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ESTIMATES FOR SCHRÖDINGER MAXIMAL OPERATORS ALONG CURVE WITH COMPLEX TIME

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Abstract. In the present paper, we give some characterization of the L^2 maximal estimate for the operator $P^t_{a,\gamma}f\big(\Gamma(x,t)\big)$ along curve with complex time, which is defined by

$$P^t_{a,\gamma}f\big(\Gamma(x,t)\big)=\int_{\mathbb{R}}e^{i\Gamma(x,t)\xi}e^{it|\xi|^a}e^{-t^\gamma|\xi|^a}\hat{f}(\xi)d\xi,$$

where $t,\gamma>0$ and $a\geq 2$, curve Γ is a function such that $\Gamma:\mathbb{R}\times[0,1]\to\mathbb{R}$, and satisfies Hölder's condition of order σ and bilipschitz conditions. The authors extend the results of the Schrödinger type with complex time of Bailey [1] and Cho, Lee and Vargas [3] to Schrödinger operators along the curves.

1. Introduction

Suppose $f\in\mathcal{S}(\mathbb{R})$, the Schwartz class on $\mathbb{R}$, for $a>1$, time parameter $t>0$, define

$$S^t_af(x)=(2\pi)^{-1}\int_{\mathbb{R}}e^{ix\cdot\xi+it|\xi|^a}\hat{f}(\xi)d\xi,\quad (x,t)\in\mathbb{R}\times\mathbb{R}^+,$$

where $\hat{f}(\xi)=\int_{\mathbb{R}}e^{-i\xi\cdot x}f(x)dx$. When $a=2$, it is well known that $e^{it\Delta}f(x):=S^t_2f(x)$ is the solution of the Schrödinger equation

$$\begin{cases} i\partial_tu-\Delta u=0,\\ u(x,0)=f(x).\end{cases}\tag{1.1}$$

Define the maximal operator S^*_af associated with the family of operators $\{S^t_a\}_{0<t<1}$ by

$$S^*_af(x)=\sup_{0<t<1}|S^t_af(x)|,\quad x\in\mathbb{R}.$$

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In 1979, Carleson [2] proposed a problem: Determining the optimal exponent s for which

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R} \tag{1.2}$$

holds whenever $f \in H^s(\mathbb{R})$. Here $H^s(\mathbb{R})\,(s \in \mathbb{R})$ denotes the non-homogeneous Sobolev space, which is defined by

$$H^s(\mathbb{R}) = \left\{ f \in \mathcal{S}' : \|f\|_{H^s} = \left(\int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty \right\}.$$

Carleson first considered this problem for one spatial dimension in the context of Hölder continuous functions, an immediate consequence of his work was that when f is taken in the Sobolev space $H^{\frac{1}{4}}(\mathbb{R})$, the following estimate may be established

$$\|S_2^* f\|_{L^2([-1,1])} \le C \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}.$$

Hence, Carleson in [2] showed that the pointwise convergence (1.2) holds for data in $H^s(\mathbb{R})$ with $s \ge \frac{1}{4}$, which is sharp was proved by Dahlberg and Kenig in [4]. On the other hand, Vega in [12] showed that

$$\|S_2^* f\|_{L^2(\mathbb{R})} \le C \|f\|_{H^s(\mathbb{R})} \tag{1.3}$$

holds for $s > \frac{1}{2}$, and that this estimate fails for $s < \frac{1}{2}$. The estimate (1.3) for $s = \frac{1}{2}$ is open. Moreover, for $a > 1$, Sjölin [7] proved the following global estimate

$$\|S_a^* f\|_{L^2(\mathbb{R})} \le C \|f\|_{H^s(\mathbb{R})} \tag{1.4}$$

holds for $s > \frac{a}{4}$, and (1.4) fails for $s < \frac{a}{4}$. The problem of boundedness for $s = \frac{a}{4}$ remains open.

When the definition of the solution operator for the Schrödinger equation is extended to allow complex-valued time with positive imaginary part, then for $t \ge 0$, the operator S_2^{it} is the solution operator for the heat equation

$$\partial_t u(t,x) = \partial_x^2 (t,x).$$

Since S_2^{it} is a convolution operator with a Gaussian multiplier, by the boundedness of the Hardy-Littlewood maximal function, the estimate yields

$$\left\| \sup_{0<t<1} |S_2^{it} f(x)| \right\|_{L^2(\mathbb{R})} \le C \|f\|_{L^2(\mathbb{R})}.$$

By this result and Vega's result of estimate (1.3), naturally, one may consider a problem: for which maps $g : [0,1] \to [0,1]$ with $\lim_{t\to 0} g(t) = 0$ and for which $s > 0$, such that

$$\left\| \sup_{0<t<1} |S_2^{t+ig(t)} f(x)| \right\|_{L^2(\mathbb{R})} \le C \|f\|_{H^s(\mathbb{R})}.$$

This interesting question was posed and partially answered by Sjölin in [8]. For $t, \gamma > 0$ and $a > 1$, the operator $P^t_{a,\gamma} f$ with complex time is defined by

$$P^t_{a,\gamma} f(x) = S^{t+it^\gamma}_a f(x) = \int_{\mathbb{R}^n} e^{ix\xi} e^{it|\xi|^a} e^{-t^\gamma |\xi|^a} \hat{f}(\xi) d\xi,$$

with the corresponding maximal operator $P^*_{a,\gamma}$ is defined by

$$P^*_{a,\gamma} f(x) = \sup_{0<t<1} |P^t_{a,\gamma} f(x)|, \quad x \in \mathbb{R}.$$

The global estimate is defined by

$$\|P^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})}. \tag{1.5}$$

When $\gamma > 0$ and $a > 1$, we denote by E_γ the set of all $s > 0$ such that (1.5) holds, and set

$$s_a(\gamma) = \inf E_\gamma.$$

When $a = 2$, Sjölin in [8], Sjölin and Soria in [10] obtained some results of the estimate (1.5). Recently, Bailey in [1] improved and extended above results. More precisely, Bailey obtained the following results.

Theorem A ([1])**.** *For $\gamma \in (0, \infty)$ and $a > 1$, $s_a(\gamma) = \max\{\frac{a}{4}(1 - \frac{1}{\gamma}), 0\}$.*

On the other hand, one may consider the solution $e^{it\Delta} f$ of equation (1.1) converges to f nontangentially for a.e. $x \in \mathbb{R}^n$. That is, for $\alpha > 0$ and $f \in H^s(\mathbb{R}^n)$, for which s such that

$$\lim_{\substack{(y,t)\in\Gamma_\alpha(x)\\(y,t)\to(x,0)}} e^{it\Delta} f(y) = f(x), \text{ a.e. } x \in \mathbb{R}^n, \tag{1.6}$$

where $\Gamma_\alpha(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |y - x| < \alpha t\}$. If $s > \frac{n}{2}$, then by Sobolev imbedding,

$$\sup_{x\in\mathbb{R}^n, t\in\mathbb{R}} |e^{it\Delta} f(x)| \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$

By a standard argument, (1.6) holds for $s > \frac{n}{2}$. However, Sjögren and Sjölin [9] proved that (1.6) fails for $s \leq \frac{n}{2}$. In fact, in [9], they proved that there is an $f \in H^{\frac{n}{2}}(\mathbb{R}^n)$ and a strictly increasing function Γ with $\Gamma(0) = 0$, such that for all $x \in \mathbb{R}^n$,

$$\limsup_{\substack{(y,t)\to(x,0), t>0\\|x-y|<\Gamma(t)}} |e^{it\Delta} f(y)| = \infty.$$

Hence, an interest problem is whether there exists an appropriate curve $\Gamma(x, t)$ and $s > 0$, such that for $f \in H^s(\mathbb{R})$, the following pointwise convergence holds along the curve $(\Gamma(x, t), t)$:

$$\lim_{t\to 0} e^{it\Delta} f\big(\Gamma(x, t)\big) = f(x), \qquad \text{a.e. } x \in \mathbb{R}, \tag{1.7}$$

where

$$e^{it\Delta} f\big(\Gamma(x, t)\big) = (2\pi)^{-1} \int_{\mathbb{R}} e^{i\Gamma(x,t)\cdot\xi + it|\xi|^2} \hat{f}(\xi) d\xi.$$

Recently, Lee and Rogers [6], Cho, Lee and Vargas [3] considered this problem and gave an affirmative answer. Assume that Γ is a function such that $\Gamma : \mathbb{R}\times[-1,1] \to \mathbb{R}$, $\Gamma(x,0) = x$, and Γ satisfies the following conditions: There exist constants C_i (i=1, 2, 3), independent of x, y and t, t', such that

(A1) *Hölder condition of order* σ in t:

$$|\Gamma(x,t) - \Gamma(x,t')| \le C_1|t-t'|^{\sigma};$$

(A2) *Bilipschitz condition* in x:

$$C_2|x-y| \le |\Gamma(x,t) - \Gamma(y,t)| \le C_3|x-y|.$$

Here σ is essentially the degree of tangential convergence. For $x_0, t_0 \in \mathbb{R}$ and $R, T > 0$, denote

$$B(x_0,R) := \{x \in \mathbb{R}; |x-x_0| \le R\}, \qquad I_T(t_0) := \{t \in \mathbb{R}; |t-t_0| \le T\}.$$

The authors in [3] obtained the following results:

Theorem B ([3])**.** *Assume* Γ *satisfies the conditions* (A1), (A2) *for* $0 < \sigma \le 1$ *and* $x, y \in B(x_0,R)$, $t, t' \in I_T(t_0)$. *If* $s > \max\{\frac{1}{2} - \sigma, \frac{1}{4}\}$, *then*

$$\left\| \sup_{t\in I_T(t_0)} \left| e^{it\Delta} f\big(\Gamma(x,t)\big)\right| \right\|_{L^2(B(x_0,R))} \le C\|f\|_{H^s(\mathbb{R})}. \tag{1.8}$$

Inspired by the above works, we will consider a class of oscillatory integral operators along curve Γ with complex time. For $t, \gamma > 0$ and $a > 1$, the operator $P^t_{a,\gamma} f\big(\Gamma(x,t)\big)$ along curve Γ with complex time, which is defined by

$$P^t_{a,\gamma} f\big(\Gamma(x,t)\big) = S_a^{t+it^{\gamma}} f\big(\Gamma(x,t)\big) = \int_{\mathbb{R}} e^{i\Gamma(x,t)\xi} e^{it|\xi|^a} e^{-t^{\gamma}|\xi|^a} \hat{f}(\xi) d\xi,$$

and with the corresponding maximal operator $P^*_{a,\gamma,\Gamma} f$ is defined by

$$P^*_{a,\gamma,\Gamma} f(x) = \sup_{t\in[0,1]} |P^t_{a,\gamma} f\big(\Gamma(x,t)\big)|, \quad x \in \mathbb{R}.$$

We will consider the global estimate

$$\|P^*_{a,\gamma,\Gamma} f\|_{L^2(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})} \tag{1.9}$$

and the local estimate

$$\|P^*_{a,\gamma,\Gamma} f\|_{L^2([-1,1])} \le C\|f\|_{H^s(\mathbb{R})}. \tag{1.10}$$

When $\gamma > 0$ and $a > 1$, we denote by $E_{\Gamma,\gamma}$ the set of all $s > 0$ such that (1.9) holds, and set

$$s_{a,\Gamma}(\gamma) = \inf E_{\Gamma,\gamma}.$$

And denote by $F_{\Gamma,\gamma}$ the set of all $s > 0$ such that (1.10) holds, and set

$$s^{loc}_{a,\Gamma}(\gamma) = \inf F_{\Gamma,\gamma}.$$

In the present paper, we will give some characterization of the global L^2 maximal estimate (1.9) and the local L^2 maximal estimate (1.10). Now we state our main results in this paper as follows.

