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CHARACTERIZING ALMOST PERFECT RINGS BY COVERS AND ENVELOPES

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Dedicated to Luigi Salce on his 70th birthday

Abstract. Characterizations of almost perfect domains by certain covers and envelopes, due to Bazzoni–Salce [7] and Bazzoni [4], are generalized to almost perfect commutative rings (with zero-divisors). These rings were introduced recently by Fuchs–Salce [14], showing that the new rings share numerous properties of the domain case. In this note, it is proved that admitting strongly flat covers characterizes the almost perfect rings within the class of commutative rings (Theorem 3.7). Also, the existence of projective dimension 1 covers characterizes the same class of rings within the class of commutative rings admitting the cotorsion pair $(\mathcal{P}_1, \mathcal{D})$ (Theorem 4.1). Similar characterization is proved concerning the existence of divisible envelopes for h-local rings in the same class (Theorem 5.3). In addition, Bazzoni's characterization *via* direct sums of weak-injective modules [4] is extended to all commutative rings (Theorem 6.4). Several ideas of the proofs known for integral domains are adapted to rings with zero-divisors.

1. Introduction

Almost perfect domains were defined by Bazzoni–Salce [8] as integral domains R such that R/I is a perfect ring (in the sense of Bass [3]) for every ideal $I \neq 0$ of R. These domains have attracted much attention, they have been investigated extensively, and it was shown that they can be characterized in various ways, both ring- and module-theoretically (see [4], [8]).

The theory of almost perfect domains has been extended to a class of rings with divisors of zero by Fuchs–Salce [14]. By definition, a commutative ring R is *almost perfect* if

- R is an order in a perfect ring Q; and
- R/Rr is a perfect ring for each non-zero-divisor $r \in R$ (equivalently, R/I is perfect for every ideal I containing such an r).

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It was shown in [14] that several characterizations of almost perfect domains carry over practically without change to almost perfect rings. (For the extension of results in [14] to the non-commutative case, see Facchini–Nazemian [11].) However, characterizations by covers and envelopes have not as yet been discussed for these rings, so it is natural to raise the question whether or not these too extend to almost perfect rings. In this note, we wish to give an affirmative answer to this question.

In the case of strongly flat covers, most of the old techniques (developed by Bazzoni–Salce [8]) could be applied *mutatis mutandis,* combined with some additional arguments needed (Theorems 3.7). As pointed out by Silvana Bazzoni, a characterization in terms of projective dimension 1 covers can be derived from recent results by Angeleri Hügel–Šaroch–Trlifaj [2], requiring more sophisticated machinery; see Theorem 4.1. Since covers and envelopes are primarily investigated for cotorsion pairs, in dealing with projective dimension 1 covers and divisible envelopes we will assume to start with that $(\mathcal{P}_1, \mathcal{D})$ consisting of the corresponding classes is a cotorsion pair. This pair is known to be a cotorsion pair for all integral domains (see Bazzoni–Herbera [6]) (but for general rings all what can be asserted is that the kernels of covers and the cokernels of envelopes belong to the orthogonal classes; cf. Wakamatsu Lemma, [15, Lemma 2.1.13]). For the existence of divisible envelopes we have succeeded in settling the problem only for h-local rings (Theorem 5.3).

In the final section, another major characterization of almost perfect domains, due to Bazzoni [4], is extended to rings with zero-divisors. It is shown that the almost perfect rings are distinguished by the property that the class of weak-injective modules is closed under the formation of (countable) direct sums.

Since the noetherian almost perfect rings are nothing else than one-dimensional Cohen-Macaulay rings [14, Theorem 5.8], our results provide characterizations of such Cohen-Macaulay rings as well.

2. Preliminaries

In this note, we consider only commutative rings R with identity. Q always denotes the classical ring of quotients of R, and K the factor module Q/R. $r \in R$ is called *regular* if it is a non-zero-divisor. The symbol $R^\times$ will stand for the set (monoid) of regular elements of R. An ideal is *regular* if it contains a regular element. R-Mod will denote the class of R-modules.

An R-module T is a *torsion module* if every $x \in T$ is annihilated by a suitable $r \in R^\times$. A module is *torsion-free* if it does not contain such an $x \neq 0$. D is called *divisible* if $rD = D$ for each $r \in R^\times$, i.e., $\mathrm{Ext}^1_R(R/Rr, D) = 0$ if $r \in R^\times$. A module H is *h-divisible* if every homomorphism $R \to H$ extends to a homomorphism $Q \to H$; equivalently, H is an epic image of some direct sum $\oplus Q$. The h-divisible torsion-free R-modules are precisely the Q-modules. An R-module M is said to be *weak-injective* if $\mathrm{Ext}^1_R(A, M) = 0$ for all R-modules

A with w.d.$A \leq 1$ (Lee [16]). (The notations p.d. and w.d. will be used for the projective, resp. for the weak (flat) dimension.) *h-reduced* means no divisible submodule $\neq 0$.

