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# THE METHOD OF LOWER AND UPPER SOLUTIONS FOR IMPULSIVE FRACTIONAL EVOLUTION EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we investigate the existence of mild solutions for a class of fractional impulsive evolution equation with periodic boundary condition by means of the method of upper and lower solutions and monotone iterative method. Using the theory of Kuratowski measure of noncompactness, a series of results about mild solutions are obtained. Finally, two examples are given to illustrate our results.

## 1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and they have been emerging as an important area of investigation in the last few decades; see [1, 3–5, 7, 8, 10, 12, 16, 22, 24, 31–33].

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical, population dynamics, ecology, chemical, biological systems, and engineering background. Therefore, it has been an object of intensive investigation in recent years, some basic results on impulsive differential equations have been obtained and applications to different areas have been considered by many authors, see [22, 24, 25, 27, 28, 30].

The monotone iterative technique in the presence of lower and upper solutions is an important method for seeking solutions of differential equations in abstract spaces. Recently, Chen and Li [9], Du and Lakshmikantham [11], Sun and Zhao [23] investigated the existence of minimal and maximal solutions to

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initial value problem of ordinary differential equation without impulse by using the method of upper and lower solutions and the monotone iterative technique.

In [19,20], Mu *et al.* use the monotone iterative technique to investigate the existence and uniqueness of mild solutions of the impulsive fractional evolution equations in an order Banach space $E$:

$$\begin{cases} D^{\alpha}u(t) + Au(t) = f(t, u(t)), \ t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \ k = 1, 2, \dots, m, \\ u(0) = x_0 \in E, \end{cases}$$

and the problem

$$\begin{cases} D^{\alpha}u(t) + Au(t) = f(t, u(t)), \ t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \ k = 1, 2, \dots, m, \\ u(0) + g(u) = x_0 \in E, \end{cases}$$

where $D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a $C_0$-semigroup $T(t)(t \geq 0)$. Furthermore, the theory of boundary value problems for nonlinear impulsive fractional evolution equations is still in the initial stages and many aspects of this theory need to be explored.

On the other hand, due to the periodic boundary problems for fractional differential equations serve as a class of important models to study the dynamics of processes that are subject to periodic changes in their initial state and final state. In [17], Li *et al.* use a monotone iterative method in the presence of lower and upper solutions to discuss the existence and uniqueness of mild solutions for the boundary value problem of impulsive evolution equation in an ordered Banach space $E$:

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), Fu(t), Gu(t)), \ t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \ k = 1, 2, \dots, m, \\ u(0) = u(\omega), \end{cases}$$

where $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a $C_0$-semigroup $T(t)(t \geq 0)$ in $E$. Under wide monotonicity conditions and the non-compactness measure condition of the nonlinearity $f$, authors obtain the existence of extremal mild solutions and a unique mild solution between lower and upper solutions requiring only that $-A$ generates a $C_0$-semigroup.

However, there are few results on the theory on periodic boundary problems for fractional evolution equations in infinite dimensional spaces. Since the unbounded operator is involved in the fractional evolution equations, it is obvious that periodic boundary problems for fractional evolution equations are much more difficult than the same problems for fractional differential equations. Furthermore, to the best of our knowledge, the theory of periodic boundary value problems for nonlinear impulsive fractional evolution equations is still in the initial stages and many aspects of this theory need to be explored, motivated by the above those works, in this paper, we use a monotone iterative method in the presence of lower and upper solutions to discuss the existence and uniqueness of mild solutions for the periodic boundary value problem (PBVP) of impulsive fractional evolution equations of Volterra type in an ordered Banach space $E$

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t) + Au(t) = f(t, u(t), Fu(t), Gu(t)), \ t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \ k = 1, 2, \ldots, m, \\ u(0) = u(\omega), \end{cases} \tag{1}$$

where ${}^cD_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$ with the lower limit zero, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a $C_0$-semigroup $T(t)(t \geq 0)$ in $E$; $f \in C(J \times E \times E \times E, E)$, $I_k \in C(E, E)$ is an impulsive function, $k = 1, 2, \ldots, m$; $J = [0, \omega], J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, the $\{t_k\}$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = \omega$, $m \in N$; $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$ respectively, the operators $Fu$ and $Gu$ are given by

$$Fu(t) = \int_0^t K(t, s)u(s)\mathrm{d}s, \ K \in C(D, R^+),$$
$$Gu(t) = \int_0^{\omega} H(t, s)u(s)\mathrm{d}s, \ H \in C(D_0, R^+),$$

$D = \{(t, s) \in R^2 : 0 \leq s \leq t \leq \omega\}$, $D_0 = \{(t, s) \in R^2 : 0 \leq t, s \leq \omega\}$.

