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# COHOMOLOGY RING OF THE TENSOR PRODUCT OF POISSON ALGEBRAS

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Abstract. In this paper, we study the Poisson cohomology ring of the tensor product of Poisson algebras. Explicitly, it is proved that the Poisson cohomology ring of tensor product of two Poisson algebras is isomorphic to the tensor product of the respective Poisson cohomology ring of these two Poisson algebras as Gerstenhaber algebras.

## 1. Introduction

Given a Poisson algebra, its Poisson cohomology, as introduced by Lichnerowicz in [12], provides important information about the Poisson structure. The Poisson cohomology $\mathrm{HP}^*(R)$ of a Poisson algebra $R$ equipped with the wedge product and the Schouten bracket is a Gerstenhaber algebra [10]. It plays a crucial role in the study of deformations of the Poisson structure, just as the Hochschild cohomology ring for the deformation theory of associative algebras [5].

The tensor product of Poisson structure appeared in Poisson Hopf algebras [3]. The tensor product of Poisson structure makes the category $\mathrm{Poi}(\Bbbk)$, the category of Poisson algebras over $\Bbbk$, to be a tensor category [2]. Cherkashin researched the cohomologies, deformations and homotopies of tensor product of Poisson algebras in [1]. In this present paper, we study the Poisson cohomology ring of the tensor product of Poisson algebras. Let $(R, \pi_R)$ and $(T, \pi_T)$ be two Poisson algebras over $\Bbbk$. Then there is a Poisson bracket $\pi$, called the product Poisson bracket, on $R \otimes T$ by

$$\pi(a \otimes b, c \otimes d) := \pi_R(a, c) \otimes bd + ac \otimes \pi_T(b, d)$$

for all $a, c \in R$ and $b, d \in T$. The Poisson algebra $(R \otimes T, \pi)$ is called the Poisson tensor product of $R$ and $T$. Our main result is the following (Theorem 3.3):

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**Theorem.** *Let $R$ and $T$ be two Poisson algebras. There is an isomorphism of Gerstenhaber algebras*

$$\mathrm{HP}^*(R \otimes T) \cong \mathrm{HP}^*(R) \otimes \mathrm{HP}^*(T).$$

Note that the tensor product of two Gerstenhaber algebras in the above theorem is defined by Le and Zhou in [11]. They also proved that, given two associative algebras $A$ and $B$ over a field $\Bbbk$, at least one of which is finite dimensional, the Hochschild cohomology of the tensor product algebra $A \otimes_{\Bbbk} B$ is isomorphic to the tensor product of the respective Hochschild cohomologies of $A$ and of $B$, as Gerstenhaber algebras. Hence, our result is a generalization of this to Poisson framework.

The paper is organized as follows. In Section 2, we recall some preliminary definitions and results mainly on Poisson algebra and cohomology ring, Gerstenhaber algebra, tensor product of Poisson algebras, etc.. The main result is proved in Section 3. Throughout, $\Bbbk$ is a field and all algebras are $\Bbbk$-algebras unless specialized and unadorned $\otimes$ means $\otimes_{\Bbbk}$.

## 2. Preliminaries

In this section, we recall some definitions and results which is essential for our purpose.

### 2.1. Poisson algebra and cohomology

**Definition.** A commutative $\Bbbk$-algebra $R$ equipped with a bilinear map $\{-,-\} : R \times R \to R$ is called a *Poisson algebra* if

(1) $(R, \{-,-\})$ is a Lie algebra;
(2) $\{-,-\}$ is a derivation in each argument with respect to multiplication of $R$.

Usually, the Poisson bracket is denoted by $\pi$. Then, the Poisson algebra is denoted by $(R, \pi)$, or $(R, \{-,-\})$. Next, we recall the notion of Poisson cohomology. Denote by $\mathfrak{X}^k(R)$ the space of all skew-symmetric $\Bbbk$-linear maps $R^{\otimes k} \to R$ that are derivations in each arguments. Here, a map $f : R^k \to R$ is called skew-symmetric if $f(a_1, \dots, a_k) = \mathrm{sgn}(\sigma) f(a_{\sigma_1}, \dots, a_{\sigma_k})$ for any permutation $\sigma \in S_k$, where $\mathrm{sgn}(\sigma)$ denotes its sign and $\sigma_i$ means $\sigma(i)$. Then there is a cochain complex $(\mathfrak{X}^*(R), \delta_\pi^*)$ ([7, 12]), where $\delta_\pi^k : \mathfrak{X}^k(R) \to \mathfrak{X}^{k+1}(R)$ is defined by

$$\begin{aligned} \delta_\pi^k(P)(y_0, y_1, \dots, y_k) := & \sum_{0 \le i \le k} (-1)^i \{y_i, P(y_0, y_1, \dots, \widehat{y_i}, \dots, y_k)\} \\ & + \sum_{0 \le i < j \le k} (-1)^{i+j} P(\{y_i, y_j\}, y_0, y_1, \dots, \widehat{y_i}, \dots, \widehat{y_j}, \dots, y_k) \end{aligned}$$

for all $P \in \mathfrak{X}^k(R)$. It is not hard to check that $\delta_\pi^k(P)$ belongs to $\mathfrak{X}^{k+1}(R)$ and that $\delta_\pi^{k+1}\delta_\pi^k = 0$. The $k$-th cohomology of this complex is denoted by $\mathrm{HP}^k(R)$ and is called the $k$-th Poisson cohomology of the Poisson algebra $R$.

Recall that $\Omega^1(R)$ is the module of Kähler differentials of $R$ by definition the $R$-module and for each $p \in \mathbb{N}$, the Kähler $p$-forms is the $R$-module $\Omega^p(R) := \bigwedge^p \Omega^1(R)$, where $\bigwedge$ is the wedge product over $R$; for $p = 0$, $\Omega^0(R) = R$. As an $R$-module, $\Omega^p(R)$ is generated by the elements of the form $dx_1 \wedge \cdots \wedge dx_p$, where $x_1, \dots, x_p \in R$. There is a natural isomorphism of $R$-modules

$$\mathfrak{X}^k(R) \cong \mathrm{Hom}_R(\Omega^k(R), R), \tag{1}$$

which is given by $P \mapsto \hat{P}$, where $\hat{P}(x dx_1 \wedge \cdots \wedge dx_k) := xP(x_1, \dots, x_k)$ for all $P \in \mathfrak{X}^k(R)$ and $x, x_1, \dots, x_k \in R$.

## 2.2. Shuffles

As a matter of convenience, we recall the notion of a shuffle and make some more preliminary on it. For $p, q \in \mathbb{N}$ and $\Sigma = \{i_1, i_2, \dots, i_{p+q}\}$ a subset of $\mathbb{N}$ with $i_1 < i_2 < \cdots < i_{p+q}$, a permutation $\sigma$ on $\Sigma$ is called a $(p,q)$-*shuffle* on $\Sigma$ if $\sigma(i_1) < \cdots < \sigma(i_p)$ and $\sigma(i_{p+1}) < \cdots < \sigma(i_{p+q})$. The set of all $(p,q)$-shuffle is denoted by $S^{\Sigma}_{p,q}$. Unless specialized, a $(p,q)$-*shuffle* is defined on the set $\{1, 2, \dots, p+q\}$. The set of all $(p,q)$-shuffle is just denoted by $S_{p,q}$ briefly.