For a class $\mathcal{C}$ of R-modules, define

$$\mathcal{C}^{\perp} = \{M \in R\text{–Mod} \mid \operatorname{Ext}^1_R(C, M) = 0 \ \forall C \in \mathcal{C}\},$$

$$^{\perp}\mathcal{C} = \{M \in R\text{–Mod} \mid \operatorname{Ext}^1_R(M, C) = 0 \ \forall C \in \mathcal{C}\}.$$

A pair $(\mathcal{A}, \mathcal{B})$ of R-module classes is said to be a *cotorsion pair* if both $\mathcal{A} =^{\perp} \mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$ hold. If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, then by an $\mathcal{A}$-*precover* of an R-module M is meant a module $A \in \mathcal{A}$ along with a homomorphism $\alpha : A \to M$ such that every homomorphism $A' \to M$ from any $A' \in \mathcal{A}$ factors through α. An $\mathcal{A}$-precover A is an $\mathcal{A}$-*cover* if it is minimal in the sense that every endomorphism η of A satisfying $\alpha\eta = \alpha$ is an automorphism. $\mathcal{B}$-*(pre)envelopes* are defined dually. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ of R-modules is *perfect* if R-modules admit both $\mathcal{A}$-covers and $\mathcal{B}$-envelopes.

The symbol $\mathcal{P}_1$ denotes the class of R-modules of p.d.≤ 1, and $\mathcal{F}_1$ the class of modules of w.d.≤ 1. $\mathcal{D}$ is the class of divisible, $\mathcal{HD}$ is the class of h-divisible, and $\mathcal{WI}$ is the class of weak-injective modules. We have $\mathcal{WI} \subseteq \mathcal{HD} \subseteq \mathcal{D}$, the inclusions are in general proper. The pair $(\mathcal{F}_1, \mathcal{WI})$ is a perfect cotorsion pair over every (associative) ring (see Göbel–Trlifaj [15]); so it admits envelopes and covers. The pair $(\mathcal{P}_1, \mathcal{D})$ is known to be a cotorsion pair over integral domains, but only $\mathcal{P}_1 =^{\perp} \mathcal{HD}$ holds over all commutative rings; for details, we refer to the paper Bazzoni–Herbera [6].

In [13], a commutative ring R with p.d.$Q \leq 1$ was named a *Matlis ring*. Observe that R is a Matlis ring if and only if it satisfies one of the following equivalent properties:

- *divisible R-modules are h-divisible*;
- $\operatorname{Ext}^1_R(Q, D) = 0$ *for all divisible R-modules D*;
- *a divisible torsion R-module has* p.d.≤ 1 *if and only if it is a summand of a direct sum of copies of $K = Q/R$.*

By an *h-local ring* we mean a ring R for which the module K decomposes into the direct sum of its localizations:

$$K = \oplus_P K_P,$$

where P ranges over the set of regular maximal ideals P of R.

The class $\mathcal{MC}$ of *Matlis-cotorsion* modules is defined to consist of modules M satisfying $\operatorname{Ext}^1_R(Q, M) = 0$. An R-module S is *strongly flat* if $\operatorname{Ext}^1_R(S, M) = 0$ holds for all Matlis-cotorsion M (Bazzoni–Salce [8]). Strongly flat modules are flat. $(\mathcal{SF}, \mathcal{MC})$ is a cotorsion pair, where $\mathcal{SF}$ denotes the class strongly flat modules.

Proposition 2.1. *Over any commutative ring, the class $\mathcal{SF}$ consists of summands of modules N that fit into an exact sequence of the form*

$$0 \to F \to N \to G \to 0,$$

where F is free and G is a direct sum $\oplus Q$. A divisible strongly flat module is therefore a summand of a direct sum of copies of Q.

Proof. Göbel–Trlifaj [15, Corollary 3.2.3] establishes the existence of such an exact sequence with a free module F and a $\{Q\}$-filtered module G. Using the simple fact that $\mathrm{Ext}^1_R(Q, D) = 0$ holds for all torsion-free divisible module D over any commutative ring [13, Lemma 3.8], we argue that in this case Q splits off any such filtered module, and hence 'filtered' may be replaced by 'direct sum.' □

Perfect rings can be defined in different ways; as their definition we shall use the property that their modules admit projective covers; see Bass [3]. They admit various other characterizations, e.g. that flat modules are projective, or that the descending chain condition holds for principal ideals. A commutative ring is perfect if and only if it is the direct sum of a finite number of local rings, each with T-nilpotent maximal ideal. (T-nilpotent means that for every sequence $a_0, \dots, a_n, \dots$ in the ideal there is an index m such that $a_0 \cdots a_m = 0$.)