The paper is organized as follows: In Section 2 we recall some basic known results and introduce some notations. In Section 3 we discuss the existence theorem for periodic boundary value problem (1). Two examples will be presented in Section 4 illustrating our results.

## 2. Preliminaries

Let $E$ be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{x \in E : x \geq 0\}$ is normal with normal constant $N$. Let $C(J, E)$ denote the Banach space of all continuous $E$-value functions on interval $J$ with the norm $\|u\|_C = \max_{t \in J} \|u(t)\|$. Evidently, $C(J, E)$ is also an ordered Banach space reduced by the convex cone $P' = \{u \in E \,|\, u(t) \geq 0, \ t \in J\}$, and $P'$ is also a normal cone.

Let $PC(J, E) = \{u : J \to E, u(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$. Evidently, $PC(J, E)$ is a Banach space with the norm $\|u\|_{PC} = \sup_{t \in J} \|u(t)\|$. $PC(J, E)$ is also an ordered Banach space with the partial order $\leq$ induced by the positive cone $K_{PC} = \{u \in PC(J, E) : u(t) \geq 0, t \in J\}$ which is also normal with the same normal constant $N$. We use $E_1$ to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. Denote $C^{\alpha}(J, E) = \{x \in C(J, E) : D^{\alpha}x$ exists and $D^{\alpha}x \in C(J, E)\}$. Obviously, $C^{\alpha}(J, E)$ is a Banach space whose norm is

$$\|x\| = \sup_{t \in J}\{\|x(t)\| + \|D^{\alpha}x(t)\|\}.$$

An abstract function $u \in PC(J, E) \cap C^{\alpha}(J, E) \cap C(J, E_1)$ is called a classical solution of (1) if $u(t)$ satisfies equalities (1).

For completeness we recall the definition of the Caputo derivative of fractional order.

**Definition 2.1.** The fractional integral of order $\gamma$ of a function $f:[0,\infty)\to\mathbb{R}$ is defined as

$$I_{0+}^{\gamma}f(t)=\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f(s)ds,\ t>0,\ \alpha>0,$$

provided the right side is point-wise defined on $(0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

**Definition 2.2.** The Riemann-Liouville derivative of order $\gamma$ with the lower limit zero for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$$D_{0+}^{\gamma}f(t)=\frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}ds,\ t>0,\ n-1<\gamma<n.$$

**Definition 2.3.** The Caputo fractional derivative of order $\gamma$ for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$${}^cD_{0+}^{\gamma}f(t)=D_{0+}^{\gamma}\Big[f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\Big],\ t>0,\ n-1<\gamma<n,$$

where $n=[\gamma]+1$ and $[\gamma]$ denotes the integer part of $\gamma$.

*Remark* 2.4. In the case $f(t)\in C^n[0,\infty)$, then

$$\begin{aligned}{}^cD_{0+}^{\gamma}f(t)&=\frac{1}{\Gamma(n-\gamma)}\int_0^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds\\&=I_{0+}^{n-\gamma}f^n(t),\ t>0,\ n-1<\gamma<n.\end{aligned}$$

*Remark* 2.5. If $u$ is an abstract function with values in $E$, then the integrals which appear in Definitions 2.2 and 2.3 are taken in Bochner's sense.

Now, we recall some properties of the measure of noncompactness will be used later. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For any $B\subset C(J,E)$ and $t\in J$, set $B(t)=\{u(t):u\in B\}\subset E$. If $B$ is bounded in $C(J,E)$, then $B(t)$ is bounded in $E$, and $\alpha(B(t))\le\alpha(B)$, for more detail see [2, 6].