Theorem 1.1. *Let* $\gamma > 1$ *and* $a \geq 2$. *Let* Γ *be a function such that* $\Gamma : \mathbb{R} \times [0,1] \to \mathbb{R}$ *and* $\Gamma(x,0) = x$. *Assume that* Γ *satisfies the conditions* (A1) *for* $\frac{1}{a} \leq \sigma \leq 1$ *and* (A2) *for* $x, y \in \mathbb{R}$ *and* $t, t' \in [0,1]$. *Then*

$$s_{a,\Gamma}(\gamma) = \frac{a}{4}(1 - \frac{1}{\gamma}).$$

Remark 1.2. In fact, from the proof of Theorem 1.1, when $a > 1$ and $\gamma > 1$, we proved that the global estimate (1.9) holds for $s > \frac{a}{4}(1 - \frac{1}{\gamma})$. Moreover, when $a \geq 2$ and $\gamma > 1$, we showed that the global estimate (1.9) fails if $s < \frac{a}{4}(1 - \frac{1}{\gamma})$. However, when $1 < a < 2$, we cannot proved that the global estimate (1.9) fails if $s < \frac{a}{4}(1 - \frac{1}{\gamma})$.

Theorem 1.3. *Let* $\gamma > 1$ *and* $a \geq 2$. *Let* Γ *be a function such that* $\Gamma : \mathbb{R} \times [0,1] \to \mathbb{R}$ *and* $\Gamma(x,0) = x$. *Assume that* Γ *satisfies the conditions* (A1) *for* $\frac{1}{a} \leq \sigma \leq 1$ *and* (A2) *for* $x, y \in [-1,1]$ *and* $t, t' \in [0,1]$. *Then*

$$s^{\text{loc}}_{a,\Gamma}(\gamma) = \min\big\{\frac{a}{4}(1 - \frac{1}{\gamma}), \frac{1}{4}\big\}.$$

Remark 1.4. In the case of $\gamma > 1$ and $a \geq 2$. Let $\Gamma(x,t) = x$ for any $t \in \mathbb{R}$, then (1.9) is just (1.5). In this sense, Theorem 1.1 is an extension of Theorem A in [1].

Remark 1.5. From the result in [3], there are curves Γ satisfying the conditions (A1), (A2) for $0 < \sigma \leq 1$ and $x, y \in B(x_0, R)$, $t, t' \in I_T(t_0)$, but the local maximal estimate (1.8) fails if $s < \max\{\frac{1}{2} - \sigma, \frac{1}{4}\}$. Hence, when $B(x_0, R) = [-1,1]$, $I_T(t_0) = [0,1]$, and Γ satisfies the conditions (A1) for $\frac{1}{2} \leq \sigma \leq 1$ and (A2) for $x, y \in [-1,1]$ and $t, t' \in [0,1]$, the local maximal estimate (1.8) fails if $s < \frac{1}{4}$. However, by the results of Theorem 1.3, when $a = 2$, $1 < \gamma < 2$, and Γ satisfies the conditions (A1) for $\frac{1}{2} \leq \sigma \leq 1$ and (A2) for $x, y \in [-1,1]$ and $t, t' \in [0,1]$, the local estimate (1.10) holds for $s > \frac{1}{2} - \frac{1}{2\gamma}$ $(\frac{1}{2} - \frac{1}{2\gamma} < \frac{1}{4})$.

As an application of the local maximal estimate (1.10), we give the pointwise convergence along curve Γ. More precisely, we have the following corollary.

Corollary 1.6. *Let* $\gamma > 1$, $a > 1$ *and* $\frac{1}{a} \leq \sigma \leq 1$. *Suppose that for every* $x_0 \in \mathbb{R}$, *there exists a neighborhood* V *of* $(x_0, 0)$ *such that* (A1) *holds for* (x,t), $(x,t') \in V$ *and* (A2) *holds for all* (x,t), $(y,t) \in V$. *Then for* $f \in H^s(\mathbb{R})$, *if* $s > \frac{a}{4}(1 - \frac{1}{\gamma})$ *when* $1 < \gamma < \frac{a}{a-1}$ *or* $s \geq \frac{1}{4}$ *when* $\gamma \geq \frac{a}{a-1}$,

$$\lim_{t \to 0} P^t_{a,\gamma} f\big(\Gamma(x,t)\big) = f(x), \qquad \text{a.e. } x \in \mathbb{R}. \tag{1.11}$$

This paper is organized as follows. The proof of Theorem 1.1 is given in Section 2. In the proof of Theorem 1.1, Lemma 2.3 plays an important role, whose proof will be given in Section 3. The proof of Theorem 1.3 is given in Section 4.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we first prove the following results.

Theorem 2.1. *Let $\gamma > 1$ and $a \geq 2$. Assume that Γ satisfies the condition* (A1) *for $\frac{1}{a} \leq \sigma \leq 1$, $x \in \mathbb{R}$ and $t, t' \in [0,1]$. Then*

$$\|P^*_{a,\gamma,\Gamma} f\|_{L^2(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})} \tag{2.1}$$

fails if $s < \frac{a}{4}(1-\frac{1}{\gamma})$.

Proof of Theorem 2.1. The proof of Theorem 2.1 given here follows the similar strategy used to prove that the estimate (1.5) cannot hold for $s < \frac{a}{4}(1-\frac{1}{\gamma})$ in [1] when $a > 1$ and $\gamma > 1$. Fix $\gamma > 1$, $a \geq 2$ and $s < \frac{a}{4}(1-\frac{1}{\gamma})$. And for each $v \in (0,1)$, we choose a positive, even and real valued function $g_v \in \mathcal{S}(\mathbb{R})$, such that $\operatorname{supp} g_v \subset [-v^{(a-1)-\frac{a}{\gamma}}, v^{(a-1)-\frac{a}{\gamma}}]$, and $g_v(\xi) = 1$ if $\xi \in [-\frac{1}{2}v^{(a-1)-\frac{a}{\gamma}}, \frac{1}{2}v^{(a-1)-\frac{a}{\gamma}}]$. Define the function f_v such that $\widehat{f_v}(\xi) = v g_v(v\xi + \frac{1}{v})$. Note that the following fact: when $s < \frac{a}{4}(1-\frac{1}{\gamma})$,

$$\|f_v\|_{H^s(\mathbb{R})} \to 0 \quad (\text{as} \quad v \to 0). \tag{2.2}$$

In fact, by the support of g_v, $v \in (0,1)$ and $(a-1) - \frac{a}{\gamma} > -1$, it follows that $\xi + \frac{1}{v} \in [-v^{(a-1)-\frac{a}{\gamma}}, v^{(a-1)-\frac{a}{\gamma}}]$, and $|\xi| \leq v^{(a-1)-\frac{a}{\gamma}} + \frac{1}{v} \leq 2\frac{1}{v}$. Hence, by a straightforward calculation, we have

$$\begin{aligned}
\|f_v\|^2_{H^s(\mathbb{R})} &= v^2 \int_{\mathbb{R}} \left| g_v(v\xi + \frac{1}{v}) \right|^2 (1+|\xi|^2)^s d\xi \\
&= v \int_{\mathbb{R}} \left| g_v(\xi + \frac{1}{v}) \right|^2 \left(\frac{v^2 + |\xi|^2}{v^2} \right)^s d\xi \\
&\leq C v^{1-2s} v^{(a-1)-\frac{a}{\gamma}} v^{-2s} = C v^{a-4s-\frac{a}{\gamma}}.
\end{aligned}$$

Thus, when $s < \frac{a}{4}(1-\frac{1}{\gamma})$, (2.2) holds. Observe that

$$P^t_{a,\gamma} f_v\big(\Gamma(x,t)\big) = \int_{\mathbb{R}} e^{i\Gamma(x,t)\xi} e^{it|\xi|^a} e^{-t^\gamma |\xi|^a} v g_v(v\xi + \frac{1}{v}) d\xi.$$

Let $\eta = v\xi + \frac{1}{v}$, we have

$$\left| P^t_{a,\gamma} f_v\big(\Gamma(x,t)\big) \right| = \left| \int_{\mathbb{R}} e^{i(\Gamma(x,t)\frac{\eta}{v} + t|\frac{\eta}{v} - \frac{1}{v^2}|^a)} e^{-t^\gamma |\frac{\eta}{v} - \frac{1}{v^2}|^a} g_v(\eta) d\eta \right|.$$

Define

$$F_{x,t,v}(\eta) = \Gamma(x,t)\frac{\eta}{v} + t\Big|\frac{\eta}{v} - \frac{1}{v^2}\Big|^a - \frac{t}{v^{2a}},$$

$$G_{t,v}(\eta) = t^\gamma \Big|\frac{\eta}{v} - \frac{1}{v^2}\Big|^a.$$

Since $e^{-i\frac{t}{v^{2a}}}$ is unimodular and does not depend on η, so $e^{i(\Gamma(x,t)\frac{\eta}{v} + t\left|\frac{\eta}{v} - \frac{1}{v^2}\right|^a)}$ in above integral can be replaced with $e^{iF_{x,t,v}(\eta)}$. Noting that

$$\operatorname{supp} g_v \subset [-v^{(a-1)-\frac{a}{\gamma}}, v^{(a-1)-\frac{a}{\gamma}}],$$

it follows that

$$
\begin{aligned}
|P^t_{a,\gamma} f_v\big(\Gamma(x,t)\big)| &\geq |\mathrm{Re}\big(P^t_{a,\gamma} f_v\big(\Gamma(x,t)\big)\big)| \\
&\geq \bigg| \int_{-v^{(a-1)-\frac{a}{\gamma}}}^{v^{(a-1)-\frac{a}{\gamma}}} \cos(F_{x,t,v}(\eta)) e^{-G_{t,v}(\eta)} g_v(\eta) d\eta \bigg|.
\end{aligned}
\tag{2.3}
$$