A ring R will be called *subperfect* if its ring of quotients Q is a perfect ring (i.e., it is an order in a perfect ring). E.g., integral domains, Cohen-Macaulay rings are subperfect. The nilradical of a subperfect ring is T-nilpotent. Recall: an *almost perfect* ring R is a subperfect ring such that R/Rr is a perfect ring for every $r \in R^\times$. Some key properties of almost perfect rings are listed next.

Proposition 2.2. (Fuchs–Salce [14, part of Theorem 6.1]) *Suppose R is a subperfect ring. Then the following conditions are equivalent*:

(i) *R is an almost perfect ring*;
(ii) *R-modules of* w.d.≤ 1 *are of* p.d.≤ 1;
(iii) *$(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair and equals $(\mathcal{F}_1, \mathcal{WI})$*;
(iv) *divisible* (*h-divisible*) *R-modules are weak-injective*;
(v) *flat R-modules are strongly flat.*

For more on almost perfect rings, for their structure, and for their relation to almost perfect domains, we refer to [14], where also a great variety of examples is exhibited. For unexplained terminology and basic facts on cotorsion pairs, envelopes, covers, etc. we refer to Göbel–Trlifaj [15] and Enochs–Jenda [10].

3. Strongly flat covers

In our search for rings with covers or envelopes, we first treat the case of rings R over which the modules admit strongly flat covers — this was the deciding property that led Bazzoni and Salce to the discovery of almost perfect domains.

We will be working with $\mathcal{SF}$-cover sequences. Recall that an exact sequence

$$0 \to C \to S \xrightarrow{\phi} M \to 0 \tag{1}$$

is an $\mathcal{SF}$-cover sequence for an R-module M if and only if S is strongly flat, C is Matlis-cotorsion (often conveniently viewed as a submodule of S), and every endomorphism ρ of S satisfying $\phi = \phi\rho$ is an automorphism.

We now turn to the proofs of preliminary lemmas.

Lemma 3.1. *If the divisible modules over a ring R admit strongly flat covers, then R is a Matlis ring.*

Proof. Let (1) be an $\mathcal{SF}$-cover sequence for a divisible module M. First we show that the strongly flat module S is divisible. As M is divisible, for every $r \in R^{\times}$ we have $rS + C = S$. From the exact sequence $0 \to rS \cap C \to C \to S/rS \to 0$ we conclude that $rS \cap C$ is Matlis-cotorsion (use the functor $\mathrm{Hom}(Q, -)$). Therefore, there exists a map $\alpha : S \to rS$ making the upper squares commute in the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & S & \xrightarrow{\phi} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \| & & \\
0 & \longrightarrow & rS \cap C & \longrightarrow & rS & \xrightarrow{\phi\restriction_{rS}} & M & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle \beta}\downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & S & \xrightarrow{\phi} & M & \longrightarrow & 0.
\end{array}$$

With the embedding map $\beta : rS \to S$, the lower squares commute. The diagram shows that $\phi = \phi\beta\alpha$, whence by the cover property of S it follows that $\beta\alpha$ is an automorphism of S. Consequently, β is surjective, and $rS = S$, as claimed.

Thus the strongly flat module S is divisible, hence also h-divisible by Proposition 2.1. We conclude that the divisible module M is h-divisible. If every divisible R-module is h-divisible, then R is a Matlis ring (cf. Fuchs–Lee [13, Theorem 6.4]). □

Lemma 3.2. *For any ring R, if the torsion-free divisible R-modules admit strongly flat covers, then the ring of quotients Q of R is a perfect ring.*

Proof. Let M be a Q-module, viewed as a torsion-free divisible R-module, and let (1) be an $\mathcal{SF}$-cover sequence for M. By the proof of Lemma 3.1, S is divisible, thus S as a divisible strongly flat module is a summand of a direct sum of copies of Q (Proposition 2.1), i.e., S is projective as a Q-module. The module C is also a Q-module (the rest of the exact sequence does not change if tensored by Q). Therefore, the given exact sequence may be treated as one in Q-Mod, where it is a Q-projective resolution of M. By the choice of the above exact sequence, it must be a cover sequence for such a resolution. We conclude that Q-modules admit projective covers, and consequently, Q is a perfect ring (Bass [3]). □

We will also need the following lemmas. In what follows, $\widetilde{M}$ will denote the completion of M in the $R^{\times}$-topology (the completion is Hausdorff, so the kernel of the canonical homomorphism $M \to \widetilde{M}$ is the divisible submodule of M). For the exchange property of modules with local endomorphism rings, see e.g. Warfield [17].