**Lemma 2.6** ([18])**.** *Let $E$ be a Banach space, and let $D\subset E$ be bounded. Then there exists a countable set $D_0\subset D$ such that $\alpha(D)\le 2\alpha(D_0)$.*

**Lemma 2.7** ([14])**.** *Let $E$ be a Banach space, and let $D\subset C(J,E)$ is equicontinuous and bounded. Then $\alpha(D(t))$ is continuous on $J$, and*

$$\alpha(D)=\max_{t\in J}\alpha(D(t)).$$

**Lemma 2.8** ([15])**.** *Let $B=\{u_n\}\subset PC(J,E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on $J$, and*

$$\alpha\Big(\Big\{\int_J u_n(t)dt:n\in\mathbb{N}\Big\}\Big)\le 2\int_J\alpha(B(t))dt.$$

In order to prove the main results, we also need the following Lemma 2.9.

**Lemma 2.9.** *Assume that* $\alpha > 0$, $m \in C(J, R^+)$ *satisfies*

$$\begin{aligned} m(t) \leq M_1 \int_0^t (t-s)^{\alpha-1} m(s)ds + M_2 \int_0^t (t-s)^{\alpha-1} m(s)ds \\ + M_3 \int_0^\omega (t-s)^{\alpha-1} m(s)ds, \ t \in J, \end{aligned} \tag{2}$$

*where* $M_i \geq 0$ $(i = 1, 2, 3)$ *are constants. Then* $m(t) \equiv 0$ *for* $t \in J$ *provided the following condition hold*: (i) $\frac{(M_1+M_2+M_3)\omega^\alpha}{\alpha} < 1$.

*Proof.* Let us suppose that (i) holds. Then, from (2)

$$m(t) \leq (M_1 + M_2 + M_3) \int_0^\omega (t-s)^{\alpha-1} m(s)ds, \ t \in J.$$

If follows by integrating the above inequality that

$$\int_0^\omega m(s)ds \leq \frac{(M_1 + M_2 + M_3)\omega^\alpha}{\alpha} \int_0^\omega m(s)ds,$$

and by assumption (i), implies

$$\int_0^\omega m(s)ds = 0,$$

and so $m(t) \equiv 0$, $t \in J$. The proof of this Lemma is complete. □

**Lemma 2.10** ([13]). *Let* $P$ *be a normal cone of the Banach space* $E$ *and* $v_0$, $w_0 \in E$ *with* $v_0 \leq w_0$, *Suppose that* $Q : [v_0, w_0] \to E$ *is a nondecreasing strict set contraction operator such that* $v_0 \leq Qv_0$ *and* $Qw_0 \leq w_0$. *Then* $Q$ *has a minimal fixed point* $\underline{u}$ *and a maximal fixed point* $\overline{u}$ *in* $[v_0, w_0]$; *moreover,* $v_n \to \underline{u}$ *and* $w_n \to \overline{u}$, *where* $v_n = Qv_{n-1}$ *and* $w_n = Qw_{n-1} (n = 1, 2, \ldots)$ *which satisfy* $v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq \underline{u} \leq \overline{u} \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0$.

Let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a $C_0$-semigroup $T(t)(t \geq 0)$ in $E$. Then there exist constants $D > 0$ and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leq De^{\delta t}, \quad t \geq 0.$$

**Definition 2.11.** A $C_0$-semigroup $T(t)(t \geq 0)$ in $E$ is called to be positive, if order inequality $T(t)x \geq \theta$ holds for each $x \geq \theta, x \in E$ and $t \geq 0$.

*Remark* 2.12. It is easy to see that for any $C \geq 0$, $-(A + CI)$ also generates a $C_0$-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$ in $E$. And $S(t)(t \geq 0)$ is a positive $C_0$-semigroup if $T(t)(t \geq 0)$ is a positive $C_0$-semigroup (about the positive $C_0$-semigroup, see [21]).

Now, we give a mild solution for the initial value problem of impulsive fractional evolution equations, which can be found in [26].