Note that $(a-1)-\frac{a}{\gamma} > -1$. Choosing c_0 is a small positive constant, thus by Taylor' formula, for $|\eta| \leq c_0 v^{(a-1)-\frac{a}{\gamma}}$, we have

$$
\left|\frac{\eta}{v} - \frac{1}{v^2}\right|^a = \left(\frac{1}{v^2} - \frac{\eta}{v}\right)^a = \frac{1}{v^{2a}} - \frac{a\eta}{v^{2(a-1)+1}} + O\bigg(\frac{\eta^2}{v^{2(a-2)+2}}\bigg).
$$

Since $\Gamma(x,0) = x$, we get

$$
F_{x,t,v}(\eta) = \big(\Gamma(x,t) - \Gamma(x,0)\big)\frac{\eta}{v} + x\frac{\eta}{v} - ta\frac{\eta}{v^{2a-1}} + O\big(\frac{t\eta^2}{v^{2(a-1)}}\big).
$$

For a fixed $x \in [0, v^{\frac{2a}{\gamma}-2(a-1)}]$, we choose $t = \frac{xv^{2(a-1)}}{a} \in [0,1]$. Then

$$
F_{x,t,v}(\eta) = \big(\Gamma(x,t) - \Gamma(x,0)\big)\frac{\eta}{v} + O(x\eta^2).
$$

When Γ satisfies the conditions (A1) with $\frac{1}{a} \leq \sigma \leq 1$, that is $\big|\Gamma(x,t) - \Gamma(x,0)\big| \leq C_1 t^\sigma$ with $\frac{1}{a} \leq \sigma \leq 1$, it follows that

$$
|F_{x,t,v}(\eta)| \leq C_1 t^\sigma \frac{\eta}{v} + \frac{a(a-1)}{2} x\eta^2.
$$

Noting that $x \in [0, v^{\frac{2a}{\gamma}-2(a-1)}]$, $|\eta| \leq c_0 v^{(a-1)-\frac{a}{\gamma}}$, c_0 and v are small positive constants, $a\sigma \geq 1$, and $a \geq 2$, we have

$$
\begin{aligned}
|F_{x,t,v}(\eta)| &\leq C_1 c_0 (\frac{1}{a})^\sigma v^{[\frac{2a}{\gamma}-2(a-1)]\sigma} v^{2(a-1)\sigma} v^{(a-1)-\frac{a}{\gamma}-1} \\
&\quad + \frac{a(a-1)c_0}{2} v^{\frac{2a}{\gamma}-2(a-1)+2(a-1)-\frac{2a}{\gamma}} \\
&= C_1 c_0 (\frac{1}{a})^\sigma v^{\frac{2a\sigma-a}{\gamma}+a-2} + \frac{a(a-1)c_0}{2} v^0 \\
&\leq C_1 c_0 (\frac{1}{a})^\sigma v^{(a-2)(1-\frac{1}{\gamma})} + \frac{a(a-1)c_0}{2} \\
&\leq C_1 c_0 (\frac{1}{a})^\sigma + \frac{a(a-1)c_0}{2} \leq 1.
\end{aligned}
$$

Here, we used the fact c_0 is a small positive constant. By Taylor' formula, for $|\eta| \leq c_0 v^{(a-1)-\frac{a}{\gamma}}$, we have

$$
\big|\frac{\eta}{v} - \frac{1}{v^2}\big|^a = (\frac{1}{v^2} - \frac{\eta}{v})^a = \frac{1}{v^{2a}} + O\bigg(\frac{\eta}{v^{2(a-1)+1}}\bigg) = O(\frac{1}{v^{2a}}).
$$

It follows that

$$
G_{t,v}(\eta) = x^\gamma v^{2\gamma(a-1)} a^{-\gamma} O(\frac{1}{v^{2a}}) = O(x^\gamma v^{2a\gamma-2\gamma-2a}).
$$

Since $x \in [0, v^{\frac{2a}{\gamma}-2(a-1)}]$, it follows that

$$G_{t,v}(\eta) \leq C v^{2a-2\gamma(a-1)+2a\gamma-2\gamma-2a} \leq C.$$

Hence, for $|\eta| \leq c_0 v^{(a-1)-\frac{a}{\gamma}}$, $\cos(F_{x,t,v}(\eta)) \geq C$ and $e^{-G_{t,v}(\eta)} \geq C$. Thus, by (2.3), for $x \in [0, v^{\frac{2a}{\gamma}-2(a-1)}]$, we have

$$|P^t_{a,\gamma} f_v\big(\Gamma(x,t)\big)| \geq C v^{(a-1)-\frac{a}{\gamma}}.$$

Thus, we have

$$\|P^t_{a,\gamma} f_v\|^2_{L^2(\mathbb{R})} \geq C v^{\frac{2a}{\gamma}-2(a-1)} v^{2(a-1)-\frac{2a}{\gamma}} \geq C.$$

It follows that

$$\|P^*_{a,\gamma} f_v \Gamma(\cdot,t)\|_{L^2(\mathbb{R})} \geq C \tag{2.4}$$

uniformly v. Hence, by estimates (2.2) and (2.4), when $\gamma > 1$, $a \geq 2$ and Γ satisfies the conditions (A1) for $\frac{1}{a} \leq \sigma \leq 1$, $x \in \mathbb{R}$ and $t, t' \in [0,1]$, the global L^2 maximal estimate (2.1) does not hold for $s < \frac{a}{4}(1-\frac{1}{\gamma})$. Thus, we complete the proof of Theorem 2.1. □

Hence, to complete the proof of Theorem 1.1, we will prove that the following result.

Theorem 2.2. *Let $\gamma > 1$ and $a > 1$. Assume that Γ satisfies the condition* (A1) *for $\frac{1}{a} \leq \sigma \leq 1$ and* (A2) *for $x, y \in \mathbb{R}$ and $t, t' \in [0,1]$. Then*

$$\|P^*_{a,\gamma,\Gamma} f\|_{L^2(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})} \tag{2.5}$$

holds for $s > \frac{a}{4}(1-\frac{1}{\gamma})$.

To prove Theorem 2.2, we need an important lemma (i.e., Lemma 2.3 below), which plays a key role in proving Theorem 1.1. Lemma 2.3 is based on Lemma 2.1 in [1] and Lemma 2.1 in [5], and its proof given follows a similar strategy used to prove that Lemma 2.1 in [1] and Lemma 2.1 in [5]. The proof of Lemma 2.3 will be given in Section 3.

Lemma 2.3. *Let $a > 1$. Assume that Γ satisfies the conditions* (A1) *with $\frac{1}{a} \leq \sigma \leq 1$ and* (A2) *for $x, y \in \mathbb{R}$ and $t, t' \in (0,1)$. Assume $t(x) : \mathbb{R} \to [0,1]$ is a measurable function. Let $\gamma > 1$ and if $\gamma < \frac{a}{a-1}$, assume that $\alpha < \frac{1}{2}$. Let μ be a positive, even and $\mu \in C_0^\infty(\mathbb{R})$. If $\alpha > \frac{1}{2}a(1-\frac{1}{\gamma})$, then there exists $K \in L^1(\mathbb{R})$, such that*

$$\left|\int_{\mathbb{R}} e^{i[(\Gamma(y,t(y))-\Gamma(x,t(x)))\xi+(t(y)-t(x))|\xi|^a]}(1+|\xi|^2)^{-\frac{\alpha}{2}} e^{-(t(y)^\gamma+t(x)^\gamma)|\xi|^a} \mu\big(\frac{\xi}{N}\big) d\xi\right|$$
$$\leq CK(x-y)$$

for all $x, y \in \mathbb{R}$, and $N \in \mathbb{N}$, the constant $C > 0$ depends only on a, α, σ, γ and μ.

Proof of Theorem 2.2. Let $t(x) : \mathbb{R} \to [0,1]$ be a measurable function. Denote

$$T_{a,\gamma,\Gamma}f(x) = \int_{\mathbb{R}} e^{i\Gamma(x,t(x)\xi}e^{it(x)|\xi|^a}e^{-t(x)^\gamma|\xi|^a}\widehat{f}(\xi)d\xi, \qquad f \in \mathcal{S}(\mathbb{R}).$$

By linearizing the maximal operator, to prove (2.5) it suffices to prove that

$$\big\|T_{a,\gamma,\Gamma}f\big\|_{L^2(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})} \tag{2.6}$$

holds for $s > \frac{1}{4}a(1-\frac{1}{\gamma})$. Set

$$R_{a,\gamma,\Gamma}g(x) = \int_{\mathbb{R}} e^{i\Gamma(x,t(x))\cdot\xi}e^{it(x)|\xi|^a}e^{-t(x)^\gamma|\xi|^a}(1+|\xi|^2)^{-s/2}g(\xi)d\xi, \quad g \in \mathcal{S}(\mathbb{R}).$$

We first assume the estimate

$$\|R_{a,\gamma,\Gamma}g\|_{L^2(\mathbb{R})} \le C\|g\|_{L^2(\mathbb{R})} \tag{2.7}$$

holds and finish the proof of the estimate (2.6). Noticing that $T_{a,\gamma,\Gamma}f(x) = R_{a,\gamma,\Gamma}\big((1+|\cdot|^2)^{s/2}\hat{f}(\cdot)\big)(x)$, by (2.7) we get

$$\begin{aligned}\|T_{a,\gamma,\Gamma}g\|_{L^2(\mathbb{R})} &= \bigg(\int_{\mathbb{R}}\big|R_{a,\gamma,\Gamma}\big((1+|\cdot|^2)^{s/2}\hat{f}(\cdot)\big)(x)\big|^2dx\bigg)^{1/2}\\ &\le C\bigg(\int_{\mathbb{R}}(1+|\xi|^2)^s|\hat{f}(\xi)|^2d\xi\bigg)^{1/2}\\ &= C\|f\|_{H^s(\mathbb{R})}.\end{aligned}$$