Let $A_1 = \{2, 3, 4, \dots, i+j\}$. Then, there is a map

$$\begin{aligned} \chi : S^{A_1}_{i-1,j} \bigcup S^{A_1}_{i,j-1} &\to S_{i,j} \\ \tau \quad &\mapsto \overline{\tau} \end{aligned}$$

where $\overline{\tau}$ is defined as follows; if $\tau \in S^{A_1}_{i-1,j}$, then

$$\overline{\tau}(l) := \begin{cases} 1, & l = 1, \\ \tau(l), & l \in \{2, 3, \dots, i+j\}; \end{cases}$$

if $\tau \in S^{A_1}_{i,j-1}$, then

$$\overline{\tau}(l) := \begin{cases} \tau(l+1), & l \in \{1, 2, \dots, i\}, \\ 1, & l = i+1, \\ \tau(l), & l \in \{i+2, \dots, i+j\}. \end{cases}$$

**Proposition 2.1.** *Keep the notation as above. The map* $\chi$

$$\chi : S^{A_1}_{i-1,j} \bigcup S^{A_1}_{i,j-1} \to S_{i,j} \tag{2}$$

*is a bijection. Further,*

$$\mathrm{sgn}(\overline{\tau}) := \begin{cases} \mathrm{sgn}(\tau), & \tau \in S^{A_1}_{i-1,j}; \\ (-1)^i \,\mathrm{sgn}(\tau), & \tau \in S^{A_1}_{i,j-1}. \end{cases}$$

*Proof.* Note that for an $(i,j)$-shuffle $\nu$ on the set $\{1, 2, \dots, i+j\}$, we have either $\nu(1) = 1$ or $\nu(i+1) = 1$. ∎

For $i, j, k, l \in \mathbb{N}$, we define an $(i, j, k, l)$-shuffle by a permutation $\sigma$ on the set $\{1, 2, \ldots, i + j + k + l\}$, such that

$$\begin{aligned}
&\sigma(1) < \sigma(2) < \cdots < \sigma(i),\\
&\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(i+j),\\
&\sigma(i+j+1) < \sigma(i+j+2) < \cdots < \sigma(i+j+k) \quad \text{and}\\
&\sigma(i+j+k+1) < \sigma(i+j+k+2) < \cdots < \sigma(i+j+k+l).
\end{aligned}$$

The set of all $(i, j, k, l)$-shuffle is denoted by $S_{i,j,k,l}$.

For each $(i + j, k + l)$-shuffle $\sigma$, consider the set

$$S_\sigma = \left\{ (\tau, \eta) \;\middle|\; \tau \in S^{A_2}_{i,j}, \eta \in S^{A_3}_{k,l} \right\},$$

where $A_2 = \{\sigma(1), \sigma(2), \ldots, \sigma(i+j)\}$ and $A_3 = \{\sigma(i+j+1), \ldots, \sigma(i+j+k+l)\}$

For a given $(i+j, k+l)$-shuffle $\sigma$ and a pair $(\tau, \eta) \in S_\sigma$, define a permutation $\delta$ by

$$\delta(n) := \begin{cases} \tau\sigma(n), & 1 \le n \le i + j; \\ \eta\sigma(n), & i + j + 1 \le n \le i + j + k + l. \end{cases} \tag{3}$$

Then $\delta$ is an $(i, j, k, l)$-shuffle. Thus, we get a map $\varrho : \bigcup_{\sigma \in S_{i+j,k+l}} S_\sigma \to S_{i,j,k,l}$ by sending the triple $(\sigma, \tau, \eta)$ to $\delta$ which defined above.

**Proposition 2.2.** *Keeping the notation as above, the map* $\varrho$ *is a bijection and* $\operatorname{sgn}(\delta) = \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \operatorname{sgn}(\eta)$.

*Proof.* The proof is direct. □

For a $(p, q)$-shuffle $\sigma$, consider the permutation $\sigma'$ by

$$\sigma'(i) := \begin{cases} \sigma(i + p), & 1 \le i \le q; \\ \sigma(i - q), & q + 1 \le i \le q + p. \end{cases}$$

Then, $\sigma'$ is a $(q, p)$-shuffle. Thus, we get a map $\vartheta : S_{p,q} \to S_{q,p}, \sigma \mapsto \sigma'$.

**Proposition 2.3.** *Keeping the notation as above, the map* $\vartheta$ *is a bijection and* $\operatorname{sgn}(\sigma) = (-1)^{pq} \operatorname{sgn}(\sigma')$.

*Proof.* The proof is direct. □

*Remark* 2.4. The similar results hold for $(i, j, k, l)$-shuffles. For example, there is a bijection between $S_{i,j,k,l}$ and $S_{k,j,i,l}$ by send $\tau$ to $\tau'$ in a similar way. Moreover, $\operatorname{sgn}(\tau) = (-1)^{k(i+j)+ij} \operatorname{sgn}(\tau')$.

For an $(i,j,k-1,l)$-shuffle $\sigma$ and each integer $p$ with $i+1 \le p \le i+j$, define an integer $q$ by

$$q := \begin{cases} i+j, & \text{if } \sigma(p) < \sigma(i+j+1); \\ t, & \text{if } \sigma(t) < \sigma(p) < \sigma(t+1) \\ & \text{for some } t \text{ with } i+j+1 \le t \le j+j+k-1; \\ i+j+k-1, & \text{if } \sigma(i+j+k-1) < \sigma(p). \end{cases}$$

Then, define a permutation $\tau$ by

$$\tau(l) := \begin{cases} \sigma(l), & \text{if } 1 \le l \le p-1; \\ \sigma(l+1), & \text{if } p \le l \le q-1; \\ \sigma(p), & \text{if } l = q; \\ \sigma(l), & \text{if } q+1 \le l \le i+j+k+l. \end{cases}$$

It is easy to see that $\tau$ is an $(i,j-1,k,l)$-shuffle. That is, we get a map

$$\begin{array}{ccc} \varsigma : \{i+1, i+2, \ldots, i+j\} \times S_{i,j,k-1,l} & \to & \{i+j, \ldots, i+j+k-1\} \times S_{i,j-1,k,l} \\ (p, \sigma) & \mapsto & (q, \tau). \end{array}$$

**Proposition 2.5.** *Keeping the notation as above, the map $\varsigma$ is a bijection*:

$$\varsigma : \{i+1, i+2, \ldots, i+j\} \times S_{i,j,k-1,l} \to \{i+j, i+j+1, \ldots, i+j+k-1\} \times S_{i,j-1,k,l}.$$

*And,* $\operatorname{sgn}(\tau) = (-1)^{p+q+1} \operatorname{sgn}(\sigma)$ *if* $\varsigma(p,\sigma) = (q,\tau)$.

*Proof.* It is easy to see that these two sets have same cardinality and that this map is injective. □

### 2.3. Gerstenhaber algebra

**Definition.** A Gerstenhaber algebra is a $\mathbb{Z}$-graded $\Bbbk$-vector space $H^\bullet := \bigoplus_{n\in\mathbb{Z}} H^n$ equipped with two linear maps

$$\wedge : H^m \times H^n \to H^{m+n}, (a,b) \mapsto a \wedge b$$

and

$$[-,-] : H^m \times H^n \to H^{m+n-1}, (a,b) \mapsto [a,b]$$

such that

(1) $(H^\bullet, \wedge)$ is a graded commutative algebra, i.e., $a \wedge b = (-1)^{mn} b \wedge a, \forall a \in H^m, b \in H^n$;

(2) $(H^\bullet, [-,-])$ is a graded Lie algebra of degree $-1$, i.e.,

$$[a,b] = -(-1)^{(m-1)(n-1)}[b,a], \forall a \in H^m, b \in H^n$$

and

$$(-1)^{(m-1)(l-1)}[[a,b],c] + \text{cyclic permutation of } a, b \text{ and } c = 0$$

for all $a \in H^m, b \in H^n, c \in H^l$;

(3) two operations satisfy the compatible condition:

$$[a \wedge b, c] = a \wedge [b, c] + (-1)^{(m-1)n} b \wedge [a, c], \forall a \in H^m, b \in H^n, c \in H.$$

**Definition.** Let $(H^\bullet, \wedge_H, [-,-]_H)$ and $(K^\bullet, \wedge_K, [-,-]_K)$ be two Gerstenhaber algebras over $\Bbbk$. Then $H$ is said to isomorphic to $K$ as Gerstenhaber algebras if there is a $\Bbbk$-linear bijection map $\Phi : H \to K$ of degree 0 with the following properties

$$\Phi(a \wedge_H b) = \Phi(a) \wedge_K \Phi(b), \tag{4}$$

$$\Phi([a, b]_H) = [\Phi(a), \Phi(b)]_K. \tag{5}$$

Let $R$ be a Poisson algebra and $\mathfrak{X}^*(R) := \bigoplus_i \mathfrak{X}^i(R)$. The wedge product $\wedge$ on $\mathfrak{X}^*(R)$:

$$\begin{aligned} &(P \wedge Q)(a_1, \dots, a_{m+n}) \\ := &\sum_{\theta \in S_{m,n}} \operatorname{sgn}(\theta) P(a_{\theta(1)}, \dots, a_{\theta(m)}) Q(a_{\theta(m+1)}, \dots, a_{\theta(m+n)}) \end{aligned}$$

and the Schouten bracket on $\mathfrak{X}^*(R)$

$$[P, Q]_S := P \circ Q - (-1)^{(m-1)(n-1)} Q \circ P,$$

where the product $\circ$ is defined by:

$$\begin{aligned} &(P \circ Q)(a_1, \dots, a_{m+n-1}) \\ := &\sum_{\theta \in S_{n,m-1}} \operatorname{sgn}(\theta) P(Q(a_{\theta(1)}, \dots, a_{\theta(n)}), a_{\theta(n+1)}, \dots, a_{\theta(m+n-1)}) \end{aligned}$$

for all $P \in \mathfrak{X}^m(R)$ and $Q \in \mathfrak{X}^n(R)$, make the triple $(\mathfrak{X}^*(R), \wedge, [-,-]_S)$ to be a Gerstenhaber algebra. Further, both the wedge product and Schouten bracket induce products in Poisson cohomology, which still denote by $\wedge$ and $[-,-]_S$. Moreover, the triple $(\mathrm{HP}^*(R), \wedge, [-,-]_S)$ is a Gerstenhaber algebra [10, Proposition 4.9].