**Lemma 2.13** ([26]). *Let $E$ be a Banach space, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a $C_0$-semigroup $T(t)(t \geq 0)$ in $E$. For any $f \in C(J \times E \times E \times E, E)$, $u_0 \in E$ and $I_k \in C(E, E), k = 1, 2, \ldots, m$, then the initial value problem*

$$(3) \qquad \begin{cases} {}^cD^{\alpha}_{0+}u(t) + Au(t) = f(t, u(t), Fu(t), Gu(t)), \ t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \ k = 1, 2, \ldots, m, \\ u(0) = u_0, \end{cases}$$

*has a unique mild solution $u \in PC(J, E) \cap C^{\alpha}(J, E) \cap C(J, E_1)$ given by*

$$(4) \quad u(t) = \begin{cases} \mathscr{T}_{\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha}(t-s)f(s, u(s), Fu(s), Gu(s))ds, \ t \in [0, t_1], \\ \mathscr{T}_{\alpha}(t)u_0 + \mathscr{T}_{\alpha}(t - t_1)I_1(u(t_1)) \\ + \int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha}(t-s)f(s, u(s), Fu(s), Gu(s))ds, \ t \in (t_1, t_2], \\ \vdots \\ \mathscr{T}_{\alpha}(t)u_0 + \sum\limits_{i=1}^{k} \mathscr{T}_{\alpha}(t - t_i)I_i(u(t_i)) \\ + \int_0^t (t-s)^{\alpha-1}\mathscr{S}_{\alpha}(t-s)f(s, u(s), Fu(s), Gu(s))ds, \ t \in (t_k, t_{k+1}], \end{cases}$$

*where*

$$(5) \qquad \begin{aligned} \mathscr{T}_{\alpha}(t) &= \int_0^{\infty} \theta_{\alpha}(\sigma)T(t^{\alpha}\sigma)d\sigma, \ \mathscr{S}_{\alpha}(t) = \alpha \int_0^{\infty} \sigma\theta_{\alpha}(\sigma)T(t^{\alpha}\sigma)d\sigma, \\ \theta_{\alpha}(\sigma) &= \frac{1}{\pi\alpha}\sum_{n=1}^{\infty}(-\sigma)^{n-1}\frac{\Gamma(n\alpha + 1)}{n!}\sin(n\pi\alpha), \ \sigma \in (0, \infty) \end{aligned}$$

*are the functions of Wright type defined on $(0, \infty)$ which satisfies*

$$\theta_{\alpha}(\sigma) \geq 0, \ \sigma \in (0, \infty), \ \int_0^{\infty} \theta_{\alpha}(\sigma)d\sigma = 1$$

*and*

$$\int_0^{\infty} \sigma^{v}\theta_{\alpha}(\sigma)d\sigma = \frac{\Gamma(1 + v)}{\Gamma(1 + \alpha v)}, \ v \in [0, 1].$$

*Clearly, if the semigroup $T(t)(t \geq 0)$ is positive, then the operators $\mathscr{T}_{\alpha}(t)$ and $\mathscr{S}_{\alpha}(t)$ are also positive for all $t \geq 0$.*

**Definition 2.14.** By a mild solution of the initial value problem (3) has a unique mild solution $u \in PC(J, E) \cap C^{\alpha}(J, E) \cap C(J, E_1)$ given by (4).

First, we give the following lemmas to be used in proving our main results, which can be found in [26].

**Lemma 2.15.** *The operators $\mathscr{T}_{\alpha}(t)$ and $\mathscr{S}_{\alpha}(t)(t \geq 0)$ have the following properties*:

(i) *For any fixed $t \geq 0$, $\mathscr{T}_{\alpha}(t)$ and $\mathscr{S}_{\alpha}(t)$ are linear and bounded operators, i.e., for any $u \in E$,*

$$\|\mathscr{T}_{\alpha}(t)u\| \leq M\|u\|, \ \|\mathscr{S}_{\alpha}(t)u\| \leq \frac{M}{\Gamma(\alpha)}\|u\|,$$

*where $M = \sup_{t \in J} \|T(t)\|$, which is a finite number.*

(ii) *For every* $u \in E$, $t \to \mathscr{T}_\alpha(t)u$ *and* $t \to \mathscr{S}_\alpha(t)u$ *are continuous functions from* $[0, \infty)$ *into* $E$.
(iii) *The operators* $\mathscr{T}_\alpha(t)$ *and* $\mathscr{S}_\alpha(t)$ *are strongly continuous for all* $t \geq 0$.
(iv) *If* $T(t)(t \geq 0)$ *is an equicontinuous semigroup,* $\mathscr{T}_\alpha(t)$ *and* $\mathscr{S}_\alpha(t)$ *are equicontinuous in* $E$ *for* $t > 0$.
(v) *For every* $t > 0$, $\mathscr{T}_\alpha(t)$ *and* $\mathscr{S}_\alpha(t)$ *are compact operators if* $T(t)$ *is compact.*

Suppose that here the bounded operator $B : E \to E$ exists given by

$$B = [I - \mathscr{T}_\alpha(\omega)]^{-1}. \tag{6}$$

We present sufficient conditions for the existence and boundedness of the operator $B$.