Thus, to obtain (2.6) it remains to prove (2.7). Taking $\rho \in C_0^\infty(\mathbb{R})$ such that $\rho(x) = 1$ if $|x| \le 1$, and $\rho(x) = 0$ if $|x| \ge 2$. For $N > 2$, let

$$R_{a,\gamma,\Gamma,N}g(x) = \rho(\frac{x}{N})\int_{\mathbb{R}} e^{i\Gamma(x,t(x)\cdot\xi}e^{it(x)|\xi|^a}e^{-t(x)^\gamma|\xi|^a}\rho(\frac{\xi}{N})(1+|\xi|^2)^{-s/2}g(\xi)d\xi.$$

It is easy to see that the adjoint operator $R'_{a,\gamma,\Gamma,N}$ of $R_{a,\gamma,\Gamma,N}$ is given by

$$\begin{aligned}&R'_{a,\gamma,\Gamma,N}h(\xi)\\ &= (1+|\xi|^2)^{-s/2}\rho(\frac{\xi}{N})\int_{\mathbb{R}}\rho(\frac{x}{N})e^{-i\Gamma(x,t(x))\cdot\xi}e^{-it(x)|\xi|^a}e^{-t(x)^\gamma|\xi|^a}h(x)dx.\end{aligned}$$

By direct calculation, we have

$$\begin{aligned}&\|R'_{a,\gamma,\Gamma,N}h\|^2_{L^2(\mathbb{R})}\\ &= \int_{\mathbb{R}}\overline{\bigg((1+|\xi|^2)^{-s/2}\rho(\frac{\xi}{N})\int_{\mathbb{R}}\rho(\frac{x}{N})e^{-i\Gamma(x,t(x))\cdot\xi}e^{-it(x)|\xi|^a}e^{-t(x)^\gamma|\xi|^a}h(x)dx\bigg)}\\ &\quad\times\bigg(1+|\xi|^2)^{-s/2}\rho(\frac{\xi}{N})\int_{\mathbb{R}}\rho(\frac{y}{N})e^{-i\Gamma(y,t(y))\cdot\xi}e^{-it(y)|\xi|^a}e^{-t(y)^\gamma|\xi|^a}h(y)dy\bigg)d\xi\\ &=: \int_{\mathbb{R}}\int_{\mathbb{R}}K_N(x,y)h(x)h(y)dxdy,\end{aligned} \tag{2.8}$$

where

$$K_N(x,y) := \rho(\frac{x}{N})\rho(\frac{y}{N}) \int_{\mathbb{R}} e^{i[(\Gamma(y,t(y))-\Gamma(x,t(x)))\xi+(t(y)-t(x))|\xi|^a]}(1+|\xi|^2)^{-s} e^{-(t(y)^\gamma+t(x)^\gamma)|\xi|^a}\rho^2\big(\frac{\xi}{N}\big)d\xi.$$

Since $s > \frac{1}{4}a(1-\frac{1}{\gamma})$, γ and Γ satisfy the conditions in Lemma 2.3, thus by Lemma 2.3 we have

$$|K_N(x,y)| \le CK(x-y), \tag{2.9}$$

where $K \in L^1(\mathbb{R})$. Thus applying Hölder's inequality and Young's inequality, we have

$$\|R'_{a,\gamma,\Gamma,N}h\|^2_{L^2(\mathbb{R})} \le C\int_{\mathbb{R}}\int_{\mathbb{R}} k(x-y)h(x)h(y)dxdy \tag{2.10}$$

$$= C\int_{\mathbb{R}} k*h(x)h(x)dx \le C\|k*h\|_{L^2(\mathbb{R})}\|h\|_{L^2(\mathbb{R})} \tag{2.11}$$

$$\le C\|k\|_{L^1}\|h\|^2_{L^2(\mathbb{R})} \le C\|h\|^2_{L^2(\mathbb{R})}.$$

Obviously, here the constant C is independent of N. Hence, by duality, and letting $N\to\infty$, we obtain the asserted inequality (2.7).

Summing up above estimates, to complete the proof of Theorem 2.2, it remains to prove Lemma 2.3. □

3. The proof of Lemma 2.3

Let us begin with recalling a variant of van der Corput's lemma.

Lemma 3.1 (see [11, p. 309–312]). *Assume that $a < b$ and set $I = [a,b]$. Let $F \in C^\infty(I)$ be real-valued and assume that $\psi \in C^\infty(I)$.*

(i) *Assume that $|F'(x)| \ge \lambda > 0$ for $x \in I$ and that F' is monotonic on I. Then*

$$\Big|\int_a^b e^{iF(x)}\psi(x)dx\Big| \le C\frac{1}{\lambda}\{|\psi(b)| + \int_a^b |\psi'(x)|dx\},$$

where C does not depend on F, ψ or I.

(ii) *Assume that $|F''(x)| \ge \lambda > 0$ for $x \in I$. Then*

$$\Big|\int_a^b e^{iF(x)}\psi(x)dx\Big| \le C\frac{1}{\lambda^{1/2}}\{|\psi(b)| + \int_a^b |\psi'(x)|dx\},$$

where C does not depend on F, ψ or I.

We now return to the proof of Lemma 2.3. Define

$$I = \int_{\mathbb{R}} e^{i[(\Gamma(y,t(y))-\Gamma(x,t(x)))\xi+(t(y)-t(x))|\xi|^a]}(1+|\xi|^2)^{-\frac{\alpha}{2}} e^{-(t(y)^\gamma+t(x)^\gamma)|\xi|^a}\mu\big(\frac{\xi}{N}\big)d\xi.$$

Without loss of generality, we may assume $x \ne y$ and $t(y)-t(x) > 0$. Define

$$F(\xi) = (\Gamma(y,t(y)) - \Gamma(x,t(x)))\xi + (t(y)-t(x))|\xi|^a$$

and

$$\psi(\xi) = (1+|\xi|^2)^{-\frac{\alpha}{2}} e^{-(t(y)^\gamma + t(x)^\gamma)|\xi|^a} \mu(\frac{\xi}{N}).$$

We rewrite

$$I = \int_{\mathbb{R}} e^{iF(\xi)} \psi(\xi).$$

To prove Lemma 2.3 it suffices to prove that

$$|I| \le CK(x-y), \tag{3.1}$$

where $K \in L^1(\mathbb{R})$, $C > 0$ may depend on γ, α, a, σ, and μ, but not on x, y, $t(x)$, $t(y)$ or N. Let $M = \max\{2\delta^{1-a}, \frac{1}{C_4}\}$, where C_4 and δ are both small positive constants, such that

$$C_4 \le \min\Big\{\frac{C_2}{4C_1}, \frac{C_2}{8a}, \frac{1}{2}\Big\} \quad \text{and} \quad \delta < \frac{1}{2}(\frac{C_2}{4a})^{\frac{1}{a-1}}. \tag{3.2}$$

Now, we divide the verification of (3.1) into two cases $|x-y| \le M$ and $|x-y| > M$ according to the value of $|x-y|$.

3.1. Proof of (3.1) when $|x-y| \le M$

Rewrite

$$I = \int_{|\xi| \le |x-y|^{-1}} e^{iF(\xi)} \psi(\xi) d\xi + \int_{|\xi| \ge |x-y|^{-1}} e^{iF(\xi)} \psi(\xi) d\xi =: I_1 + I_2.$$

Thus, to get (3.1) it suffices to give the following estimates:

$$|I_1| \le CK(x-y) \tag{3.3}$$

and

$$|I_2| \le CK(x-y), \tag{3.4}$$

where $K \in L^1(\mathbb{R})$, the constant C is independent on x, y, $t(x)$, $t(y)$ and N. The estimate of (3.3) is simple. Note that $|x-y|^{-1} \ge \frac{1}{M}$ by $|x-y| \le M$, $\mu \in C_0^\infty(\mathbb{R})$ and $1-\alpha < 1$. We have

$$\begin{aligned}
|I_1| &\le \int_{|\xi| \le |x-y|^{-1}} (1+|\xi|^2)^{-\frac{\alpha}{2}} d\xi \\
&\le \int_{|\xi| \le \frac{1}{M}} (1+|\xi|^2)^{-\frac{\alpha}{2}} d\xi + \int_{\frac{1}{M} \le |\xi| \le |x-y|^{-1}} (1+|\xi|^2)^{-\frac{\alpha}{2}} d\xi \\
&\le \frac{2}{M} + \int_{\frac{1}{M} \le |\xi| \le |x-y|^{-1}} |\xi|^{-\alpha} d\xi \\
&\le C\big(1 + \frac{1}{|x-y|^{1-\alpha}}\big) \le CK(x-y),
\end{aligned}$$

where $K(x) = \big(1 + \frac{1}{|x|^{1-\alpha}}\big)\chi_{\{|x| \le M\}}$ and $K \in L^1(\mathbb{R})$. As for (3.4), it suffices to show that the following estimates:

$$\Big| \int_{|x-y|^{-1}}^{\infty} e^{iF(\xi)} \psi(\xi) d\xi \Big| \le CK(x-y) \tag{3.5}$$

and

$$\left|\int_{-\infty}^{-|x-y|^{-1}} e^{iF(\xi)}\psi(\xi)d\xi\right| \le CK(x-y), \tag{3.6}$$

where $K \in L^1(\mathbb{R})$, the constant C is independent on x, y, $t(x)$, $t(y)$ and N. By symmetry, it will suffice to verify the estimate (3.5). We choose $C_4 > 0$ such that

$$C_4 \le \min\Big\{\frac{C_2}{4C_1}, \frac{C_2}{8a}, \frac{1}{2}\Big\}. \tag{3.7}$$

Case (I-a): $\big(t(y)-t(x)\big)^\sigma \ge C_4|x-y|$. Note that $\xi \ge |x-y|^{-1} > 0$. We have

$$F'(\xi) = \Gamma(y,t(y)) - \Gamma(x,t(x)) + a(t(y)-t(x))\xi^{a-1}$$

and

$$F''(\xi) = a(a-1)(t(y)-t(x))\xi^{a-2}.$$

To verify (3.5), we need the following estimate:

$$\max_{\xi\ge|x-y|^{-1}} |\psi(\xi)| + \int_{|x-y|^{-1}}^{\infty} |\psi'(\xi)|d\xi \le C|x-y|^\alpha. \tag{3.8}$$