### 2.4. Tensor product of Poisson algebras

Let $(R, \{-,-\})$ and $(T, \{-,-\})$ be two Poisson algebras. Then $R \otimes T$ is a commutative associative algebra by

$$(a \otimes b) \cdot (c \otimes d) := ac \otimes bd.$$

**Proposition 2.6** ([1, 3, 10])**.** *There exists a unique Poisson bracket*

$$\{a \otimes b, c \otimes d\} := \{a, c\} \otimes bd + ac \otimes \{b, d\}$$

*such that $(R \otimes T, \cdot, \{-,-\})$ is a Poisson algebra, making the canonical inclusions $R \to R \otimes T$ and $T \to R \otimes T$ are Poisson algebra homomorphism.*

The Poisson algebra $(R \otimes T, \cdot, \{-,-\})$ is called the *Poisson tensor product* of $R$ and $T$, while the Poisson bracket $\{-,-\}$ is called the *product Poisson structure*.

## 3. The main result and its proof

Let $R$ and $T$ be two Poisson algebras. In this section, we firstly recall that the tensor product of two Gerstenhaber algebras is still a Gerstenhaber algebra. Then, we prove that $\mathrm{HP}^*(R) \otimes \mathrm{HP}^*(T)$ and $\mathrm{HP}^*(R \otimes T)$ are isomorphic as Gerstenhaber algebras.

**Proposition-Definition 3.1** ([11, Proposition-Definition 2.2])**.** *Let* $(H^\bullet, \wedge_H, [-,-]_H)$ *and* $(K^\bullet, \wedge_K, [-,-]_K)$ *be two Gerstenhaber algebras over* $\Bbbk$. *Then there is a new Gerstenhaber algebra* $(L^\bullet, \wedge, [-,-])$ *over* $\Bbbk$ *given as follows*

(1) $L^n := \oplus_{i+j=n} H^i \otimes K^j$ *as a* $\Bbbk$*-vector space for each* $n \in \mathbb{Z}$;
(2) $(a \otimes b) \wedge (a' \otimes b') := (-1)^{|a'||b|}(a \wedge_H a') \otimes (b \wedge_K b')$;
(3) $$[a \otimes b, a' \otimes b'] := (-1)^{(|a|+|b|-1)|b'|}[a, a']_H \otimes (b \wedge_K b') + (-1)^{|a|(|a'|+|b'|-1)}(a \wedge_H a') \otimes [b, b']_K.$$

*The Gerstenhaber algebra* $(L^\bullet, \wedge, [-,-])$ *is called the tensor product of the two Gerstenhaber algebras* $H^\bullet$ *and* $K^\bullet$*, and denote it by* $H^\bullet \otimes K^\bullet$*.*

In order to prove the main result of this section, we retrieve the relation between the Kähler differentials of $R \otimes T$ and that of $R$ and $T$. We rewrite the following result here for convenience though it is standard.

**Lemma 3.2.** *Let* $R$ *and* $T$ *be two commutative algebras. Then*

$$\Omega^*(R) \otimes \Omega^*(T) \simeq \Omega^*(R \otimes T)$$

*as complexes.*

*Proof.* For each $p, q$, the map

$$\varepsilon : (a_0 da_1 \wedge \cdots \wedge da_p) \otimes (b_0 db_1 \wedge \cdots \wedge db_q) \to (a_0 \otimes b_0) d(a_1 \otimes 1) \wedge \cdots \wedge d(a_p \otimes 1) \wedge d(1 \otimes b_1) \wedge \cdots \wedge d(1 \otimes b_q)$$

is well-defined and induces a morphism of $R \otimes T$-modules (still denoted it by $\varepsilon$),

$$\varepsilon : \quad \Omega^p(R) \otimes \Omega^q(T) \to \Omega^{p+q}(R \otimes T). \tag{6}$$

Moreover, it is bijective and a morphism of complexes. □

**Theorem 3.3.** *Let* $R$ *and* $T$ *be two Poisson algebras. There is an isomorphism of Gerstenhaber algebras*

$$\mathrm{HP}^*(R \otimes T) \cong \mathrm{HP}^*(R) \otimes \mathrm{HP}^*(T).$$

*Proof.* The left hand side of the above isomorphism is computed by the complex $\mathfrak{X}^*(R \otimes T)$, while the right hand side is computed by the complex $\mathfrak{X}^*(R) \otimes \mathfrak{X}^*(T)$. We will take in three steps to prove that there is an isomorphism

$$\mathfrak{X}^*(R) \otimes \mathfrak{X}^*(T) \simeq \mathfrak{X}^*(R \otimes T) \tag{7}$$

as Gerstenhaber algebras

**Step 1:** construct an isomorphism of complexes $\Phi : \mathfrak{X}^*(R) \otimes \mathfrak{X}^*(T) \to \mathfrak{X}^*(R \otimes T)$;

**Step 2:** prove that $\Phi$ preserves the wedge product

$$\Phi((f_1 \otimes g_1) \wedge (f_2 \otimes g_2)) = \Phi(f_1 \otimes g_1) \wedge \Phi(f_2 \otimes g_2)$$

for all $f_1 \in \mathfrak{X}^{i_1}(R)$, $f_2 \in \mathfrak{X}^{i_2}(R)$, $g_1 \in \mathfrak{X}^{j_1}(T)$ and $g_2 \in \mathfrak{X}^{j_2}(T)$;

**Step 3:** prove that $\Phi$ preserves the Schouten product

$$\Phi[f_1 \otimes g_1, f_2 \otimes g_2]_S = [\Phi(f_1 \otimes g_1), \Phi(f_2 \otimes g_2)]_S.$$

Then by taking cohomology we get the desired conclusion.

**Step 1:** In [1], a morphism between the above complexes was constructed in the following way. Note that $\mathfrak{X}^k(R) = \mathrm{Hom}_R(\Omega^k(R), R)$ by equation (1) for each $k$. For $f \in \mathrm{Hom}_R(\Omega^p(R), R)$ and $g \in \mathrm{Hom}_T(\Omega^q(T), T)$, define

$$\Phi_{pq} : \mathrm{Hom}_R(\Omega^p(R), R) \otimes \mathrm{Hom}_T(\Omega^q(T), T) \to \mathrm{Hom}_{R\otimes T}(\Omega^{p+q}(R \otimes T), R \otimes T)$$

by

$$\begin{aligned}\Phi_{pq}(f \otimes g) : (d(a_1 \otimes b_1) \wedge \cdots \wedge d(a_{p+q} \otimes b_{p+q})) \mapsto \\ \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma) a_{\sigma(p+1)} \cdots a_{\sigma(p+q)} f(da_{\sigma(1)} \wedge \cdots \wedge da_{\sigma(p)}) \\ \otimes b_{\sigma(1)} \cdots b_{\sigma(p)} g(db_{\sigma(p+1)} \wedge \cdots \wedge db_{\sigma(p+q)}).\end{aligned}$$

Then $\Phi := \bigoplus_{p,q} \Phi_{pq}$ is a morphism from the complex $\mathrm{Hom}_R(\Omega^*(R), R) \otimes \mathrm{Hom}_T(\Omega^*(T), T)$ to the complex $\mathrm{Hom}_{R\otimes T}(\Omega^*(R \otimes T), R \otimes T)$.