**Lemma 2.16** (see [29, Theorem 3.3 and Remark 3.4])**.** *The operator* $B$ *defined in* (6) *exists and is bounded, if one of the following three conditions holds*:

(i) $T(t)$ *is compact for each* $t > 0$ *and the homogeneous linear nonlocal problem*

$$\begin{cases} {}^cD^\alpha_{0^+}u(t) = Au(t), \ t \in J, \\ u(0) = u(\omega), \end{cases}$$

*has no non-trivial mild solutions.*

(ii) *If* $\|\mathscr{T}_\alpha(\omega)\| < 1$, *then the operator* $I - \mathscr{T}_\alpha(\omega)$ *is invertible and* $[I - \mathscr{T}_\alpha(\omega)]^{-1} \in L_b(E)$.

(iii) *If* $\|T(t)\| < 1$ *for* $t \in (0, \omega]$, *then* $\mathscr{T}_\alpha(n\omega) \to 0$ *as* $n \to \infty$ *and the operator* $I - \mathscr{T}_\alpha(\omega)$ *is invertible and* $[I - \mathscr{T}_\alpha(\omega)]^{-1} \in L_b(E)$, *where* $L_b(E)$ *denote the space of bounded linear operators from* $E$ *to* $E$.

**Lemma 2.17.** *Let* $T(t)(t \geq 0)$ *be a compact* $C_0$*-semigroup in* $E$ *generated by* $-A$. *Then the PBVP* (1) *has a unique mild solution* $u \in PC(J, E) \cap C^\alpha(J, E) \cap C(J, E_1)$ *given by*

$$u(t) = \begin{cases} \mathscr{T}_\alpha(t)B\Big[\int_0^\omega (\omega - s)^{\alpha-1}\mathscr{S}_\alpha(\omega - s)f(s, u(s), Fu(s), Gu(s))ds\Big] \\ +\int_0^t (t - s)^{\alpha-1}\mathscr{S}_\alpha(t - s)f(s, u(s), Fu(s), Gu(s))ds, \ t \in [0, t_1], \\ \mathscr{T}_\alpha(t)B\Big[\int_0^\omega (\omega - s)^{\alpha-1}\mathscr{S}_\alpha(\omega - s)f(s, u(s), Fu(s), Gu(s))ds \\ +\mathscr{T}_\alpha(\omega - t_1)I_1(u(t_1))\Big] + \mathscr{T}_\alpha(t - t_1)I_1(u(t_1)) \\ +\int_0^t (t - s)^{\alpha-1}\mathscr{S}_\alpha(t - s)f(s, u(s), Fu(s), Gu(s))ds, \ t \in (t_1, t_2], \\ \vdots \\ \mathscr{T}_\alpha(t)B\Big[\int_0^\omega (\omega - s)^{\alpha-1}\mathscr{S}_\alpha(\omega - s)f(s, u(s), Fu(s), Gu(s))ds \\ +\sum\limits_{i=1}^k \mathscr{T}_\alpha(\omega - t_i)I_i(u(t_i))\Big] + \sum\limits_{i=1}^k \mathscr{T}_\alpha(t - t_i)I_i(u(t_i)) \\ +\int_0^t (t - s)^{\alpha-1}\mathscr{S}_\alpha(t - s)f(s, u(s), Fu(s), Gu(s))ds, \ t \in (t_m, \omega], \end{cases} \tag{7}$$

*where* $\mathscr{T}_\alpha(t), \mathscr{S}_\alpha(t)(t > 0)$ *are given by* (5).

*Proof.* For any $u \in PC(J, E)$, by Definition 2.14 and Lemma 2.13, we know easily that the initial value problem of impulsive fractional evolution equation (3) has a unique mild solution $u \in PC(J, E)$ given by (4).