For each $\varepsilon > 0$, define function $h_\varepsilon(\xi) = e^{-\varepsilon|\xi|^a}$. For $\xi \ne 0$, we have

$$|h'_\varepsilon(\xi)| \le C\frac{1}{|\xi|} \tag{3.9}$$

and

$$|h''_\varepsilon(\xi)| \le C\frac{1}{|\xi|^2}. \tag{3.10}$$

In fact, since $h'_\varepsilon(\xi) = -sgn(\xi)\varepsilon a|\xi|^{a-1}e^{-\varepsilon|\xi|^a}$, it follows that

$$|h'_\varepsilon(\xi)| \le \frac{a}{|\xi|}\max_{y\in\mathbb{R}^+} ye^{-y} \le C\frac{1}{|\xi|},$$

$$|h''_\varepsilon(\xi)| \le \frac{1}{|\xi|^2}\max_{y\in\mathbb{R}^+} ye^{-y} + \frac{1}{|\xi|^2}\max_{y\in\mathbb{R}^+} y^2e^{-y} \le C\frac{1}{|\xi|^2}.$$

Thus, let $h_{\varepsilon_0}(\xi) = e^{-\varepsilon_0|\xi|^a}$, where $\varepsilon_0 = t(y)^\gamma + t(x)^\gamma$. It follows that $\psi(\xi) = (1+|\xi|^2)^{-\frac{\alpha}{2}}h_{\varepsilon_0}(\xi)\mu\big(\frac{\xi}{N}\big)$, $\mu \in C_0^\infty(\mathbb{R})$ and $\alpha > 0$. For $\xi \ge |x-y|^{-1}$, we obtain

$$|\psi(\xi)| \le (1+|\xi|^2)^{-\frac{\alpha}{2}} \le |x-y|^\alpha. \tag{3.11}$$

On the other hand, since

$$\begin{aligned}\psi'(\xi) = {} & 2\xi(-\frac{\alpha}{2})(1+\xi^2)^{-\frac{\alpha}{2}-1}h_{\varepsilon_0}(\xi)\mu\big(\frac{\xi}{N}\big) + (1+\xi^2)^{-\frac{\alpha}{2}}h'_{\varepsilon_0}(\xi)\mu\big(\frac{\xi}{N}\big) \\ & + (1+\xi^2)^{-\frac{\alpha}{2}}h_{\varepsilon_0}(\xi)\frac{1}{N}\mu'\big(\frac{\xi}{N}\big).\end{aligned}$$

By (3.9) and $\mu \in C_0^\infty(\mathbb{R})$, we get

$$|\psi'(\xi)| \le \alpha\xi(1+\xi^2)^{-\frac{\alpha}{2}-1}h_{\varepsilon_0}(\xi)\mu\big(\frac{\xi}{N}\big) + (1+\xi^2)^{-\frac{\alpha}{2}}|h'_{\varepsilon_0}(\xi)|\mu\big(\frac{\xi}{N}\big)$$

$$+ (1+\xi^2)^{-\frac{\alpha}{2}} h_{\varepsilon_0}(\xi) \frac{1}{N} |\mu'(\frac{\xi}{N})|$$
$$\le C\xi^{-\alpha-1} + \xi^{-\alpha-1} \frac{\xi}{N} |\mu'(\frac{\xi}{N})|$$
$$\le C\xi^{-\alpha-1}.$$

Hence, we have

$$\int_{|x-y|^{-1}}^{\infty} |\psi'(\xi)| d\xi \le \int_{|x-y|^{-1}}^{\infty} \xi^{-\alpha-1} d\xi \le C|x-y|^{\alpha}. \tag{3.12}$$

Thus, (3.8) follows from (3.11) and (3.12).

Now, we verify the estimate (3.5). We choose a positive constant C_5 such that

$$C_5 > \max\Big\{ \frac{1}{C_4} \Big(\frac{2(C_3 + C_1 C_4)}{aC_4} \Big)^{\frac{1}{a-1}}, 1 \Big\}.$$

We rewrite

(3.13)

$$\int_{|x-y|^{-1}}^{\infty} e^{iF(\xi)} \psi(\xi) d\xi = \int_{|x-y|^{-1}}^{C_5|x-y|^{-1}} e^{iF(\xi)} \psi(\xi) d\xi + \int_{C_5|x-y|^{-1}}^{\infty} e^{iF(\xi)} \psi(\xi) d\xi$$
$$=: L_1 + L_2.$$

The estimate of L_1 is simple. Note that $\mu \in C_0^{\infty}(\mathbb{R})$. It follows that

$$|L_1| \le \int_{|x-y|^{-1}}^{C_5|x-y|^{-1}} (1+|\xi|^2)^{-\frac{\alpha}{2}} d\xi \le C \frac{1}{|x-y|^{1-\alpha}}. \tag{3.14}$$

Next we give estimate of L_2. Since $\xi \ge C_4 C_5 \big(t(y)-t(x)\big)^{-\sigma}$ by $\xi \ge C_5|x-y|^{-1}$ and $\big(t(y)-t(x)\big)^{\sigma} \ge C_4|x-y|$. And note that $0 < t(y) - t(x) < 1$, $a\sigma \ge 1$ and

$$C_5 > \max\Big\{ \frac{1}{C_4} \Big(\frac{2(C_3 + C_1 C_4)}{aC_4} \Big)^{\frac{1}{a-1}}, 1 \Big\}.$$

Thus, for $\xi \in [C_5|x-y|^{-1}, \infty)$, we obtain

$$a\big(t(y)-t(x)\big)\xi^{a-1} \ge a\big(t(y)-t(x)\big)(C_4C_5)^{a-1}\big(t(y)-t(x)\big)^{-a\sigma+\sigma}$$
$$= a(C_4C_5)^{a-1}\big(t(y)-t(x)\big)^{1-a\sigma+\sigma}$$
$$\ge a(C_4C_5)^{a-1}\big(t(y)-t(x)\big)^{\sigma}$$
$$\ge 2\Big(\frac{C_3}{C_4} + C_1\Big)\big(t(y)-t(x)\big)^{\sigma}. \tag{3.15}$$

Since $|x-y| < \frac{1}{C_4}\big(t(y)-t(x)\big)^{\sigma}$ by $\big(t(y)-t(x)\big)^{\sigma} > C_4|x-y|$ and Γ satisfies the conditions (A1) for $\frac{1}{a} \le \sigma \le 1$ and (A2). We have

$$\big|\Gamma(y,t(y)) - \Gamma(x,t(x))\big| \le \big|\Gamma(y,t(y)) - \Gamma(x,t(y))\big| + \big|\Gamma(x,t(y)) - \Gamma(x,t(x))\big|$$
$$\le C_3|x-y| + C_1\big(t(y)-t(x)\big)^{\sigma}$$

$$\leq \Big(\frac{C_3}{C_4}+C_1\Big)\big(t(y)-t(x)\big)^{\sigma}. \tag{3.16}$$

Thus, by (3.15), (3.16) and $\big(t(y)-t(x)\big)^{\sigma} > C_4|x-y|$, we get

$$\begin{aligned}|F'(\xi)| &\geq a\big(t(y)-t(x)\big)\xi^{a-1} - \big|\Gamma(y,t(y))-\Gamma(x,t(x))\big|\\ &\geq \Big(\frac{C_3}{C_4}+C_1\Big)\big(t(y)-t(x)\big)^{\sigma}\\ &\geq \big(C_3+C_1C_4\big)|x-y|. \qquad (3.17)\end{aligned}$$

Observe that F' is monotonic on $[|x-y|^{-1},\infty)$. Applying (i) of Lemma 3.1 with (3.17) and (3.8), we obtain

$$|L_2| \leq C|x-y|^{-1}|x-y|^{\alpha} = C\frac{1}{|x-y|^{1-\alpha}}. \tag{3.18}$$

Hence, (3.5) holds from (3.13), (3.14) and (3.18).

Case (I-b): $\big(t(y)-t(x)\big)^{\sigma} < C_4|x-y|$. Let $\rho = \big(\frac{|x-y|}{t(y)-t(x)}\big)^{\frac{1}{a-1}}$. We choose $\delta, \lambda > 0$ such that $\delta < \frac{1}{2}(\frac{C_2}{4a})^{\frac{1}{a-1}}$ and

$$\lambda \geq \max\Big\{2\Big(\frac{2(C_3+C_1C_4)}{a}\Big)^{\frac{1}{a-1}}, 4(\frac{C_2}{4a})^{\frac{1}{a-1}}\Big\}.$$

Denote

$$\begin{aligned}B_1 &= \Big\{\xi \geq |x-y|^{-1}:\ \xi \leq \delta\rho\Big\},\\ B_2 &= \Big\{\xi \geq |x-y|^{-1}:\ \delta\rho \leq \xi \leq \lambda\rho\Big\},\\ B_3 &= \Big\{\xi \geq |x-y|^{-1}:\ \xi \geq \lambda\rho\Big\}.\end{aligned}$$

Hence, we may write

$$\begin{aligned}\int_{|x-y|^{-1}}^{\infty} e^{iF(\xi)}\psi(\xi)\big)d\xi &= \int_{B_1} e^{iF(\xi)}\psi(\xi)d\xi + \int_{B_2} e^{iF(\xi)}\psi(\xi)d\xi + \int_{B_3} e^{iF(\xi)}\psi(\xi)d\xi\\ &=: I_{2,1}+I_{2,2}+I_{2,3}. \qquad (3.19)\end{aligned}$$

We first consider $I_{2,1}$. For $\xi \in B_1$, we obtain

$$a\big(t(y)-t(x)\big)\xi^{a-1} \leq a\delta^{a-1}|x-y| \leq \frac{C_2|x-y|}{4}. \tag{3.20}$$

Because $C_4 \leq \frac{C_2}{4C_1}$, $\big(t(y)-t(x)\big)^{\sigma} < C_4|x-y|$ and Γ satisfies the conditions (A1) for $\frac{1}{a} \leq \sigma \leq 1$ and (A2). We have

$$\begin{aligned}\big|\Gamma(y,t(y))-\Gamma(x,t(x))\big| &\geq \big|\Gamma(y,t(y))-\Gamma(x,t(y))\big| - \big|\Gamma(x,t(y))-\Gamma(x,t(x))\big|\\ &\geq C_2|x-y| - C_1\big|t(y)-t(x)\big|^{\sigma}\\ &\geq C_2|x-y| - C_1C_4|x-y| \geq \frac{3C_2}{4}|x-y|. \qquad (3.21)\end{aligned}$$

By (3.20) and (3.21), we get

$$|F'(\xi)| \geq \left|\Gamma(y,t(y)) - \Gamma(x,t(x))\right| - a\big(t(y)-t(x)\big)\xi^{a-1} \geq \frac{C_2}{2}|x-y|. \tag{3.22}$$