Conversely, we firstly construct

$$F : \mathrm{Hom}_{R\otimes T}(\Omega^r(R \otimes T), R \otimes T) \to \bigoplus_{t=0}^{r} \mathrm{Hom}_{R\otimes T}(\Omega^t(R) \otimes \Omega^{r-t}(T), R \otimes T)$$

by $F := \mathrm{Hom}_{R\otimes T}(\varepsilon, R \otimes T)$. It is easy to see that $F$ is an isomorphism. Recall that it is known that if $M$ and $P$ are finite generated modules, then the map

$$\begin{aligned} G : \mathrm{Hom}_A(M, N) \otimes \mathrm{Hom}_B(P, Q) &\to \mathrm{Hom}_{A\otimes B}(M \otimes P, N \otimes Q) \\ \alpha \otimes \beta &\longmapsto G_{\alpha\otimes\beta}, \end{aligned}$$

where $G_{\alpha\otimes\beta}(m \otimes p) := \alpha(m) \otimes \beta(p)$ for each $m \in M, p \in P$, is an isomorphism. Thus, we get an isomorphism

$$G : \mathrm{Hom}_R(\Omega^p(R), R) \otimes \mathrm{Hom}_T(\Omega^q(T), T) \to \mathrm{Hom}_{R\otimes T}(\Omega^p(R) \otimes \Omega^q(T), R \otimes T)$$

since both $\Omega^p(R)$ and $\Omega^q(T)$ are finite generated for all $p, q$. Define

$$\begin{aligned}\Psi = G^{-1}F : &\mathrm{Hom}_{R\otimes T}(\Omega^r(R \otimes T), R \otimes T) \\ &\to \bigoplus_{t=0}^{r} \mathrm{Hom}_R(\Omega^t(R), R) \otimes \mathrm{Hom}_T(\Omega^{r-t}(T), T).\end{aligned}$$

Therefore, $\Psi$ is bijective and $F\Phi = G$, i.e., $\Psi\Phi = id$. What we have done can be described as the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{R\otimes T}(\Omega^r(R\otimes T), R\otimes T) & \underset{\Phi}{\overset{\Psi=G^{-1}F}{\rightleftarrows}} & \bigoplus_{t=0}^r \mathrm{Hom}_R(\Omega^t(R), R)\otimes \mathrm{Hom}_T(\Omega^{r-t}(T), T) \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \bigoplus_{t=0}^r \mathrm{Hom}_{R\otimes T}(\Omega^t(R)\otimes\Omega^{r-t}(T), R\otimes T) & \end{array}$$

Henceforth, $\Phi$ is an isomorphism of complexes.

**Step 2:** Next, we prove that the isomorphism $\Phi$ preserves the wedge product

$$\Phi((f_1\otimes g_1)\wedge(f_2\otimes g_2)) = \Phi(f_1\otimes g_1)\wedge\Phi(f_2\otimes g_2) \tag{8}$$

for all $f_1 \in \mathfrak{X}^{i_1}(R)$, $f_2 \in \mathfrak{X}^{i_2}(R)$, $g_1 \in \mathfrak{X}^{j_1}(T)$ and $g_2 \in \mathfrak{X}^{j_2}(T)$. To simplify the notation, set $I = i_1 + i_2$, $J = j_1 + j_2$, $I' = i_1 + j_1$ and $J' = i_2 + j_2$.

Recall that

$$(f_1\otimes g_1)\wedge(f_2\otimes g_2) = (-1)^{i_2j_1}(f_1\wedge f_2)\otimes(g_1\wedge g_2).$$

Hence, the left hand side of (8) is the map by sending $(a_1\otimes b_1)\otimes\cdots\otimes(a_{I+J}\otimes b_{I+J})$ to

$$\sum_{\sigma\in S_{I,J}}(-1)^{i_2j_1}\,\mathrm{sgn}(\sigma)a_{\sigma(I+1)}\cdots a_{\sigma(I+J)}f_1\wedge f_2(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(I)})$$
$$\otimes b_{\sigma(1)}\cdots b_{\sigma(I)}g_1\wedge g_2(b_{\sigma(I+1)}\otimes\cdots\otimes b_{\sigma(I+J)}). \tag{E1}$$

By the definition of the wedge product on $\mathfrak{X}^*(R)$,

$$\begin{aligned} & f_1\wedge f_2(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(I)}) \\ = & \sum_{\tau\in S_{i_1,i_2}^{A_4}}\mathrm{sgn}(\tau)f_1(a_{\tau(1)}\otimes\cdots\otimes a_{\tau(i_1)})f_2(a_{\tau(i_1+1)}\otimes\cdots\otimes a_{\tau(I)}), \end{aligned}$$

where the set $A_4 = \{\sigma(1), \sigma(2), \ldots, \sigma(I)\}$. Similarly,

$$\begin{aligned} & g_1\wedge g_2(b_{\sigma(I+1)}\otimes\cdots\otimes b_{\sigma(I+J)}) \\ = & \sum_{\eta\in S_{j_1,j_2}^{A_5}}\mathrm{sgn}(\eta)g_1(b_{\eta(1)}\otimes\cdots\otimes b_{\eta(j_1)})g_2(b_{\eta(j_1+1)}\otimes\cdots\otimes b_{\eta(J)}), \end{aligned}$$

where the set $A_5 = \{\sigma(I+1), \ldots, \sigma(I+J)\}$. By Proposition 2.2,

$$\begin{aligned} (E1) = \sum_{\sigma\in S_{i_1,i_2,j_1,j_2}} & (-1)^{i_2j_1}\,\mathrm{sgn}(\sigma)a_{\sigma(I+1)}\cdots a_{\sigma(I+J)}f_1(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(i_1)}) \\ & f_2(a_{\sigma(i_1+1)}\otimes\cdots\otimes a_{\sigma(I)})\otimes b_{\sigma(1)}\cdots b_{\sigma(I)} \\ & g_1(b_{\sigma(I+1)}\otimes\cdots\otimes b_{\sigma(I+j_1)})g_2(b_{\sigma(I+j_1+1)}\otimes\cdots\otimes b_{\sigma(I+J)}). \end{aligned} \tag{9}$$

For the right hand side of (8), $\Phi(f_1 \otimes g_1) \wedge \Phi(f_2 \otimes g_2)$ is the map sending $(a_1 \otimes b_1) \otimes \cdots \otimes (a_{I+J} \otimes b_{I+J})$ to

$$\sum_{\sigma \in S_{I',J'}} \operatorname{sgn}(\sigma)\Phi(f_1 \otimes g_1)\Big((a_{\sigma(1)} \otimes b_{\sigma(1)}) \otimes \cdots \otimes (a_{\sigma(I')} \otimes b_{\sigma(I')})\Big)$$

$$\otimes \Phi(f_2 \otimes g_2)\Big((a_{\sigma(I'+1)} \otimes b_{\sigma(I'+1)}) \otimes \cdots \otimes (a_{\sigma(I'+J')} \otimes b_{\sigma(I'+J')})\Big)$$

$$= \sum_{\sigma \in S_{I',J'}} \operatorname{sgn}(\sigma)\Big( \sum_{\tau \in S^{A_6}_{i_1,j_1}} \operatorname{sgn}(\tau) a_{\tau(i_1+1)} \cdots a_{\tau(I')} f_1(a_{\tau(1)} \otimes \cdots \otimes a_{\tau(i_1)})$$

$$\otimes b_{\tau(1)} \cdots b_{\tau(i_1)} g_1(b_{\tau(i_1+1)} \otimes \cdots \otimes b_{\tau(I')})\Big)$$

$$\otimes \Big( \sum_{\eta \in S^{A_7}_{i_2,j_2}} \operatorname{sgn}(\eta) a_{\eta(i_2+1)} \cdots a_{\eta(J')} f_2(a_{\eta(1)} \otimes \cdots \otimes a_{\eta(i_2)})$$

$$\otimes b_{\eta(1)} \cdots b_{\eta(i_2)} g_2(b_{\eta(i_2+1)} \otimes \cdots \otimes b_{\eta(J')})\Big)$$

$$= \sum_{\substack{\sigma \in S_{I',J'} \\ \tau \in S^{A_6}_{i_1,j_1},\, \eta \in S^{A_7}_{i_2,j_2}}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\eta)\operatorname{sgn}(\tau)\Big(a_{\tau(i_1+1)} \cdots a_{\tau(I')} a_{\eta(i_2+1)} \cdots a_{\eta(J')}$$

$$f_1(a_{\tau(1)} \otimes \cdots \otimes a_{\tau(i_1)}) f_2(a_{\eta(1)} \otimes \cdots \otimes a_{\eta(i_2)})\Big) \otimes \Big(b_{\tau(1)} \cdots b_{\tau(i_1)}$$