We show that the PBVP (1) has a unique mild solution $u \in PC(J,E)$ given by (7). If a function $u \in PC(J,E)$ defined by (4) is a solutions of the PBVP (1) and $u_0 = u(\omega)$, then

$$
\begin{aligned}
u_0 = &\int_0^\omega (\omega - s)^{\alpha-1}\mathscr{S}_\alpha(\omega - s) f(s, u(s), Fu(s), Gu(s))ds \\
&+ \sum_{i=1}^{k} \mathscr{T}_\alpha(\omega - t_i) I_i(u(t_i)),\ t \in J_k,\ k = 1, 2, \ldots, m.
\end{aligned} \tag{8}
$$

By (v) of Lemma 2.15, $\mathscr{T}_\alpha(\omega)$ is a compact operator. By the Fredholm alternative theorem, $[I - \mathscr{T}_\alpha(\omega)]^{-1}$ exists and is bounded. Since the periodic boundary value problem

$$
\begin{cases}
{}^cD_{0+}^\alpha u(t) + Au(t) = 0, & t \in J,\ t \neq t_k, \\
\Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \ldots, m, \\
u(0) = u(\omega),
\end{cases}
$$

has no non-trivial mild solution, the operator equation (8) has an unique solution

$$
\begin{aligned}
u_0 = B\Big[&\int_0^\omega (\omega - s)^{\alpha-1}\mathscr{S}_\alpha(\omega - s) f(s, u(s), Fu(s), Gu(s))ds \\
&+ \sum_{i=1}^{m} \mathscr{T}_\alpha(\omega - t_i) I_i(u(t_i))\Big].
\end{aligned}
$$

Then $u_0$ is the unique initial value of the problem (3) in $E$, which satisfies $u(0) = u_0 = u(\omega)$. It follows that the mild solution $u$ of the problem (3) corresponding to initial value

$$
\begin{aligned}
u(0) = u_0 = B\Big[&\int_0^\omega (\omega - s)^{\alpha-1}\mathscr{S}_\alpha(\omega - s) f(s, u(s), Fu(s), Gu(s))ds \\
&+ \sum_{i=1}^{m} \mathscr{T}_\alpha(\omega - t_i) I_i(u(t_i))\Big]
\end{aligned}
$$

is just the mild solution of the PBVP (1). Therefore, the conclusion of Lemma 2.17 holds. □

*Remark* 2.18. By Lemma 2.16, we can replace the assumption of $\{T(t)\}_{t\geq 0}$ being compact by $\|T(t)\| < 1$ for $t \in (0, \omega]$ or $\|\mathscr{T}_\alpha(\omega)\| \leq 1$ directly. It is obvious that we have the following the result.

**Corollary 2.19.** *Let $T(t)(t \geq 0)$ be a $C_0$-semigroup in $E$ generated by $-A$, and $\|T(t)\| < 1$ for $t \in (0, \omega]$. then the PBVP (1) has a unique mild solution $u \in PC(J,E) \cap C^\alpha(J,E) \cap C(J,E_1)$ given by (7).*

*Proof.* For any $u \in PC(J,E)$, by Definition 2.14 and Lemma 2.15, we know easily that the initial value problem of impulsive fractional evolution equation (3) has a unique mild solution $u \in PC(J,E)$ given by (4).

We show that the PBVP (1) has a unique mild solution $u \in PC(J,E)$ given by (7). If a function $u \in PC(J,E)$ defined by (4) is a solutions of the PBVP (1) and $u_0 = u(\omega)$, then the operator equation (8) hold. If $\|T(t)\| < 1$ for $t \in (0,\omega]$, i.e., $\|\mathscr{T}_\alpha(\omega)\| < 1$, by (ii) of Lemma 2.16, then the operator $I - \mathscr{T}_\alpha(\omega)$ is invertible and is bounded, the operator equation (8) has an unique solution

$$u_0 = B\Big[\int_0^\omega (\omega - s)^{\alpha-1}\mathscr{S}_\alpha(\omega - s)f(s,u(s),Fu(s),Gu(s))ds + \sum_{i=1}^m \mathscr{T}_\alpha(\omega - t_i)I_i(u(t_i))\Big].$$

