathrm{lex}} u_0$ in C_1) divides $\mathrm{ev}_1(\rho(y_1^{a_1}y_0^{a_2}x_1^{a_3}x_0^{a_4})) = \mathrm{ev}_1(v_1^{a_1}v_0^{r(a_1+a_2)-a_1}u_1^{a_3}u_0^{s(a_3+a_4)-a_3}) = v_0^{r(a_1+a_2)}u_0^{s(a_3+a_4)}$. Thus, it implies that $trs \leq r(a_1+a_2)$; hence, we have $ts \leq a_1+a_2$. For $0 \leq j_1,\dots,j_t \leq s$, the pairs $(j_1+\cdots+j_t, ts-(j_1+\cdots+j_t))$ range over all possible pairs of nonnegative integers whose sum equals ts. Thus some monomial in $\{y_1^{j_1+\cdots+j_t}y_0^{ts-(j_1+\cdots+j_t)}\}_{0\leq j_1,\dots,j_t\leq s}$ (which is a subset of $\textsc{lt}(\mathfrak{B})$, take $\ell = t$) divides the term $y_1^{a_1}y_0^{a_2}$, and hence also the term $y_1^{a_1}y_0^{a_2}x_1^{a_3}x_0^{a_4}$. ☐

7.15. Let us prove Theorem 7.6.

For every integer $i \in \{1,\dots,t\}$, we set $f_i = x_0^r - \lambda_i y_0^s \in B_0$, $f_{cusp} = \prod_{i=1}^t f_i \in B_0$, $g_i = \rho(f_i) = u_0^{rs} - \lambda_i v_0^{rs} \in C_0$ and $g_{cusp} = \rho(f_{cusp}) \in C_0$. The corresponding affine plane k-curves are respectively denoted by $\mathscr{C}_i = \mathrm{Spec}(B_0/\langle f_i\rangle)$ and $\mathscr{C}_{cusp} = \mathrm{Spec}(B_0/\langle f_{cusp}\rangle)$. We write $\mathscr{C}_x$ (resp. $\mathscr{C}_y$) for the affine plane k-curve attached to the datum of x_0^{ε} (resp. $y_0^{\varepsilon'}$). By applying the Kolchin theorem as in the proof of Proposition 7.12, we deduce that $\mathcal{N}_1(\mathscr{C}) = \mathcal{N}_1(\mathscr{C}_x) \cap \mathscr{N}_1(\mathscr{C}_y) \cap \mathcal{N}_1(\mathscr{C}_{cusp})$. Then by the injectivity of ρ and Remark 7.13 applied to $\mathcal{C}_{cusp}$, we deduce that, if we take a polynomial P in $\mathcal{N}_1(\mathcal{C})$, then $\mathrm{ev}_1(\rho(P))$ belongs to $\langle u_0^{s\varepsilon}\rangle \cap \langle v_0^{r\varepsilon'}\rangle \cap \langle g_{cusp}\rangle$. We can write $\mathrm{ev}_1(\rho(P))$ in the form $\mathrm{ev}_1(\rho(P)) = Qg_{cusp}$ for a polynomial $Q \in C_0$, and $u_0^{s\varepsilon}$ and $v_0^{r\varepsilon'}$ divide Qg_{cusp}. Let us recall that

$g_{cusp} = \prod_{i=1}^{t} g_i = \prod_{i=1}^{t} u_0^{rs} - \lambda_i v_0^{rs} = (-1)^t \lambda_1 \cdots \lambda_t v_0^{trs} + \cdots + u_0^{trs}$. A direct calculation then proves that $u_0^{s\varepsilon} v_0^{r\varepsilon'}$ divides Q. So, the polynomial $\mathrm{ev}_1(\rho(P))$ can be written as $Q' u_0^{s\varepsilon} v_0^{r\varepsilon'} g_{cusp}$. We conclude that the polynomial $\mathrm{ev}_1(\rho(P))$ belongs to the ideal $\langle u_0^{s\varepsilon} v_0^{r\varepsilon'} g_{cusp} \rangle = \langle \rho(f) \rangle$. It is clear that $\mathfrak{B} \subset \mathcal{N}_1(\mathscr{C})$. By [2, Proposition 5.38], in order to show that $\mathfrak{B}$ is a Groebner basis of $\mathcal{N}_1(\mathscr{C})$ it is sufficient to prove that every element in $\mathcal{N}_1(\mathscr{C})$ has some term in $\langle \mathrm{LT}(\mathfrak{B}) \rangle$. From the computations in the first part of the proof of Proposition 7.14 we deduce that

$$\langle \mathrm{LT}(\mathfrak{B}) \rangle = \left\langle y_1 x_0, \{ x_{h_1}^{\varepsilon} y_{h_2}^{\varepsilon'} y_1^{t-\ell+j_{i_1}+\cdots+j_{i_\ell}} y_0^{\ell s-(j_{i_1}+\cdots+j_{i_\ell})} x_0^{t-\ell} \}_{\substack{0 \leq \ell \leq t \\ 0 \leq j_{i_1},\ldots,j_{i_\ell} \leq s \\ 0 \leq h_1, h_2 \leq 1}} \right\rangle .$$

Let P be a polynomial in $\mathcal{N}_1(\mathcal{C})$. We have already observed that $\mathrm{ev}_1(\rho(P))$ belongs to $\langle u_0^{s\varepsilon} v_0^{r\varepsilon'} g_{cusp} \rangle$. Then, we finish the proof in an analogous way as we did in the proof of Proposition 7.14; the arguments are indeed the same as in the last part of that proof.

Remark 7.16*.* We observe that we can take only some of the elements in $\mathfrak{B}$. For example, the following family is a Groebner basis of $\mathcal{N}_1(\mathscr{C})$ (we keep the notation and assumptions of Theorem 7.6):

$$\mathfrak{B} = \{ s y_1 x_0 - r y_0 x_1, x_{h_1}^{\varepsilon} y_{h_2}^{\varepsilon'} \widetilde{D}_{j_1,\ldots,j_t} : h_1, h_2 \in \{0,1\}, j_i \in \{0,\ldots,s\}, i \in \{1,\ldots,t\} \}$$

where j_i is zero unless $j_m = s$ for every integer $m < i$.

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Mario Morán Cañón
Université de Rennes
CNRS, IRMAR - UMR 6625
F-35000 Rennes, France
Email address: `mario.moran-canon@univ-rennes1.fr`

Julien Sebag
Université de Rennes
CNRS, IRMAR - UMR 6625
F-35000 Rennes, France
Email address: `julien.sebag@univ-rennes1.fr`