(E2)

$$b_{\eta(1)} \cdots b_{\eta(i_2)} g_1(b_{\tau(i_1+1)} \otimes \cdots \otimes b_{\tau(I')}) g_2(b_{\eta(i_2+1)} \otimes \cdots \otimes b_{\eta(J')})\Big),$$

where the set $A_6 = \{\sigma(1), \sigma(2), \ldots, \sigma(I')\}$ and the set $A_7 = \{\sigma(I'+1), \ldots, \sigma(I' + J')\}$. By Proposition 2.2,

(10)
$$(E2) = \sum_{\delta \in S_{i_1,j_1,i_2,j_2}} \operatorname{sgn}(\delta)\Big(a_{\delta(i_1+1)} \cdots a_{\delta(I')} a_{\delta(I'+i_2+1)} \cdots a_{\delta(I'+J')}$$

$$f_1(a_{\delta(1)} \otimes \cdots \otimes a_{\delta(i_1)}) f_2(a_{\delta(I'+1)} \otimes \cdots \otimes a_{\delta(I'+i_2)})\Big)$$

$$\otimes \Big(b_{\delta(1)} \cdots b_{\delta(i_1)} b_{\delta(I'+1)} \cdots b_{\delta(I'+i_2)}$$

$$g_1(b_{\delta(i_1+1)} \otimes \cdots \otimes b_{\delta(I')}) g_2(b_{\delta(I'+i_2+1)} \otimes \cdots \otimes b_{\delta(I'+J')})\Big).$$

Finally, interchanging $S_{i_1,i_2,j_1,j_2}$ and $S_{i_1,j_1,i_2,j_2}$ by using the argument in Remark 2.4 we get that

$$\text{RHS of (10)} = \text{RHS of (9)}.$$

That is, two sides of (8) define the same map. Henceforth, $\Phi$ preserves the wedge product.

**Step 3:** We will prove that the isomorphism $\Phi$ preserves the Schouten product

(11)
$$\Phi[f_1 \otimes g_1, f_2 \otimes g_2] = [\Phi(f_1 \otimes g_1), \Phi(f_2 \otimes g_2)]$$

for all $f_1 \in \mathfrak{X}^{i_1}(R)$, $f_2 \in \mathfrak{X}^{i_2}(R)$, $g_1 \in \mathfrak{X}^{j_1}(T)$ and $g_2 \in \mathfrak{X}^{j_2}(T)$. Recall that

$$[f_1 \otimes g_1, f_2 \otimes g_2] = (-1)^{(I'-1)j_2}[f_1, f_2] \otimes g_1 \wedge g_2 + (-1)^{i_1(J'-1)} f_1 \wedge f_2 \otimes [g_1, g_2].$$

Hence, the left hand side of (11) is the map sending $(a_1 \otimes b_1) \otimes \cdots \otimes (a_{I+J-1} \otimes b_{I+J-1})$ to

$$\begin{aligned} &(-1)^{(I'-1)j_2}\Phi([f_1, f_2] \otimes g_1 \wedge g_2)\big((a_1 \otimes b_1) \otimes \cdots \otimes (a_{I+J-1} \otimes b_{I+J-1})\big) \\ &+(-1)^{i_1(J'-1)}\Phi(f_1 \wedge f_2 \otimes [g_1, g_2])\big((a_1 \otimes b_1) \otimes \cdots \otimes (a_{I+J-1} \otimes b_{I+J-1})\big). \end{aligned}$$

Since

$$\begin{aligned} &\Phi([f_1, f_2] \otimes g_1 \wedge g_2)\big((a_1 \otimes b_1) \otimes \cdots \otimes (a_{I+J-1} \otimes b_{I+J-1})\big) \\ = &\sum_{\sigma \in S_{I-1,J}} \operatorname{sgn}(\sigma) a_{\sigma(I)} \cdots a_{\sigma(I+J-1)}[f_1, f_2]\big(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(I-1)}\big) \\ &\qquad \otimes b_{\sigma(1)} \cdots b_{\sigma(I-1)} g_1 \wedge g_2\big(b_{\sigma(I)} \otimes \cdots \otimes b_{\sigma(I+J-1)}\big) \\ = &\sum_{\sigma \in S_{I-1,J}} \operatorname{sgn}(\sigma) a_{\sigma(I)} \cdots a_{\sigma(I+J-1)}\Big(\sum_{\tau \in S^{A_8}_{i_2, i_1-1}} \operatorname{sgn}(\tau) f_1(f_2(a_{\tau(1)} \otimes \cdots \otimes a_{\tau(i_2)}) \\ &\otimes a_{\tau(i_2+1)} \otimes \cdots \otimes a_{\tau(I-1)}) \\ &- (-1)^{(i_1-1)(i_2-1)} \sum_{\eta \in S^{A_9}_{i_1, i_2-1}} \operatorname{sgn}(\eta) f_2(f_1(a_{\eta(1)} \otimes \cdots \otimes a_{\eta(i_1)}) \\ &\otimes a_{\eta(i_1+1)} \otimes \cdots \otimes a_{\eta(I-1)})\Big) \otimes \Big(b_{\sigma(1)} \cdots b_{\sigma(I-1)} \\ &\sum_{\delta \in S^{A_10}_{j_1, j_2}} \operatorname{sgn}(\delta) g_1(b_{\delta(I)} \otimes \cdots \otimes b_{\delta(I+j_1-1)}) g_2(b_{\delta(I+j_1)} \otimes \cdots \otimes b_{\delta(I+J-1)})\Big), \end{aligned}$$

where the set $A_8 = \{\sigma(1), \sigma(2), \ldots, \sigma(I-1)\}$, the set $A_9 = \{\sigma(1), \sigma(2), \ldots, \sigma(I-1)\}$ and the set $A_{10} = \{\sigma(I), \ldots, \sigma(I+J-1)\}$. By using Proposition 2.2,

$$\begin{aligned} &\Phi([f_1, f_2] \otimes g_1 \wedge g_2)\big((a_1 \otimes b_1) \otimes \cdots \otimes (a_{I+J-1} \otimes b_{I+J-1})\big) \\ = &\sum_{\nu \in S_{i_2, i_1-1, j_1, j_2}} \operatorname{sgn}(\nu) f_1\big(f_2(a_{\nu(1)} \otimes \cdots \otimes a_{\nu(i_2)}) \otimes a_{\nu(i_2+1)} \otimes \cdots \otimes a_{\nu(I-1)}\big) \\ &\qquad a_{\nu(I)} \cdots a_{\nu(I+J-1)} \otimes b_{\nu(1)} \cdots b_{\nu(I-1)} \\ &\qquad g_1(b_{\nu(I)} \otimes \cdots \otimes b_{\nu(I+j_1-1)}) g_2(b_{\nu(I+j_1)} \otimes \cdots \otimes b_{\nu(I+J2-1)}) \\ &- \sum_{\nu \in S_{i_1, i_2-1, j_1, j_2}} \operatorname{sgn}(\nu)(-1)^{(i_1-1)(i_2-1)} f_2\big(f_1(a_{\nu(1)} \otimes \cdots \otimes a_{\nu(i_1)}) \otimes a_{\nu(i_1+1)} \\ &\qquad \otimes \cdots \otimes a_{\nu(I-1)}\big) \cdot a_{\nu(I)} \cdots a_{\nu(I+J-1)} \otimes b_{\nu(1)} \cdots b_{\nu(I-1)} \\ &\qquad g_1(b_{\nu(I)} \otimes \cdots \otimes b_{\nu(I+j_1-1)}) g_2(b_{\nu(I+j_1)} \otimes \cdots \otimes b_{\nu(I+J-1)}). \end{aligned}$$