Then $u_0$ is the unique initial value of the problem (3) in $E$, which satisfies $u(0) = u_0 = u(\omega)$. It follows that the mild solution $u$ of the problem (3) corresponding to initial value

$$u(0) = u_0 = B\Big[\int_0^\omega (\omega - s)^{\alpha-1}\mathscr{S}_\alpha(\omega - s)f(s,u(s),Fu(s),Gu(s))ds + \sum_{i=1}^m \mathscr{T}_\alpha(\omega - t_i)I_i(u(t_i))\Big]$$

is just the mild solution of the PBVP (1). Therefore, the conclusion of Corollary 2.19 holds. □

## 3. Main results

In this section, we will present some main results. Before stating and proving these results, we introduce notations which are used in this sequel.

For $v, w \in PC(J,E)$ with $v \le w$, we use $[v,w]$ to denote the order interval $\{u \in PC(J,E) : v \le u \le w\}$ in $PC(J,E)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in E : v(t) \le u(t) \le w(t), t \in J\}$ in $E$.

**Definition 3.1.** If a function $v_0 \in PC(J,E) \cap C^\alpha(J,E) \cap C(J,E_1)$ satisfies

$$\left\{\begin{array}{l} {}^cD_{0+}^\alpha v_0(t) + Av_0(t) \le f(t, v_0(t), Fv_0(t), Gv_0(t)),\ t \in J', \\ \Delta v_0|_{t=t_k} \le I_k(v_0(t_k)),\ k = 1,2,\dots,m, \\ v_0(0) \le v_0(\omega), \end{array}\right. \tag{9}$$

we call it a lower solution of the PBVP (1); if all the inequalities in (9) are reversed, we call it an upper solution of PBVP (1).

In the following we give some existence theorems of mild solutions of the PBVP (1).

**Theorem 3.2.** *Let $E$ be an ordered Banach space, whose positive cone $P$ is normal, $A : D(A) \subset E \to E$ be a closed linear operator, the positive $C_0$-semigroup $T(t)(t \ge 0)$ generated by $-A$ is compact in $E$, $f \in C(J \times E \times E \times E, E)$ and $I_k \in C(E,E)$, $k = 1,2,\dots,m$. Assume that the PBVP (1) has a lower solution $v_0 \in PC(J,E) \cap C^\alpha(J,E) \cap C(J,E_1)$ and an upper solution $w_0 \in$*

*$PC(J,E)\cap C^{\alpha}(J,E)\cap C(J,E_1)$ with $v_0\le w_0$. Suppose also that the following conditions are satisfied*:

(H1) *There exists a constant $C>0$ such that*

$$f(t,u_2,v_2,z_2)-f(t,u_1,v_1,z_1)\ge -C(u_2-u_1),$$

*for any $t\in J$, and $v_0(t)\le u_1\le u_2\le w_0(t)$, $Fv_0(t)\le v_1\le v_2\le Fw_0(t)$, $Gv_0(t)\le z_1\le z_2\le Gw_0(t)$.*

(H2) *The impulsive function $I_k(\cdot)$ satisfies*

$$I_k(u_1)\le I_k(u_2),\ \ k=1,2,\ldots,m,$$

*for any $t\in J$, and $v_0(t)\le u_1\le u_2\le w_0(t)$.*

*Then the PBVP* (1) *has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between $v_0$ and $w_0$.*

*Proof.* Let $C>\delta$, it is easy to see that $-(A+CI)$ generates an exponentially stable, positive $C_0$-semigroup $S(t)=e^{-Ct}T(t)(t\ge 0)$. Also, it is compact. Let $\Phi(t)=\int_0^{\infty}\theta_\alpha(\sigma)S(t^\alpha\sigma)d\sigma$, $\Psi(t)=\alpha\int_0^{\infty}\sigma\theta_\alpha(\sigma)S(t^\alpha\sigma)d\sigma$, by Remark 2.12 and Lemma 2.15, the operators $\Phi(t)$ and $\Psi(t)$ are also positive and compact for all $t\ge 0$. By Lemma 2.15, we have

$$\|\Phi(t)\|\le M,\ \ \|\Psi(t)\|\le \frac{M}{\Gamma(\alpha)},\ \ t\ge 0.$$

Let $J_0=[t_0,t_1]=[0,t_1]$, $J_k=(t_k,t_{k+1}]$, $k=1,2,\ldots,m$, we define the mapping $Q:[v_0,w_0]\to PC(J,E)$ by