Since that $\xi > 0$, and $t(y) - t(x) > 0$, it follows that F' is monotonic on $[|x-y|^{-1}, \infty)$. Thus, applying (i) of Lemma 3.1 with (3.22) and (3.8), we get

$$|I_{2,1}| \leq C|x-y|^{-1}|x-y|^{\alpha} = C\frac{1}{|x-y|^{1-\alpha}}. \tag{3.23}$$

For $I_{2,3}$, by $\xi \in B_3$ and $a > 1$, and $\lambda \geq \left(\frac{2(C_3+C_1C_4)}{a}\right)^{\frac{1}{a-1}}$, we have

$$a\big(t(y)-t(x)\big)\xi^{a-1} \geq a\lambda^{a-1}|x-y| \geq 2\Big(C_3 + C_1C_4\Big)|x-y|. \tag{3.24}$$

Note that $\big(t(y)-t(x)\big)^{\sigma} < C_4|x-y|$ and Γ satisfies the conditions (A1) for $\frac{1}{a} \leq \sigma \leq 1$ and (A2). We have

$$\begin{aligned}\left|\Gamma(y,t(y)) - \Gamma(x,t(x))\right| &\leq \left|\Gamma(y,t(y)) - \Gamma(x,t(y))\right| + \left|\Gamma(x,t(y)) - \Gamma(x,t(x))\right| \\ &\leq C_3|x-y| + C_1\big(t(y)-t(x)\big)^{\sigma} \\ &\leq \Big(C_3 + C_1C_4\Big)|x-y|. \end{aligned} \tag{3.25}$$

Thus, we get from (3.24) and (3.25)

$$|F'(\xi)| \geq a\big(t(y)-t(x)\big)\xi^{a-1} - \left|\Gamma(y,t(y)) - \Gamma(x,t(x))\right| \geq \Big(C_3 + C_1C_4\Big)|x-y|. \tag{3.26}$$

Since F' is monotonic on $[|x-y|^{-1}, \infty)$. Applying (i) of Lemma 3.1 with (3.26) and (3.8), we obtain

$$|I_{2,3}| \leq C|x-y|^{-1}|x-y|^{\alpha} = C\frac{1}{|x-y|^{1-\alpha}}. \tag{3.27}$$

Finally, we give estimate of $I_{2,2}$. One hand, $\big(t(y)-t(x)\big)^{a\sigma} < (C_4)^a|x-y|^a$. On the other hand, by $t(x), t(y) \in [0,1]$, $\frac{1}{a} \leq \sigma \leq 1$ and $a > 1$, we have

$$\big(t(y)-t(x)\big)^{a\sigma} \leq t(y) - t(x).$$

Assume that $t(y) - t(x) \geq (C_4)^a|x-y|^a$. Thus, for $\xi \in B_2$, by $F''(\xi) = a(a-1)\big(t(y)-t(x)\big)\xi^{a-2}$, we get

$$\begin{aligned}|F''(\xi)| &\geq C\big(t(y)-t(x)\big)\left(\frac{|x-y|}{t(y)-t(x)}\right)^{\frac{a-2}{a-1}} \\ &= C\big(t(y)-t(x)\big)^{\frac{1}{a-1}}|x-y|^{\frac{a-2}{a-1}} \\ &\geq C|x-y|^{\frac{a}{a-1}}|x-y|^{\frac{a-2}{a-1}} = C|x-y|^2. \end{aligned} \tag{3.28}$$

Thus, using (ii) of Lemma 3.1 with the estimates (3.28) and (3.8), we obtain

$$|I_{2,2}| \leq C(|x-y|^2)^{-\frac{1}{2}}|x-y|^{\alpha} = C\frac{1}{|x-y|^{1-\alpha}}.$$

If $t(y)-t(x)<(C_4)^a|x-y|^a$, then $|F''(\xi)|\geq a(a-1)\big(t(y)-t(x)\big)\xi^{a-2}$. Thus, for $\xi\in B_2$, we have

$$
\begin{aligned}
|F''(\xi)| &\geq a(a-1)\delta^{a-2}\big(t(y)-t(x)\big)\left(\frac{|x-y|}{t(y)-t(x)}\right)^{\frac{a-2}{a-1}} \\
&= C\big(t(y)-t(x)\big)^{\frac{1}{a-1}}|x-y|^{\frac{a-2}{a-1}}.
\end{aligned}
\tag{3.29}
$$

We first give the following estimate:

$$
\max_{\xi\in B_2}|\psi(\xi)|+\int_{B_2}|\psi'(\xi)|d\xi\leq C\left(\frac{|x-y|}{t(y)-t(x)}\right)^{\frac{-\alpha}{a-1}}. \tag{3.30}
$$

In fact, for $\xi\in B_2$, we obtain

$$
|\psi(\xi)|\leq(1+|\xi|^2)^{-\frac{\alpha}{2}}\leq C\left(\frac{|x-y|}{t(y)-t(x)}\right)^{\frac{-\alpha}{a-1}}. \tag{3.31}
$$

On the other hand, since

$$
|\psi'(\xi)|\leq C\rho^{-\alpha}|h'_{\varepsilon_0}(\xi)|+C\rho^{-\alpha-1}|h_{\varepsilon_0}(\delta\rho)|. \tag{3.32}
$$

Here, $h_{\varepsilon_0}(\xi)=e^{-\varepsilon_0|\xi|^a}$, where $\varepsilon_0=t(y)^\gamma+t(x)^\gamma$. Hence, we have

$$
\begin{aligned}
\int_{\xi\in B_2}|\psi'(\xi)|d\xi &\leq C\rho^{-\alpha}\int_{\delta\rho}^{\lambda\rho}|h'_\varepsilon(\xi)|d\xi+\int_{\delta\rho}^{\lambda\rho}\rho^{-\alpha-1}|h_\varepsilon(\delta\rho)|d\xi \\
&= -C\rho^{-\alpha}\int_{\delta\rho}^{\lambda\rho}h'_\varepsilon(\xi)d\xi+\int_{\delta\rho}^{\lambda\rho}\rho^{-\alpha-1}|h_\varepsilon(\delta\rho)|d\xi \\
&\leq C\rho^{-\alpha}e^{-\delta^a(t(y)^\gamma+t(x)^\gamma)\rho^a}\leq C\left(\frac{|x-y|}{t(y)-t(x)}\right)^{\frac{-\alpha}{a-1}}.
\end{aligned}
\tag{3.33}
$$

Thus, (3.30) follows from (3.31) and (3.33). Applying (ii) of Lemma 3.1 with (3.29) and (3.30), we obtain

$$
\begin{aligned}
|I_{2,2}| &\leq C\left(\big(t(y)-t(x)\big)^{\frac{1}{a-1}}|x-y|^{\frac{a-2}{a-1}}\right)^{-\frac{1}{2}}\left(\frac{|x-y|}{t(y)-t(x)}\right)^{\frac{-\alpha}{a-1}} \\
&= C\big(t(y)-t(x)\big)^{\frac{1}{a-1}(\alpha-\frac{1}{2})}|x-y|^{\frac{1}{a-1}(1-\frac{a}{2}-\alpha)}.
\end{aligned}
\tag{3.34}
$$

In case $\gamma\geq\frac{a}{a-1}$, it is necessary $\alpha>\frac{1}{2}$. Note that $t(y)-t(x)<(C_4)^a|x-y|^a$ and $\alpha>\frac{1}{2}$, $a>1$. Thus by (3.34), we get

$$
|I_{2,2}|\leq C|x-y|^{\frac{a}{a-1}(\alpha-\frac{1}{2})}|x-y|^{\frac{1}{a-1}(1-\frac{a}{2}-\alpha)}=C\frac{1}{|x-y|^{1-\alpha}}. \tag{3.35}
$$

In case $\gamma<\frac{a}{a-1}$. Note that the following estimate:

$$
\max_{\xi\in B_2}|\psi(\xi)|+\int_{B_2}|\psi'(\xi)|d\xi\leq C\rho^{-\alpha}e^{-\delta^a(t(y)^\gamma+t(x)^\gamma)\rho^a}. \tag{3.36}
$$

In fact, for $\xi \in B_2$, we obtain

$$|\psi(\xi)| \le (1+|\xi|^2)^{-\frac{\alpha}{2}} \le C\rho^{-\alpha} e^{-\delta^a (t(y)^\gamma + t(x)^\gamma)\rho^a}. \tag{3.37}$$

Thus, (3.36) follows from (3.37) and (3.33). Applying (ii) of Lemma 3.1 with (3.29) and (3.36), we obtain

$$\begin{aligned}|I_{2,2}| &\le C\Big(\big(t(y)-t(x)\big)^{\frac{1}{a-1}}|x-y|^{\frac{a-2}{a-1}}\Big)^{-\frac{1}{2}} \rho^{-\alpha} e^{-\delta^a (t(y)^\gamma + t(x)^\gamma)\rho^a} \\ &= C\big(t(y)-t(x)\big)^{\frac{1}{a-1}(\alpha-\frac{1}{2})}|x-y|^{\frac{1}{a-1}(1-\frac{a}{2}-\alpha)} \\ &\quad e^{-\delta^a (t(x)^\gamma + t(y)^\gamma)|x-y|^{\frac{a}{a-1}}\left(t(y)-t(x)\right)^{-\frac{a}{a-1}}}.\end{aligned}$$

Since

$$t(x)^\gamma + t(y)^\gamma \ge 2^{-\gamma}\big(t(x)+t(y)\big)^\gamma \ge 2^{-\gamma}\big(t(y)-t(x)\big)^\gamma.$$

Thus, we have

$$|I_{2,2}| \le C\big(t(y)-t(x)\big)^{\frac{1}{a-1}(\alpha-\frac{1}{2})}|x-y|^{\frac{1}{a-1}(1-\frac{a}{2}-\alpha)} e^{-\delta^a 2^{-\gamma}\left(t(y)-t(x)\right)^{\gamma-\frac{a}{a-1}}|x-y|^{\frac{a}{a-1}}}. \tag{3.38}$$

Because for any $y, \beta > 0$, the inequality $e^{-y} \le C_\beta y^{-\beta}$ holds, it follows that

$$\begin{aligned}|I_{2,2}| &\le C\big(t(y)-t(x)\big)^{\frac{1}{a-1}(\alpha-\frac{1}{2})}|x-y|^{\frac{1}{a-1}(1-\frac{a}{2}-\alpha)} \\ &\quad \big(t(y)-t(x)\big)^{-\beta(\gamma-\frac{a}{a-1})}|x-y|^{-\frac{\beta a}{a-1}} \\ &= C\frac{\big(t(y)-t(x)\big)^{\frac{1}{a-1}(\alpha-\frac{1}{2})}}{\big(t(y)-t(x)\big)^{\beta(\gamma-\frac{a}{a-1})}} \frac{1}{|x-y|^{\frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a)}}.\end{aligned}$$