It is easy to see that the term

$$f_1\big(f_2(a_{\nu(1)} \otimes \cdots \otimes a_{\nu(i_2)}) \otimes a_{\nu(i_2+1)} \otimes \cdots \otimes a_{\nu(I-1)}\big) \cdot a_{\nu(I)} \cdots a_{\nu(I+J-1)}$$

$$\otimes b_{\nu(1)} \cdots b_{\nu(I-1)} \cdot g_1(b_{\nu(I)} \otimes \cdots \otimes b_{\nu(I+j_1-1)}) g_2(b_{\nu(I+j_1)} \otimes \cdots \otimes b_{\nu(I+J2-1)})$$

is completely determined by the $(i_2, i_1-1, j_1, j_2)$-shuffle $\nu$, and we write it as $f_1(f_2(\cdots)\cdots)\cdots\otimes\cdots g_1(\cdots)g_2(\cdots)$ briefly. Similarly for the second term in the last equation. Moreover,

$$\begin{aligned}
(12)\qquad & \Phi[f_1 \otimes g_1, f_2 \otimes g_2]\big((a_1 \otimes b_1) \otimes \cdots \otimes (a_{I+J-1} \otimes b_{I+J-1})\big) \\
&= \sum_{\nu \in S_{i_2, i_1-1, j_1, j_2}} K f_1(f_2(\cdots)\cdots)\cdots \otimes \cdots g_1(\cdots)g_2(\cdots) \\
&\quad - \sum_{\nu \in S_{i_1, i_2-1, j_1, j_2}} L f_2(f_1(\cdots)\cdots)\cdots \otimes \cdots g_1(\cdots)g_2(\cdots) \\
&\quad + \sum_{\nu \in S_{i_1, i_2, j_2, j_1-1}} M f_1(\cdots)f_2(\cdots)\cdots \otimes \cdots g_1(g_2(\cdots)\cdots) \\
&\quad - \sum_{\nu \in S_{i_1, i_2, j_1, j_2-1}} N f_1(\cdots)f_2(\cdots)\cdots \otimes \cdots g_2(g_1(\cdots)\cdots),
\end{aligned}$$

where the constants $K=\mathrm{sgn}(\nu)(-1)^{(I'-1)j_2}$, $L=\mathrm{sgn}(\nu)(-1)^{(I'-1)j_2+(i_1-1)(i_2-1)}$, $M=\mathrm{sgn}(\nu)(-1)^{i_1(J'-1)}$ and $N=\mathrm{sgn}(\nu)(-1)^{i_1(J'-1)+(j_1-1)(j_2-1)}$.

Now, let us compute the right hand side of (11),

$$\begin{aligned}
&[\Phi(f_1 \otimes g_1), \Phi(f_2 \otimes g_2)]\big((a_1 \otimes b_1) \otimes \cdots \otimes (a_{I+J-1} \otimes b_{I+J-1})\big) \\
&= \sum_{\sigma \in S_{J', I'-1}} \mathrm{sgn}(\sigma)\Phi(f_1 \otimes g_1)\Big(\Phi(f_2 \otimes g_2)\big((a_{\sigma(1)} \otimes b_{\sigma(1)}) \otimes \cdots \otimes (a_{\sigma(J')} \otimes b_{\sigma(J')})\big) \\
&\quad \otimes (a_{\sigma(J'+1)} \otimes b_{\sigma(J'+1)}) \otimes \cdots \otimes (a_{\sigma(I'+J'-1)} \otimes b_{\sigma(I'+J'-1)})\Big) \\
&\quad - \sum_{\sigma \in S_{I', J'-1}} \mathrm{sgn}(\sigma)(-1)^{(I'-1)(J'-1)}\Phi(f_2 \otimes g_2)\Big(\Phi(f_1 \otimes g_1)\big((a_{\sigma(1)} \otimes b_{\sigma(1)} \otimes \cdots \otimes \\
&\quad (a_{\sigma(I')} \otimes b_{\sigma(I')})\big) \otimes (a_{\sigma(I'+1)} \otimes b_{\sigma(I'+1)}) \otimes \cdots \otimes (a_{\sigma(I'+J'-1)} \otimes b_{\sigma(I'+J'-1)})\Big),
\end{aligned}$$

and denote it by $U+V$ briefly.

Denote $\Phi(f_2 \otimes g_2)\big((a_{\sigma(1)} \otimes b_{\sigma(1)}) \otimes \cdots \otimes (a_{\sigma(J')} \otimes b_{\sigma(J')})\big)$ by $a_0 \otimes b_0$ briefly, that is,

$$\begin{aligned}
(13)\qquad a_0 \otimes b_0 = \sum_{\nu \in S^{A_{11}}_{i_2, j_2}} & \mathrm{sgn}(\nu) f_2(a_{\nu(1)} \otimes \cdots \otimes a_{\nu(i_2)}) a_{\nu(i_2+1)} \cdots a_{\nu(J')} \\
& \otimes b_{\nu(1)} \cdots b_{\nu(i_2)} g_2(b_{\nu(i_2+1)} \otimes \cdots \otimes b_{\nu(J')}),
\end{aligned}$$

where the set $A_{11} = \{\sigma(1), \ldots, \sigma(J')\}$.

Then,

$$\begin{aligned}
&\Phi(f_1 \otimes g_1)\Big(\Phi(f_2 \otimes g_2)\big((a_{\sigma(1)} \otimes b_{\sigma(1)}) \otimes \cdots \otimes (a_{\sigma(J')} \otimes b_{\sigma(J')})\big) \otimes \\
&\qquad (a_{\sigma(J'+1)} \otimes b_{\sigma(J'+1)}) \otimes \cdots \otimes (a_{\sigma(I'+J'-1)} \otimes b_{\sigma(I'+J'-1)})\Big)
\end{aligned}$$

$$= \Phi(f_1 \otimes g_1)\Big((a_0 \otimes b_0) \otimes (a_{\sigma(J'+1)} \otimes b_{\sigma(J'+1)}) \otimes \cdots$$
$$\otimes (a_{\sigma(I'+J'-1)} \otimes b_{\sigma(I'+J'-1)})\Big)$$
$$= \sum_{\tau \in S^{A_{12}}_{i_1,j_1}} \operatorname{sgn}(\tau) f_1(a_{\tau(1)} \otimes \cdots \otimes a_{\tau(i_1)}) a_{\tau(i_1+1)} \cdots$$
$$a_{\tau(I')} \otimes b_{\tau(1)} \cdots b_{\tau(i_1)} g_1(b_{\tau(i_1+1)} \otimes \cdots \otimes b_{\tau(I')}), \tag{E3}$$

where the set $A_{12} = \{0, \sigma(J'+1), \ldots, \sigma(I'+J'-1)\}$. By Proposition 2.1,

$$(E3) = \sum_{\eta \in S^{A_{13}}_{i_1-1,j_1}} \operatorname{sgn}(\eta) f_1(a_0 \otimes a_{\eta(1)} \otimes \cdots \otimes a_{\eta(i_1-1)}) a_{\eta(i_1)} \cdots a_{\eta(I'-1)}$$
$$\otimes b_0 b_{\eta(1)} \cdots b_{\eta(i_1-1)} g_1(b_{\eta(i_1)} \otimes \cdots \otimes b_{\eta(I'-1)})$$
$$+ \sum_{\delta \in S^{A_{13}}_{i_1,j_1-1}} (-1)^{i_1} \operatorname{sgn}(\delta) f_1(a_{\delta(1)} \otimes \cdots \otimes a_{\delta(i_1)}) a_0 a_{\delta(i_1+1)} \cdots a_{\delta(I'-1)}$$
$$\otimes b_{\delta(1)} \cdots b_{\delta(i_1)} g_1(b_0 \otimes b_{\delta(i_1+1)} \otimes \cdots \otimes b_{\delta(I'-1)}), \tag{E4}$$

where the set $A_{13} = \{\sigma(J'+1), \ldots, \sigma(I'+J'-1)\}$. Combining with (13), we have

$$(E4) = \sum_{\substack{\nu \in S^{A_{11}}_{i_2,j_2} \\ \eta \in S^{A_{13}}_{i_1-1,j_1}}} \operatorname{sgn}(\eta) \operatorname{sgn}(\nu) f_1\Big(f_2\big(a_{\nu(1)} \otimes \cdots \otimes a_{\nu(i_2)}\big) a_{\nu(i_2+1)} \cdots a_{\nu(J')}$$
$$\otimes a_{\eta(1)} \otimes \cdots \otimes a_{\eta(i_1-1)}\Big) a_{\eta(i_1)} \cdots a_{\eta(I'-1)} \otimes b_{\nu(1)} \cdots b_{\nu(i_2)}$$
$$g_2(b_{\nu(i_2+1)} \otimes \cdots \otimes b_{\nu(J')}) b_{\eta(1)} \cdots b_{\eta(i_1-1)} g_1(b_{\eta(i_1)} \otimes \cdots \otimes b_{\eta(I'-1)})$$
$$+ \sum_{\substack{\nu \in S^{A_{11}}_{i_2,j_2} \\ \delta \in S^{A_{13}}_{i_1,j_1-1}}} (-1)^{i_1} \operatorname{sgn}(\delta) \operatorname{sgn}(\nu) f_1(a_{\delta(1)} \otimes \cdots \otimes a_{\delta(i_1)})$$
$$f_2(a_{\nu(1)} \otimes \cdots \otimes a_{\nu(i_2)}) a_{\nu(i_2+1)} \cdots a_{\nu(J')} a_{\delta(i_1+1)} a_{\delta(I'-1)} \otimes b_{\delta(1)} \cdots b_{\delta(i_1)}$$
$$g_1\Big(b_{\nu(1)} \cdots b_{\nu(i_2)} g_2\big(b_{\nu(i_2+1)} \otimes \cdots \otimes b_{\nu(J')}\big) \otimes b_{\delta(i_1+1)} \otimes \cdots \otimes b_{\delta(I'-1)}\Big).$$