Lemma 3.3. *Let R be a local Matlis ring with regular maximal ideal. Then the endomorphism ring* $\operatorname{End}_R K \cong \widetilde{R}$ *is a local ring, so the module K enjoys the exchange property.*

Proof. From the exact sequence $0 \to R \to Q \to K \to 0$ (where $K \neq 0$ since the maximal ideal is regular) we derive the exactness of the sequence

$$0 \to \operatorname{Hom}_R(K, K) \to \operatorname{Ext}^1_R(K, R) \to \operatorname{Ext}^1_R(K, Q) = 0.$$

Furthermore, we form the exact sequence $0 \to D \to R \to R/D \to 0$ (where D denotes the divisible part of R), inducing the exact sequence

$$\operatorname{Ext}^1_R(K, D) \to \operatorname{Ext}^1_R(K, R) \to \operatorname{Ext}^1_R(K, R/D) \to 0;$$

here the first Ext vanishes as p.d.$K = 1$ (R is Matlis) and D is divisible. Hence we conclude that $\operatorname{End}_R K \cong \operatorname{Ext}^1_R(K, R/D)$, where the last Ext is known to be the completion $\widetilde{R}_0$ of the h-reduced module $R_0 = R/D$. This completion is a local ring, it is the same as $\widetilde{R}$. □

Lemma 3.4. *Let R be again a local Matlis ring with regular maximal ideal. A strongly flat Matlis-cotorsion R-module M embeds in an exact sequence*

$$0 \to B \to M \to \widetilde{F} \to 0, \tag{2}$$

where B is torsion-free divisible and $\widetilde{F}$ is the completion of a free R-module F.

Proof. As a strongly flat module, M is a summand of a module N that fits into an exact sequence $0 \to H \to N \to G \to 0$ where H is a free module and G is a direct sum $\oplus Q$. Tensoring with K, we obtain $H \otimes_R K \cong N \otimes_R K$, whence we conclude that $M \otimes_R K$ is isomorphic to a summand of $H \otimes_R K \cong \oplus K$. Referring to Lemma 3.3, we argue that $M \otimes_R K$ is also a direct sum of copies of K. It follows that the h-reduced part $M' = M/B$ of the Matlis-cotorsion M (M' is likewise Matlis-cotorsion) satisfies

$$M' \cong \operatorname{Hom}_R(K, M \otimes_R K) \cong \operatorname{Hom}_R(K, \oplus K) \cong \widetilde{(\oplus R)}$$

(for the first isomorphism we have applied the Matlis category equivalence over R, see e.g. [13, Theorem 5.1]). This establishes the claim. □

The following arguments are borrowed from Bazzoni–Salce [7], with an extra consideration of rings with divisors of zero, up to reaching the conclusion that flat modules are strongly flat. Once this is done, we can then finish the proof by simply quoting [14, Theorem 6.1], a major theorem on almost perfect rings (see Proposition 2.2).

Lemma 3.5. *Let R be any commutative ring, and suppose $0 \to C \to S \to M \to 0$ is an $\mathcal{SF}$-cover sequence for an R-module M. Then*

$$C \leq PS \quad for\ every\ maximal\ ideal\ P\ of\ R.$$

Proof. By way of contradiction, assume $C \not\leq PS$ for some P. Then necessarily $PS < S$, and we can find a proper submodule $A < S$ containing PS such that $C + A = S$. The submodule $A \cap C$ is Matlis-cotorsion, since in the exact sequence $0 \to A \cap C \to C \to S/A \to 0$, C is such and S/A is semisimple (use the functor $\mathrm{Hom}(Q, -)$). If we argue as in the first part of the proof of Lemma 3.1 with rS replaced by A, then we are led to the contradiction $A = S$. □

Lemma 3.6. *Suppose R is a subperfect Matlis ring. If every flat R-module admits a strongly flat cover, then every flat R-module is strongly flat.*

Proof. By hypothesis, there exists an $\mathcal{SF}$-cover sequence $0 \to C \to S \to M \to 0$ for the flat module M. Owing to Lemma 3.5, $C \leq PS$ holds for all maximal ideals P of R. Now C is pure in S (since M is flat), so we have $PC = C \cap PS = C$ for all P. Evidently, C is also flat, so by assumption it embeds in an $\mathcal{SF}$-cover sequence $0 \to C' \to S' \to C \to 0$. Here C', C, and hence also S', are Matlis-cotorsion. By Proposition 2.1, S' is a summand of an extension N of a free R-module by a direct sum $\oplus Q$. As before, it follows that $PC' = C' \leq PS'$ for all P. Clearly, from $PC = C, PC' = C'$ we obtain $PS' = S'$ for all P.