We choose β such that $\frac{1}{a-1}(\alpha-\frac{1}{2}) = \beta(\gamma-\frac{a}{a-1})$, that is $\beta = \frac{\alpha-\frac{1}{2}}{(a-1)\gamma-a}$. Note that $\beta > 0$ by $\gamma < \frac{a}{a-1}$ and $\alpha < \frac{1}{2}$. Let

$$k = \frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a) = \frac{1}{a-1}\Big(\alpha+\frac{1}{2}(a-2)+\frac{a(\alpha-\frac{1}{2})}{(a-1)\gamma-a}\Big),$$

and rewrite

$$k = \frac{1}{a-1}\Big(\alpha\Big(\frac{(a-1)\gamma}{(a-1)\gamma-a}\Big)+\frac{1}{2}(a-2)-\frac{\frac{1}{2}a}{(a-1)\gamma-a}\Big).$$

Noting that $\alpha > \frac{1}{2}a(1-\frac{1}{\gamma})$ and $\frac{(a-1)\gamma}{(a-1)\gamma-a} < 0$ by $\gamma < \frac{a}{a-1}$, it follows that

$$k < \frac{1}{a-1}\Big(\frac{1}{2}a(1-\frac{1}{\gamma})\Big(\frac{(a-1)\gamma}{(a-1)\gamma-a}\Big)+\frac{1}{2}(a-2)-\frac{\frac{1}{2}a}{(a-1)\gamma-a}\Big) = 1.$$

Thus, there exist $k < 1$, such that

$$|I_{2,2}| \le C\frac{1}{|x-y|^k}.$$

Thus, the estimate (3.5) follows from estimates (3.19), (3.23), (3.27) and (3.35) for the case $|x-y| \le M$.

3.2. Proof of (3.1) when $|x-y|>M$

We recall that

$$\rho=\left(\frac{|x-y|}{t(y)-t(x)}\right)^{\frac{1}{a-1}} \quad \text{and} \quad C_4\le \min\Big\{\frac{C_2}{4C_1},\frac{C_2}{8a},\frac{1}{2}\Big\},$$

and

$$0<\delta\le\frac{1}{2}\Big(\frac{C_2}{4a}\Big)^{\frac{1}{a-1}} \quad \text{and} \quad \lambda\ge\max\Big\{2\Big(\frac{2(C_3+C_1C_4)}{a}\Big)^{\frac{1}{a-1}},4\Big(\frac{C_2}{4a}\Big)^{\frac{1}{a-1}}\Big\}.$$

When $|x-y|>M$, note that $\delta\rho>1$ by $M=\max\{2\delta^{1-a},\frac{1}{C_4}\}$. Define $\phi_0\in\mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}\phi_0\subset[-1,1]$ and $\phi_0(\xi)=1$ if $|\xi|\le\frac{1}{2}$. And define $\phi_2\in\mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}\phi_2\subset[\delta\rho,\lambda\rho]$ and $\phi_2(\xi)=1$ if $\xi\in[2\delta\rho,\frac{1}{2}\lambda\rho]$. Since the supports of ϕ_0 and ϕ_2 do not overlap. Define $\phi_1:=(1-\phi_2-\phi_0)\chi_{[\frac{1}{2},2\delta\rho]}$ and $\phi_3:=(1-\phi_2)\chi_{[\frac{1}{2}\lambda\rho,\infty)}$. Thus

$$\psi(\xi)=\psi(\xi)\phi_0(\xi)+\psi(\xi)\phi_1(\xi)+\psi(\xi)\phi_2(\xi)+\psi(\xi)\phi_3(\xi):=\sum_{j=0}^{3}\psi_j(\xi),$$

where $\psi_j(\xi)=\psi_j(\xi)\phi_j(\xi)$, $j=0,1,2,3$. Let I_j represent the support of ψ_j, so that

$$I_0=[-1,1],\qquad I_1=[\frac{1}{2},2\delta\rho],\qquad I_2=[\delta\rho,\lambda\rho],\qquad I_3=[\frac{1}{2}\lambda\rho,\infty).$$

To estimate $\int_{\mathbb{R}}=e^{iF(\xi)}\psi(\xi)d\xi$, by symmetry, it will suffice to estimate

$$L_j=\int_{I_j}e^{iF(\xi)}\psi_j(\xi)d\xi$$

for each $j=0,1,2,3$. Integrating by parts twice, we have

$$L_0\le\frac{1}{\big|\Gamma(y,t(y))-\Gamma(x,t(x))\big|^2}\int_{-1}^{1}\Big|\frac{d^2}{d\xi^2}(e^{i(t(y)-t(x))|\xi|^a}\psi_0(\xi))\Big|d\xi.$$

Since $|x-y|>\frac{1}{C_4}$ by $|x-y|>M$, $M=\max\{2\delta^{1-a},\frac{1}{C_4}\}$ and $C_4\le\frac{C_2}{4C_1}$. It follows that $\big(t(y)-t(x)\big)^\sigma<C_4|x-y|$. Noting that Γ satisfying the conditions (A1) for $\frac{1}{a}\le\sigma\le1$ and (A2), it follows that

$$\begin{aligned}\big|\Gamma(y,t(y))-\Gamma(x,t(x))\big|&\ge\big|\Gamma(y,t(y))-\Gamma(x,t(y))\big|-\big|\Gamma(x,t(y))-\Gamma(x,t(x))\big|\\&\ge C_2|x-y|-C_1\big(t(y)-t(x)\big)^\sigma\\&\ge C_2|x-y|-C_1C_4|x-y|\ge\frac{3C_2}{4}|x-y|.\end{aligned}\tag{3.39}$$

Observe that the following fact in [1]

$$\int_{-1}^{1}\Big|\frac{d^2}{d\xi^2}(e^{i(t(y)-t(x))|\xi|^a}\psi_0(\xi))\Big|d\xi\le C.$$

Thus, by (3.39), we get

$$L_0 \leq \frac{1}{|x-y|^2}.$$

By direct calculation,

$$F'(\xi) = \Gamma(y,t(y)) - \Gamma(x,t(x)) + a(t(y)-t(x))\xi^{a-1},$$
$$F''(\xi) = a(a-1)(t(y)-t(x))\xi^{a-2},$$

and

$$F'''(\xi) = a(a-1)(a-2)(t(y)-t(x))\xi^{a-3}.$$

For $j = 1,3$, we have

$$\begin{aligned}|L_j| &= \left|\int_{I_j} e^{iF(\xi)}\Big(-\frac{\psi_j''(\xi)}{(F'(\xi))^2} + \frac{2\psi_j'(\xi)F''(\xi)}{(F'(\xi))^3} + \frac{\psi_j(\xi)F'''(\xi)}{(F'(\xi))^3} - \frac{3\psi_j(\xi)(F''(\xi))^2}{(F'(\xi))^4}\Big)d\xi\right| \\ &\leq C\int_{I_j} \frac{1}{|F'(\xi)|^2}\Big(|\psi_j''(\xi)| + \frac{|F''(\xi)|}{|F'(\xi)|}|\psi_j'(\xi)| + \frac{|F'''(\xi)|}{|F'(\xi)|}|\psi_j(\xi)| + \frac{|F''(\xi)|^2}{|F'(\xi)|^2}|\psi_j(\xi)|\Big)d\xi.\end{aligned}$$

For $\xi \in I_1$, since $a > 1$, we obtain

$$a\big(t(y)-t(x)\big)\xi^{a-1} \leq a2^{a-1}\delta^{a-1}|x-y| \leq \frac{C_2|x-y|}{4}. \tag{3.40}$$

Because $C_4 \leq \frac{C_2}{4C_1}$, $\big(t(y)-t(x)\big)^\sigma < C_4|x-y|$ and Γ satisfies the conditions (A1) for $\frac{1}{a} \leq \sigma \leq 1$ and (A2). We have

$$\begin{aligned}\big|\Gamma(y,t(y)) - \Gamma(x,t(x))\big| &\geq \big|\Gamma(y,t(y)) - \Gamma(x,t(y))\big| - \big|\Gamma(x,t(y)) - \Gamma(x,t(x))\big| \\ &\geq C_2|x-y| - C_1\big(t(y)-t(x)\big)^\sigma \\ &\geq C_2|x-y| - C_1C_4|x-y| \geq \frac{3C_2}{4}|x-y|. \qquad (3.41)\end{aligned}$$

By (3.40) and (3.41), for $\xi \in I_1$, we get

$$|F'(\xi)| \geq \big|\Gamma(y,t(y)) - \Gamma(x,t(x))\big| - a\big(t(y)-t(x)\big)\xi^{a-1} \geq \frac{C_2}{2}|x-y|. \tag{3.42}$$

By (3.40) and (3.42), we have

$$|F'(\xi)| \geq 2a\big(t(y)-t(x)\big)\xi^{a-1}. \tag{3.43}$$

For $\xi \in I_3$, we obtain

$$a\big(t(y)-t(x)\big)\xi^{a-1} \geq a2^{1-a}\lambda^{a-1}|x-y|. \tag{3.44}$$

Since $\lambda \geq 2\Big(\frac{2(C_3+C_1C_4)}{a}\Big)^{\frac{1}{a-1}}$, we have

$$a\big(t(y)-t(x)\big)\xi^{a-1} \geq a2^{1-a}\lambda^{a-1}|x-y| \geq 2\Big(C_3 + C_1C_4\Big)|x-y|. \tag{3.45}$$

Note that $\big(t(y)-t(x)\big)^\sigma < C_4|x-y|$ and Γ satisfies the conditions (A1) for $\frac{1}{a} \leq \sigma \leq 1$ and (A2). We have

$$\big|\Gamma(y,t(y)) - \Gamma(x,t(x))\big| \leq \big|\Gamma(y,t(y)) - \Gamma(x,t(y))\big| + \big|\Gamma(x,t(y)) - \Gamma(x,t(x))\big|$$

$$\leq C_3|x-y| + C_1\big(t(y)-t(x)\big)^{\sigma}$$
$$\leq \big(C_3 + C_1C_4\big)|x-y|. \tag{3.46}$$