Now, by the construction (3) and Proposition 2.2,

$$U = \sum_{\tau \in S_{i_2,j_2,i_1-1,j_1}} \operatorname{sgn}(\tau) f_1\Big(f_2\big(a_{\tau(1)} \otimes \cdots \otimes a_{\tau(i_2)}\big) a_{\tau(i_2+1)} \cdots a_{\tau(J')}$$
$$\otimes a_{\tau(J'+1)} \otimes \cdots \otimes a_{\tau(J'+i_1-1)}\Big) a_{\tau(J'+i_1)} \cdots a_{\tau(J'+I'-1)}$$
$$\otimes b_{\tau(1)} \cdots b_{\tau(i_2)} g_2(b_{\tau(i_2+1)} \otimes \cdots \otimes b_{\tau(J')}) b_{\tau(J'+1)} \cdots b_{\tau(J'+i_1-1)}$$
$$g_1(b_{\tau(J'+i_1)} \otimes \cdots \otimes b_{\tau(J'+I'-1)})$$
$$+ \sum_{\tau \in S_{i_2,j_2,i_1,j_1-1}} (-1)^{i_1} \operatorname{sgn}(\tau) f_1(a_{\tau(J'+1)} \otimes \cdots \otimes a_{\tau(J'+i_1)})$$

$$f_2(a_{\tau(1)} \otimes \cdots \otimes a_{\tau(i_2)}) a_{\tau(i_2+1)} \cdots a_{\tau(J')} a_{\tau(J'+i_1+1)}$$
$$\cdots a_{\tau(J'+I'-1)} \otimes b_{\tau(J'+1)} \cdots b_{\tau(J'+i_1)}$$
$$g_1\Big(b_{\tau(1)} \cdots b_{\tau(i_2)} g_2\big(b_{\tau(i_2+1)} \otimes \cdots \otimes b_{\tau(J')}\big)$$
$$\otimes b_{\tau(J'+i_1+1)} \otimes \cdots \otimes b_{\tau(J'+I'-1)}\Big),$$

and denote the right hands of last equation by $U_1 + U_2$ briefly.

By Proposition 2.3 and Remark 2.4,

$$\begin{aligned}
U_1 = & \sum_{\sigma \in S_{i_2, i_1-1, j_1, j_2}} (-1)^{(i_1-1)j_2 + j_1 j_2} \operatorname{sgn}(\sigma) f_1\Big(f_2\big(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i_2)}\big) \\
& a_{\sigma(I+j_1)} \cdots a_{\sigma(I+J-1)} \otimes a_{\sigma(i_2+1)} \otimes \cdots \otimes a_{\sigma(I-1)}\Big) \\
& a_{\sigma(I)} \cdots a_{\sigma(I+j_1-1)} \otimes b_{\sigma(1)} \cdots b_{\sigma(i_2)} b_{\sigma(i_2+1)} \cdots b_{\sigma(I-1)} \\
& g_1(b_{\sigma(I)} \otimes \cdots \otimes b_{\sigma(I+j_1-1)}) g_2(b_{\sigma(I+j_1)} \otimes \cdots \otimes b_{\sigma(I+J-1)}) \\
= & \sum_{\sigma \in S_{i_2, i_1-1, j_1, j_2}} (-1)^{(i_1-1)j_2 + j_1 j_2} \operatorname{sgn}(\sigma) f_1\Big(f_2\big(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i_2)}\big) \\
& \otimes a_{\sigma(i_2+1)} \otimes \cdots \otimes a_{\sigma(I-1)}\Big) a_{\sigma(I)} \cdots a_{\sigma(I+j_1-1)} a_{\sigma(I+j_1)} \cdots a_{\sigma(I+J-1)} \\
& \otimes b_{\sigma(1)} \cdots b_{\sigma(i_2)} b_{\sigma(i_2+1)} \cdots b_{\sigma(I-1)} g_1(b_{\sigma(I)} \otimes \cdots \otimes b_{\sigma(I+j_1-1)}) \\
& g_2(b_{\sigma(I+j_1)} \otimes \cdots \otimes b_{\sigma(I+J-1)}) \\
& + \sum_{\sigma \in S_{i_2, i_1-1, j_1, j_2}} (-1)^{(i_1-1)j_2 + j_1 j_2} \operatorname{sgn}(\sigma) f_2\big(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i_2)}\big) \\
& f_1\Big(a_{\sigma(I+j_1)} \cdots a_{\sigma(I+J-1)} \otimes a_{\sigma(i_2+1)} \otimes \cdots \otimes a_{\sigma(I-1)}\Big) \\
& a_{\sigma(I)} \cdots a_{\sigma(I+j_1-1)} \otimes b_{\sigma(1)} \cdots b_{\sigma(i_2)} b_{\sigma(i_2+1)} \cdots b_{\sigma(I-1)} \\
& g_1(b_{\sigma(I)} \otimes \cdots \otimes b_{\sigma(I+j_1-1)}) g_2(b_{\sigma(I+j_1)} \otimes \cdots \otimes b_{\sigma(I+J-1)}) \\
= & \sum_{\sigma \in S_{i_2, i_1-1, j_1, j_2}} K f_1(f_2(\cdots)\cdots)\cdots \otimes \cdots g_1(\cdots) g_2(\cdots) \\
& + \sum_{\sigma \in S_{i_2, i_1-1, j_1, j_2}} \sum_{I+j_1 \le q \le I+J-1} K f_2\big(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i_2)}\big) \\
& f_1\Big(a_{\sigma(q)} \otimes a_{\sigma(i_2+1)} \otimes \cdots \otimes a_{\sigma(I-1)}\Big) \\
& a_{\sigma(I)} \cdots a_{\sigma(I+j_1-1)} a_{\sigma(I+j_1)} \cdots \widehat{a_{\sigma(q)}} \cdots a_{\sigma(I+J-1)} \\
& \otimes b_{\sigma(1)} \cdots b_{\sigma(I-1)} g_1(b_{\sigma(I)} \otimes \cdots \otimes b_{\sigma(I+j_1-1)}) \\
& g_2(b_{\sigma(I+j_1)} \otimes \cdots \otimes b_{\sigma(I+J-1)}) \\
\triangleq & \sum_{\sigma \in S_{i_2, i_1-1, j_1, j_2}} K f_1(f_2(\cdots)\cdots)\cdots \otimes \cdots g_1(\cdots) g_2(\cdots) + U_{12}.
\end{aligned}$$