We claim that $PS' = S'$ for all maximal ideals P implies that our strongly flat Matlis-cotorsion module S' is divisible. It suffices to verify this for local rings R (localizations of subperfect Matlis rings are subperfect Matlis).

First we deal with the case when the maximal ideal P is regular. We now have $N = S' \oplus S''$ for a suitable module S'', and a similar decomposition holds for the divisible hulls (modules tensored by Q): $H = D' \oplus D''$. Evidently, $H/N \cong D'/S' \oplus D''/S''$ is a direct sum of copies of the indecomposable K. As R is local Matlis, Lemma 3.3 implies that D'/S' is likewise a direct sum of copies of K. To show that $S' = D'$, we refer to the exact sequence (2) which in the present situation takes the form $0 \to B \to S' \to S'/B \to 0$ (with divisible B and completion S'/B of a free module). Since $PS' = S'$ and $PB = B$, we must have also $P(S'/B) = S'/B$. As S'/B is the completion of a free R-module, $P(S'/B) = S'/B$ must be 0, i.e., S' is divisible.

If the maximal ideal P of the local R is not regular, then P is at the same time a minimal prime in R. The ring R is subperfect, so its nilradical P is T-nilpotent. This means that R is a perfect ring. Over a perfect ring all modules are divisible, thus S' is divisible in this case as well. This completes the proof of the divisibility of S'.

Consequently, in the given exact sequence, C is likewise a Q-module. As Q is a perfect ring, by [14, Theorem 6.5] Q-modules are weak-injective as R-modules. Hence it follows that the above $\mathcal{SF}$-cover sequence for M splits, and M is strongly flat. □

We have all the ingredients to be able to verify:

Theorem 3.7. *The modules over a commutative ring R admit strongly flat covers if and only if R is almost perfect.*

Proof. One way the claim follows immediately from Proposition 2.2: over an almost perfect ring, strongly flat modules are the same as flat modules, and the existence of flat covers (see Bican–El Bashir–Enochs [9]) completes the proof of sufficiency.

Conversely, assume R-modules admit $\mathcal{SF}$-covers. By Lemma 3.1, R is a Matlis ring, and by Lemma 3.2, it is also subperfect. A reference to Lemma 3.6 shows that all flat R-modules are strongly flat. It remains to appeal to Proposition 2.2 to conclude that R is almost perfect. □

The following corollary is immediate.

Corollary 3.8. (Bazzoni–Salce [7]) *An integral domain admits strongly flat covers if and only if it is an almost perfect domain.*

Considering that a noetherian almost perfect ring is the same as a one-dimensional Cohen-Macaulay ring, we can also state:

Corollary 3.9. *The modules over a commutative noetherian ring admit strongly flat covers if and only if the ring is one-dimensional Cohen-Macaulay.*

4. $\mathcal{P}_1$-covers

Next we turn our attention to those rings over which the modules admit $\mathcal{P}_1$-covers. It makes sense to deal with this problem under the assumption that $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair. This hypothesis has a powerful consequence: recall Fuchs [12, Theorem 6.5] (it also follows from Bazzoni–Herbera [6, Theorem 6.3]) which shows that if $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair in R-Mod, then the finitistic dimension $\mathrm{Fdim}(Q) = 0$, i.e., in the category of Q-modules, all modules of finite p.d. are projective; consequently, Q is a perfect ring. Thus R is a subperfect ring.

I am indebted to Silvana Bazzoni for pointing out that from a recent result by Angeleri Hügel–Šaroch–Trlifaj [2] one can easily derive a characterization of rings whose modules admit $\mathcal{P}_1$-covers. In view of this, our original weaker theorem can be replaced by the following much stronger result:

Theorem 4.1. *Suppose R is a commutative ring such that $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair. The R-modules admit $\mathcal{P}_1$-covers if and only if R is an almost perfect ring.*

Proof. In [2, Theorem 5.2] it is shown that, over any ring R, if in a cotorsion pair $(\mathcal{A}, \mathcal{B})$ the class $\mathcal{B}$ is closed under direct limits, then the R-modules admit $\mathcal{A}$-covers if and only if also the class $\mathcal{A}$ is closed under direct limits. Since the class $\mathcal{D}$ is closed under direct limits, this theorem applies to the cotorsion pair $(\mathcal{P}_1, \mathcal{D})$. As for subperfect rings $\mathcal{F}_1$ is precisely the class of direct limits of modules in class $\mathcal{P}_1$, it follows that over a ring with $(\mathcal{P}_1, \mathcal{D})$ as cotorsion pair,

$\mathcal{P}_1$-covers exist exactly if $\mathcal{P}_1 = \mathcal{F}_1$. This equality is satisfied by the module classes of a subperfect ring if and only if the ring is almost perfect [14, Theorem 6.1]. □

As far as the hypothesis of $(\mathcal{P}_1, \mathcal{D})$ being a cotorsion pair over R is concerned, note that this is equivalent to having $\mathrm{Fdim}(Q) = 0$ plus the existence of filtrations for every module in $\mathcal{P}_1$ with countably presented factors in $\mathcal{P}_1$ (e.g. [12, Theorem 6.5]).