Thus, by (3.45) and (3.46), we get

$$|F'(\xi)| \geq a\big(t(y)-t(x)\big)\xi^{a-1} - \big|\Gamma(y,t(y)) - \Gamma(x,t(x))| \geq \big(C_3 + C_1C_4\big)|x-y|. \tag{3.47}$$

By (3.45) and (3.46), we have

$$\big|\Gamma(y,t(y)) - \Gamma(x,t(x))\big| \leq \frac{a}{2}\big(t(y)-t(x)\big)\xi^{a-1}. \tag{3.48}$$

Thus

$$|F'(\xi)| \geq a\big(t(y)-t(x)\big)\xi^{a-1} - \big|\Gamma(y,t(y)) - \Gamma(x,t(x)) \geq \frac{a}{2}\big(t(y)-t(x)\big)\xi^{a-1}. \tag{3.49}$$

Thus for $\xi \in I_1$ or $\xi \in I_3$, we have

$$\frac{|F''(\xi)|}{|F'(\xi)|} \leq C\xi^{-1} \quad \text{and} \quad \frac{|F'''(\xi)|}{|F'(\xi)|} \leq C\xi^{-2}.$$

For $j=1$ or $j=3$, we get

$$|L_j| \leq C\frac{1}{|x-y|^2}\int_{I_j}\bigg(|\psi_j''(\xi)| + |\xi|^{-1}|\psi_j'(\xi)| + 2|\xi|^{-2}|\psi_j(\xi)|\bigg)d\xi.$$

Note that

$$\psi_j(\xi) = (1+|\xi|^2)^{-\frac{\alpha}{2}}e^{-(t(y)^{\gamma}+t(x)^{\gamma})|\xi|^a}\mu\big(\frac{\xi}{N}\big)\phi_j(\xi).$$

Hence, for $j=1,2,3$, by estimate (3.9) and (3.10), we get

$$|\psi_j(\xi)| \leq C\frac{1}{|\xi|^{\alpha}}, \quad |\psi_j'(\xi)| \leq C\frac{1}{|\xi|^{\alpha+1}}, \text{ and } |\psi_j''(\xi)| \leq C\frac{1}{|\xi|^{\alpha+2}}.$$

Thus, for $j=1,3$, we get

$$|L_j| \leq C\frac{1}{|x-y|^2}\int_{I_j}\frac{1}{|\xi|^{\alpha}}d\xi \leq C\frac{1}{|x-y|^2}.$$

Finally, we give estimate of L_2. For $\xi \in I_2$, we have

$$\begin{aligned}|F''(\xi)| &\geq a(a-1)\delta^{a-2}\big(t(y)-t(x)\big)\bigg(\frac{|x-y|}{t(y)-t(x)}\bigg)^{\frac{a-2}{a-1}}\\ &= C\big(t(y)-t(x)\big)^{\frac{1}{a-1}}|x-y|^{\frac{a-2}{a-1}}.\end{aligned} \tag{3.50}$$

And the estimate

$$\max_{\xi\in I_2}|\psi_2(\xi)| + \int_{I_2}|\psi_2'(\xi)|d\xi \leq C\rho^{-\alpha}e^{-\delta^a(t(y)^{\gamma}+t(x)^{\gamma})\rho^a}. \tag{3.51}$$

Thus, similar to estimating (3.38), by (3.50) and (3.51), we have

$$\begin{aligned}|L_2| \leq\ & C\big(t(y)-t(x)\big)^{\frac{1}{a-1}(\alpha-\frac{1}{2})}|x-y|^{\frac{1}{a-1}(-\alpha-\frac{1}{2}(a-2))}\\ & e^{-\delta^a 2^{-\gamma}\big(t(y)-t(x)\big)^{\gamma-\frac{a}{a-1}}|x-y|^{\frac{a}{a-1}}}.\end{aligned}$$

When $\gamma = \frac{a}{a-1}$, since $\alpha > \frac{1}{2}$ and $\alpha + \frac{1}{2}(a-2) > 0$. We have

$$|L_2| \leq Ce^{-\delta^a 2^{-\gamma}|x-y|^{\frac{a}{a-1}}} \leq C\frac{1}{|x-y|^{\frac{a}{a-1}}}.$$

Note that for any $y, \beta > 0$, the inequality $e^{-y} \leq C_\beta y^{-\beta}$ holds. We have

$$|L_2| \leq C\frac{\big(t(y)-t(x)\big)^{\frac{1}{a-1}(\alpha-\frac{1}{2})}}{\big(t(y)-t(x)\big)^{\beta(\gamma-\frac{a}{a-1})}}\frac{1}{|x-y|^{\frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a)}}. \tag{3.52}$$

When $\gamma < \frac{a}{a-1}$, we rewrite (3.52) as

$$|L_2| \leq C\frac{\big(t(y)-t(x)\big)^{\beta(\frac{a}{a-1}-\gamma)}}{\big(t(y)-t(x)\big)^{\frac{1}{a-1}(\frac{1}{2}-\alpha)}}\frac{1}{|x-y|^{\frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a)}}. \tag{3.53}$$

Since $t(y)-t(x) < 1$, $\gamma < \frac{a}{a-1}$ and $\alpha < \frac{1}{2}$. Hence, we choose a positive constant β such that $\beta(\frac{a}{a-1}-\gamma) > \frac{1}{a-1}(\frac{1}{2}-\alpha)$ and $\frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a) > 1$. Thus, by (3.53), there exists $k > 1$, such that

$$|L_2| \leq C\frac{\big(t(y)-t(x)\big)^{\beta(\frac{a}{a-1}-\gamma)}}{\big(t(y)-t(x)\big)^{\frac{1}{a-1}(\frac{1}{2}-\alpha)}}\frac{1}{|x-y|^{\frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a)}} \leq C\frac{1}{|x-y|^k}.$$

When $\gamma > \frac{a}{a-1}$, we choose β such that $\frac{1}{a-1}(\alpha-\frac{1}{2}) = \beta(\gamma-\frac{a}{a-1})$, that is $\beta = \frac{\alpha-\frac{1}{2}}{(a-1)\gamma-a}$, noting that $\beta > 0$ by $\gamma > \frac{a}{a-1}$ and $\alpha > \frac{1}{2}$. Let

$$k = \frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a) = \frac{1}{a-1}\left(\alpha+\frac{1}{2}(a-2)+\frac{a(\alpha-\frac{1}{2})}{(a-1)\gamma-a}\right),$$

and rewrite

$$k = \frac{1}{a-1}\left(\alpha\left(\frac{(a-1)\gamma}{(a-1)\gamma-a}\right)+\frac{1}{2}(a-2)-\frac{\frac{1}{2}a}{(a-1)\gamma-a}\right).$$

Since $\alpha > \frac{1}{2}a(1-\frac{1}{\gamma})$ and $\frac{(a-1)\gamma}{(a-1)\gamma-a} > 0$ by $\gamma > \frac{a}{a-1}$, it follows that

$$k > \frac{1}{a-1}\left(\frac{1}{2}a(1-\frac{1}{\gamma})\left(\frac{(a-1)\gamma}{(a-1)\gamma-a}\right)+\frac{1}{2}(a-2)-\frac{\frac{1}{2}a}{(a-1)\gamma-a}\right) = 1.$$

Thus, there exists $k > 1$, such that

$$|I_{2,2}| \leq C\frac{1}{|x-y|^k}.$$

Summing up above all estimates, we complete the proof of estimate (3.5). Hence, we show (3.4) and complete the proof of Lemma 2.3.

4. Proof of Theorem 1.3.

Now, we prove Theorem 1.3 by considering the two cases $1 < \gamma < \frac{a}{a-1}$ and $\gamma \geq \frac{a}{a-1}$ according to the value of γ, separately.

Case (I): $1 < \gamma < \frac{a}{a-1}$. In this case, we will prove $s^{loc}_{a,\Gamma}(\gamma) = \frac{a}{4}(1 - \frac{1}{\gamma})$. Since

$$\|P^*_{a,\gamma,\Gamma} f\|_{L^2([-1,1])} \leq \|P^*_{a,\gamma,\Gamma} f\|_{L^2(\mathbb{R})},$$

the global bounds (1.9) from Theorem 1.1 imply some local bounds in (1.10), and it is thus necessarily the case that $s^{loc}_{a,\Gamma}(\gamma) \leq s_{a,\Gamma}(\gamma) = \frac{a}{4}(1 - \frac{1}{\gamma})$. On the other hand, we may obtain $s^{loc}_{a,\Gamma}(\gamma) \geq \frac{a}{4}(1 - \frac{1}{\gamma})$ for all $1 < \gamma < \frac{a}{a-1}$. In fact, we note that the counterexample given in Section 2 is also a counterexample for the local estimate (1.10) whenever the choices x are contained within $[-1, 1]$. Since x is chosen to be in $[0, v^{\frac{2a}{\gamma}-2(a-1)}]$ for some small ν, and note that $\frac{2a}{\gamma} - 2(a-1) \geq 0$ by $1 < \gamma < \frac{a}{a-1}$. Hence, $s^{loc}_{a,\Gamma}(\gamma) = \frac{a}{4}(1 - \frac{1}{\gamma})$ for all $1 < \gamma < \frac{a}{a-1}$.

Case (II): $\gamma \geq \frac{a}{a-1}$. In this case, we will prove $s^{loc}_{a,\Gamma}(\gamma) = \frac{1}{4}$. In the proof of Lemma 2.3, when $\gamma \geq \frac{a}{a-1}$, and x is small, that is $|x| \leq M$ $(M \geq 1)$, the only requirement on α with $\alpha \geq \frac{1}{2}$. Thus, for such γ, $s^{loc}_{a,\Gamma}(\gamma) \leq \frac{1}{4}$. On the other hand, $s^{loc}_{a,\Gamma}(\gamma) \geq \frac{1}{4}$ for all $\gamma \geq \frac{a}{a-1}$ can be deduced from the counterexample of Section 2 in case $\gamma = \frac{a}{a-1}$. In fact, note that $0 < t < 1$, so the function $G_{t,v}(\eta)$ from Section 2 is non-increasing in γ the counterexample for $\gamma = \frac{a}{a-1}$, which shows that $s^{loc}_{a,\Gamma}(\frac{a}{a-1}) \geq \frac{1}{4}$. Thus, for all $\gamma \geq \frac{a}{a-1}$, $s^{loc}_{a,\Gamma}(\gamma) \geq \frac{1}{4}$. Thus, $s^{loc}_{a,\Gamma}(\gamma) = \frac{1}{4}$ for all $\gamma \geq \frac{a}{a-1}$. Hence we complete the proof of Theorem 1.3.

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