By a similar process,

$$
\begin{aligned}
U_2 = & \sum_{\sigma\in S_{i_1,i_2,j_2,j_1-1}} Mf_1(\cdots)f_2(\cdots)\cdots\otimes\cdots g_1(g_2(\cdots)\cdots) \\
& + \sum_{\sigma\in S_{i_1,i_2,j_2,j_1-1}} \sum_{i_1+1\le p\le i_1+i_2} Mf_1(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(i_1)}) \\
& f_2(a_{\sigma(i_1+1)}\otimes\cdots\otimes a_{\sigma(I)})a_{\sigma(I+1)}\cdots a_{\sigma(I+J-1)} \\
& \otimes b_{\sigma(1)}\cdots b_{\sigma(i_1)}b_{\sigma(i_1+1)}\cdots\widehat{b_{\sigma(p)}}\cdots b_{\sigma(I)} \\
& g_2(b_{\sigma(I+1)}\otimes\cdots\otimes b_{\sigma(I+j_2)})g_1(b_{\sigma(p)}\otimes b_{\sigma(I+j_2+1)}\otimes\cdots\otimes b_{\sigma(I+J-1)}) \\
= & \sum_{\sigma\in S_{i_1,i_2,j_2,j_1-1}} Mf_1(\cdots)f_2(\cdots)\cdots\otimes\cdots g_1(g_2(\cdots)\cdots) \\
& + \sum_{\sigma\in S_{i_1,i_2,j_1-1,j_2}} \sum_{i_1+1\le p\le i_1+i_2} (-1)^{(j_1-1)j_2} Mf_1(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(i_1)}) \\
& f_2(a_{\sigma(i_1+1)}\otimes\cdots\otimes a_{\sigma(I)})a_{\sigma(I+1)}\cdots a_{\sigma(I+J-1)} \\
& \otimes b_{\sigma(1)}\cdots b_{\sigma(i_1)}b_{\sigma(i_1+1)}\cdots\widehat{b_{\sigma(p)}}\cdots b_{\sigma(I)} \\
& g_1(b_{\sigma(p)}\otimes b_{\sigma(I+1)}\otimes\cdots\otimes b_{\sigma(I+j_1)})g_2(b_{\sigma(I+j_1+1)}\otimes\cdots\otimes b_{\sigma(I+J-1)}) \\
\triangleq & \sum_{\sigma\in S_{i_1,i_2,j_2,j_1-1}} Mf_1(\cdots)f_2(\cdots)\cdots\otimes\cdots g_1(g_2(\cdots)\cdots) + U_{22}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
V = & -\sum_{\sigma\in S_{i_1,i_2-1,j_1,j_2}} Lf_2(f_1(\cdots)\cdots)\cdots\otimes\cdots g_1(\cdots)g_2(\cdots) \\
& -\sum_{\sigma\in S_{i_1,i_2-1,j_1,j_2}} \sum_{I\le q\le I+j_1-1} Lf_1\big(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(i_1)}\big) \\
& f_2\Big(a_{\sigma(q)}\otimes a_{\sigma(i_1+1)}\otimes\cdots\otimes a_{\sigma(I-1)}\Big) \\
& a_{\sigma(I)}\cdots\cdots\widehat{a_{\sigma(q)}}\cdots a_{\sigma(I+j_1-1)}a_{\sigma(I+j_1)}\cdots a_{\sigma(I+J-1)} \\
& \otimes b_{\sigma(1)}\cdots b_{\sigma(I-1)}g_1(b_{\sigma(I)}\otimes\cdots\otimes b_{\sigma(I+j_1-1)}) \\
& g_2(b_{\sigma(I+j_1)}\otimes\cdots\otimes b_{\sigma(I+J-1)}) \\
& -\sum_{\sigma\in S_{i_1,i_2,j_1,j_2-1}} Nf_1(\cdots)f_2(\cdots)\cdots\otimes\cdots g_2(g_1(\cdots)\cdots) \\
& -\sum_{\sigma\in S_{i_1,i_2,j_1,j_2-1}} \sum_{1\le p\le i_1} Nf_1(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(i_1)})f_2(a_{\sigma(i_1+1)}\otimes\cdots\otimes a_{\sigma(I)}) \\
& a_{\sigma(I+1)}\cdots a_{\sigma(I+J-1)}\otimes b_{\sigma(1)}\cdots\widehat{b_{\sigma(p)}}\cdots b_{\sigma(i_1)}b_{\sigma(i_1+1)}\cdots b_{\sigma(I)} \\
& g_1(b_{\sigma(I+1)}\otimes\cdots\otimes b_{\sigma(I+j_1)}) \\
& g_2(b_{\sigma(p)}\otimes b_{\sigma(I+j_2+1)}\otimes\cdots\otimes b_{\sigma(I+J-1)})
\end{aligned}
$$

$$\triangleq -\sum_{\sigma\in S_{i_1,i_2-1,j_1,j_2}} Lf_2(f_1(\cdots)\cdots)\cdots\otimes\cdots g_1(\cdots)g_2(\cdots)+V_{12}$$
$$-\sum_{\sigma\in S_{i_1,i_2,j_1,j_2-1}} Nf_1(\cdots)f_2(\cdots)\cdots\otimes\cdots g_2(g_1(\cdots)\cdots)+V_{22}.$$

To prove the equation (11), it remains to prove that

$$U_{12}+U_{22}+V_{12}+V_{22}=0$$

due to the equation (12). Now, by Proposition 2.5,

$$\begin{aligned}
U_{22}= & \sum_{\sigma\in S_{i_1,i_2,j_1-1,j_2}} \sum_{i_1+1\le p\le i_1+i_2} (-1)^{(j_1-1)j_2} Mf_1(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(i_1)}) \\
& f_2(a_{\sigma(i_1+1)}\otimes\cdots\otimes a_{\sigma(I)})\cdot a_{\sigma(I+1)}\cdots a_{\sigma(I+J-1)} \\
& \otimes b_{\sigma(1)}\cdots b_{\sigma(i_1)}b_{\sigma(i_1+1)}\cdots\widehat{b_{\sigma(p)}}\cdots b_{\sigma(I)} \\
& g_1(b_{\sigma(p)}\otimes b_{\sigma(I+1)}\otimes\cdots\otimes b_{\sigma(I+j_1)})g_2(b_{\sigma(I+j_1+1)}\otimes\cdots\otimes b_{\sigma(I+J-1)}) \\
= & \sum_{\tau\in S_{i_1,i_2-1,j_1,j_2}} \sum_{i_1+i_2\le q\le i_1+i_2+j_1-1} (-1)^{(j_1-1)j_2}\operatorname{sgn}(\tau)(-1)^{i_1(J'-1)}(-1)^{p+q+1} \\
& f_1(a_{\tau(1)}\otimes\cdots\otimes a_{\tau(i_1)})f_2(a_{\tau(i_1+1)}\otimes\cdots\otimes a_{\tau(q)}\otimes\cdots\otimes a_{\tau(I-1)}) \\
& a_{\tau(I)}\cdots a_{\tau(I+J-1)}\otimes b_{\tau(1)}\cdots b_{\tau(i_1)}b_{\tau(i_1+1)}\cdots b_{\tau(I-1)} \\
& g_1(b_{\tau(q)}\otimes b_{\tau(I)}\otimes\cdots\otimes\widehat{b_{\tau(q)}}\otimes\cdots b_{\tau(I+j_1)}) \\
& g_2(b_{\tau(I+j_1+1)}\otimes\cdots\otimes b_{\tau(I+J-1)}) \\
= & -V_{12}.
\end{aligned}$$

By a similar process, we obtain that $U_{12}=-V_{22}$. That is, we finish the proof. □

*Remark* 3.4. Finally, we talk some words on Batalin-Vilkovisky algebra. By definition, Batalin-Vilkovisky algebra is a Gerstenhaber algebra $H$ together with an operator $\Delta$ of degree $-1$ such that $\Delta^2=0$ and that the equation

$$-(-1)^{(m-1)n}[a,b]:=\Delta(a\wedge b)-\Delta(a)\wedge b-(-1)^m a\wedge\Delta(b)$$

holds for $a\in H^m, b\in H^n$. If $A$ is a Calabi-Yau or a symmetric Frobenius algebra, the Hochschild cohomology ring $\mathrm{HH}^*(A)$ is a Batalin-Vilkovisky algebra [4,6,14]. If $R$ is a unimodular smooth Poisson algebra or a unimodular Frobenius Poisson algebra, $\mathrm{HP}^*(R)$ is a Batalin-Vilkovisky algebra [15,16]. Some general cases were studied in [8,9,13]. It is proved that if both $A$ and $B$ are symmetric Frobenius algebra, then

$$\mathrm{HH}^*(A)\otimes\mathrm{HH}^*(B)\cong\mathrm{HH}^*(A\otimes B)$$

is an isomorphic of Batalin-Vilkovisky algebras in [11]. Here, if we consider the tensor product of unimodular smooth Poisson algebras or of unimodular Frobenius Poisson algebras, then the isomorphism in Theorem 3.3 is an isomorphism of Batalin-Vilkovisky algebras.

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