I wish to thank the referee for pointing out to me that, in view of Bazzoni–Herbera [6, Corollary 8.4], if R is a commutative noetherian ring, then the class $\mathcal{D}$ of divisible modules is equal to $\mathcal{B}^{\perp}$ for a specific subclass $\mathcal{B}$ of $\mathcal{P}_1$; and hence it follows that in this situation $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair. Consequently, in the noetherian case we can draw the following corollary to Theorem 4.1:

Corollary 4.2. *Modules over a noetherian ring admit $\mathcal{P}_1$-covers if and only if the ring is one-dimensional Cohen-Macaulay.*

5. Divisible envelopes

Our next objective is to characterize the rings whose modules admit $\mathcal{D}$-envelopes. Again, to start with, we assume that $(\mathcal{P}_1, \mathcal{D})$ is a genuine cotorsion pair for R-modules. As pointed out in Section 4, if $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair of R-modules, then R has to be a subperfect ring.

We begin with a lemma allowing us to restrict our search to Matlis rings. For a more general version of this lemma, see Angeleri Hügel–Herbera–Trlifaj [1, Theorem 1.1].

Lemma 5.1. *If $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair and the module R_R has a $\mathcal{D}$-envelope, then R is a Matlis ring, and Q is the $\mathcal{D}$-envelope of R.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc} & & & & D & & & & \\ & & & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle \dot r} & & & & \\ 0 & \longrightarrow & R & \xrightarrow{\delta} & D & \longrightarrow & D/R & \longrightarrow & 0 \end{array}$$

with a $\mathcal{D}$-envelope sequence for R. As $\dot r$ (multiplication by $r \in R^{\times}$) is a surjective map on D, there exists a map $\gamma : R \to D$ such that $\dot r\gamma = \delta$. The envelope property of D implies the existence of a map $\phi : D \to D$ satisfying $\phi\delta = \gamma$. Thus $\dot r\phi\delta = \dot r\gamma = \delta$, and referring again to the envelope property, it follows that $\dot r\phi$ is an automorphism of D. This can happen only if $\dot r$ is an automorphism (as ϕD is divisible, it must be all of D whenever $\dot r\phi D = D$; and that $\dot r$ must be monic is clear from the automorphism $\dot r\phi$). Hence D is torsion-free. As $\mathrm{p.d.}D/R \le 1$, we have $\mathrm{p.d.}D = 1$.

In view of the envelope property, for the embedding $R \to Q$, there exists a map $\psi : D \to Q$ that extends the identity map of R. As $\mathrm{Im}\,\psi$ is divisible and Q is a minimal divisible module containing R, ψ is surjective. $\operatorname{Ker}\psi$ is likewise a Q-module (pure in D, hence divisible), so $D \cong \operatorname{Ker}\psi \oplus Q$. Therefore p.d.$Q = 1$, and R is a Matlis ring. It also follows that $D = Q$. $\square$

The rest of the proof is an application Bazzoni's techniques [4].

Lemma 5.2. *Suppose $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair and the modules over the local Matlis ring R with regular maximal ideal P admit $\mathcal{D}$-envelopes. Then for every $r \in R^\times$, the factor ring R/Rr is perfect.*

Proof. Let $r \in R^\times$ be a non-unit, and consider the exact sequence

(3) $$0 \to Rr^{-1}/R \to Q/R \to Q/Rr^{-1} \to 0$$

which is clearly a special $\mathcal{D}$-preenvelope sequence for Rr^{-1}/R over the Matlis ring R. As $Q/Rr^{-1} \cong K$ is indecomposable for a local R, and by hypothesis Rr^{-1}/R has an envelope, (3) must be a $\mathcal{D}$-envelope sequence for Rr^{-1}/R.

In order to prove that the ring R/Rr is perfect, we show that its maximal ideal P/Rr is T-nilpotent. Since for a sequence of $t_n \in P \setminus Rr$ $(n < \omega)$, the $\mathcal{D}$-envelope of $\oplus_{n<\omega} Rt_n^{-1}/R$ is the direct sum of the $\mathcal{D}$-envelopes of the Rt_n^{-1}/R, i.e., $\oplus_{n<\omega} Q/R$, we can appeal to a theorem by Enochs–Jenda [10, Proposition 6.4.1] on the envelope of a direct sum that is isomorphic to the direct sum of the envelopes of the summands. It claims that then for every countable sequence $\eta_n : Q/R \to Q/R$ $(n < \omega)$ of homomorphisms satisfying $\eta_n(Rt_n^{-1}/R) = 0$ and for each $q \in Q/R$, there exists an index $m \geq 1$ such that $\eta_m \eta_{m-1} \cdots \eta_0(q) = 0$.

Let $q = r^{-1} + R \in Q/R$ with any non-unit $r \in R^\times$. Multiplication by t_n is an endomorphism in Q/R that annihilates Rt_n^{-1}/R, so there is an index $m \geq 1$ such that

$$t_m t_{m-1} \cdots t_0(r^{-1} + R) = 0.$$

This is equivalent to saying that $t_m t_{m-1} \cdots t_0 \in Rr$, establishing the T-nilpotency of P/Rr. $\square$

We are able to verify the following theorem only for h-local rings (we have failed to show the h-local property in the global case).

Theorem 5.3. *Suppose R is an h-local commutative ring with the cotorsion pair $(\mathcal{P}_1, \mathcal{D})$. R-modules admit divisible envelopes if and only if R is almost perfect.*

Proof. If R is almost perfect, then divisibility is tantamount to weak-injectivity, and weak-injective envelopes exist for all rings.

Conversely, it suffices to deal with local rings R. Assume the existence of $\mathcal{D}$-envelopes for R-modules. Lemma 5.1 implies that R is a Matlis ring, thus R is a subperfect Matlis ring. If the maximal ideal P of R is not regular, then all regular elements of R are units, so R is trivially almost perfect. On the other hand, if P is regular, then by Lemma 5.2, R/Rr is a perfect ring for every non-unit $r \in R^\times$, and so R is almost perfect. $\square$

In the noetherian case, we do not need the hypothesis that $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair (just as in Corollary 4.2), so we can state:

Corollary 5.4. *The modules over a commutative h-local noetherian ring admit divisible envelopes if and only if the ring is one-dimensional Cohen-Macaulay.*

6. Direct sums of weak-injectives

There is one more important characterization of almost perfect domains, due to Bazzoni. She proved that an integral domain is almost perfect if and only if direct sums of weak-injective modules are again weak-injective (see [4, Theorem 5.7]). We are looking for an analogous characterization of almost perfect rings.

For the next proof, it should be borne in mind that $(\mathcal{F}_1, \mathcal{WI})$ is a perfect cotorsion pair. Furthermore, if the class of weak-injective modules is closed under direct sums, then the $\mathcal{WI}$-envelope of a direct sum is the direct sum of the $\mathcal{WI}$-envelopes of the summands.

Lemma 6.1. *For any commutative ring R, if countable direct sums of the $\mathcal{WI}$-envelopes of cyclic torsion R-modules are again weak-injective, then R is a subperfect ring.*

Proof. It suffices to show that if R is as stated, then the principal ideals of its quotient ring Q satisfy the descending chain condition (Bass [3]). Let

$$Qt_0 > Qt_1 > \cdots > Qt_n > \cdots \tag{4}$$

be a strictly descending chain where we may assume without loss of generality that $t_n \in R$, since multiplication (or division) of a generator by a non-zero-divisor in R does not change a principal ideal of Q. We may write $t_n = s_n t_{n-1}$ $(n \geq 1)$ for some $s_n \in R$; and let $s_0 = t_0$. Form the $\mathcal{WI}$-envelopes U_n of the cyclic modules $C_n = R/Rt_n$ $(n < \omega)$. By the well-known Wakamatsu Lemma ([15, Lemma 2.1.13]) we then have $U_n/C_n \in \mathcal{F}_1$ for each n. This makes it possible to define maps

$$\phi_n : U_n \to U_{n+1} \qquad (n < \omega)$$

between these weak-injective modules (though not uniquely) as extensions of the correspondences

$$\phi_n : \ 1 + Rt_n \mapsto s_{n+1} + Rt_{n+1} \qquad (n < \omega).$$

Consider the modules $V_n = U_n/Rs_n$ $(n < \omega)$ along with the maps $\bar{\phi}_n : V_n \to V_{n+1}$ induced by the ϕ_n; thus

$$\bar{\phi}_n(1 + Rs_n) = s_{n+1} + Rs_{n+1} = 0.$$

Clearly, the module V_n is a special $\mathcal{WI}$-preenvelope of the cyclic module R/Rs_n. Evidently, the $\mathcal{WI}$-envelope of this cyclic module must be a summand V'_n of V_n. It is easily seen that the maps $\bar{\phi}_n$ induce maps $\bar{\phi}'_n$ between these envelope summands such that R/Rs_n is in the kernel of the map $\bar{\phi}'_n$. Knowing that $\oplus_{n<\omega} V'_n$ is the $\mathcal{WI}$-envelope of $\oplus_{n<\omega} R/Rs_n$, and for each n, $\bar{\phi}'_n$ annihilates

R/Rs_n, we are able to appeal again to [9, Proposition 6.4.1] on a direct sum of envelopes that is also an envelope. Hence we can conclude that for every $x \in V_0'$ there is an index $m \geq 1$ such that $\bar{\phi}_m' \cdots \bar{\phi}_0'(x) = 0 \in V_{m+1}'$. Choosing $x = r^{-1} + Rs_0$ with a non-unit $r \in R^{\times}$, a close scrutiny of the actions of the maps yields

$$t_m r^{-1} = s_m \cdots s_0(r^{-1}) = q t_{m+1}$$

for some $q \in Q$. This means $t_m \in Qt_{m+1}$, so the strictly descending chain (4) of the ideals Qt_n terminates at m. □

The proof of Theorem 6.4 relies on the following general result.

Proposition 6.2. (Bazzoni–Herbera [5, Theorem 2.5]) *Let N be a countably presented R-module, and $\mathcal{C}$ a class of R-modules such that $C \in \mathcal{C}$ implies that the countable direct sum $C^{(\aleph_0)} \in \mathcal{C}$. If*

$$\operatorname{Ext}^1_R(N, C) = 0 \quad \text{for all} \quad C \in \mathcal{C},$$

then $\operatorname{Ext}^1_R(N, D) = 0$ also for every pure submodule D of any $C \in \mathcal{C}$.

For the proof of the next lemma, we have to recall that over subperfect rings, a divisible submodule with cokernel of w.d.≤ 1 is necessarily a pure submodule. This is a simple consequence of [12, Theorem 4.1]. Hence every divisible module is pure in its weak-injective envelope.

Lemma 6.3. *Let R be a subperfect ring such that countable direct sums of copies of any weak-injective module are also weak-injective. Then every countably presented R-module of* w.d.≤ 1 *has* p.d.≤ 1.

Proof. Choose a countably presented R-module N of w.d.≤ 1. Then assuming that a countable direct sum of copies of any $C \in \mathcal{WI}$ is weak-injective, the hypotheses of Proposition 6.2 are satisfied for these N and $\mathcal{WI}$, so we can conclude that $\operatorname{Ext}^1_R(N, D) = 0$ holds for all pure submodules D of weak-injectives, and hence for all divisible R-modules D (see the remark before this lemma). Then from the fact that $\mathcal{P}_1 = ^{\perp} \mathcal{HD}$ we can deduce that p.d.$N \leq 1$. □

We are now prepared to finish the proof of:

Theorem 6.4. *Let R be a commutative ring. The class of weak-injective R-modules is closed under (countable) direct sums if and only if R is an almost perfect ring.*

Proof. Over almost perfect rings, direct sums of weak-injective modules are again weak-injective — this follows at once from the equality of divisible and weak-injective modules over almost perfect rings; see Proposition 2.2.

To verify the 'if' part, observe that any ring R satisfying the stated condition is a subperfect ring, as proved in Lemma 6.1. We show that the ring $\bar{R} = R/Rr$ is perfect for all non-units $r \in R^{\times}$ by proving that every countably presented flat $\bar{R}$-module N is projective. As a countably presented flat module, N is the direct limit over a countable index set of projective $\bar{R}$-modules. Projective

$\bar{R}$-modules are of p.d. 1 as R-modules, hence we derive that w.d.$_R N \leq 1$. N is evidently countably presented also as an R-module, so Proposition 6.2 applies to N and the class $\mathcal{C}$ of weak-injectives. Hence N satisfies the hypothesis of Lemma 6.3, so we obtain p.d.$_R N = 1$ (the case 0 is obviously ruled out). By a well-known Kaplansky formula on projective dimensions, N satisfies

$$\text{p.d.}_R N = \text{p.d.}_{\bar{R}} N + 1.$$

Consequently, N is projective as an $\bar{R}$-module, and $\bar{R}$ is a perfect ring. □

Corollary 6.5. *A commutative noetherian ring has the property that countable direct sums of weak-injective modules are again weak-injective if and only if it is a one-dimensional Cohen-Macaulay ring* (*in which case all divisible modules are weak-injective*).

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