(10)

$$Qu(t)=\begin{cases}\Phi(t)B\Big[\int_0^{\omega}(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,u(s),Fu(s),Gu(s))+Cu(s)]ds\Big]\\ +\int_0^{t}(t-s)^{\alpha-1}\Psi(t-s)[f(s,u(s),Fu(s),Gu(s))+Cu(s)]ds,\ t\in[0,t_1],\\ \Phi(t)B\Big[\int_0^{\omega}(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,u(s),Fu(s),Gu(s))+Cu(s)]ds\\ +\Phi(\omega-t_1)I_1(u(t_1))\Big]+\Phi(t-t_1)I_1(u(t_1))\\ +\int_0^{t}(t-s)^{\alpha-1}\Psi(t-s)[f(s,u(s),Fu(s),Gu(s))+Cu(s)]ds,\ t\in(t_1,t_2],\\ \vdots\\ \Phi(t)B\Big[\int_0^{\omega}(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,u(s),Fu(s),Gu(s))+Cu(s)]ds\\ +\sum\limits_{i=1}^{k}\Phi(\omega-t_i)I_i(u(t_i))\Big]+\sum\limits_{i=1}^{k}\Phi(t-t_i)I_i(u(t_i))\\ +\int_0^{t}(t-s)^{\alpha-1}\Psi(t-s)[f(s,u(s),Fu(s),Gu(s))+Cu(s)]ds,\ t\in(t_m,\omega],\end{cases}$$

Clearly, $Q:[v_0,w_0]\to PC(J,E)$ is continuous. By Lemma 2.17, the mild solution of the PBVP (1) is equivalent to the fixed point of the operator $Q$. Since $S(t)(t\ge 0)$ is a positive $C_0$-semigroup, combine this with the assumptions (H1) and (H2), $Q$ is increasing in $[v_0,w_0]$.

Now, we first show $v_0\le Qv_0$, $Qw_0\le w_0$. Let $h(t)={}^cD^{\alpha}_{0+}v_0(t)+Av_0(t)+Cv_0(t)$, by (9), $h\in PC(J,E)$ and $h(t)\le f(t,v_0(t),Fv_0(t),Gv_0(t))+Cv_0(t)$,

$t \in J$. By Lemma 2.9, the positivity of operator $\Phi(t)$ and $\Psi(t)$, for $t \in J_0$, we have

$$\begin{aligned} v_0(t) &= \Phi(t)v_0(0) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)h(s)ds \\ &\le \Phi(t)v_0(0) \\ &\quad + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)(f(s, v_0(s), Fv_0(s), Gv_0(s)) + Cv_0(s))ds. \end{aligned}$$

Especially, we have

$$\begin{aligned} v_0(\omega) &\le \Phi(\omega)v_0(0) \\ &\quad + \int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s)(f(s, v_0(s), Fv_0(s), Gv_0(s)) + Cv_0(s))ds. \end{aligned}$$

Combining this inequality with $v_0(0) = v_0(\omega)$, it follows that

$$\begin{aligned} v_0(0) \le [I-\Phi(\omega)]^{-1}\Big[\int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s)(f(s, v_0(s), Fv_0(s), Gv_0(s)) \\ + Cv_0(s))ds\Big]. \end{aligned}$$

For $t \in J_1$, we have

$$\begin{aligned} v_0(t) &= \Phi(t)v_0(0) + \Phi(t-t_1)I_1(u(t_1)) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)h(s)ds \\ &\le \Phi(t)v_0(0) + \Phi(t-t_1)I_1(u(t_1)) \\ &\quad + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)(f(s, v_0(s), Fv_0(s), Gv_0(s)) + Cv_0(s))ds. \end{aligned}$$

Especially, we have

$$\begin{aligned} v_0(\omega) &\le \Phi(\omega)v_0(0) + \Phi(\omega-t_1)I_1(u(t_1)) \\ &\quad + \int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s)(f(s, v_0(s), Fv_0(s), Gv_0(s)) + Cv_0(s))ds. \end{aligned}$$

Combining this inequality with $v_0(0) = v_0(\omega)$, it follows that

$$\begin{aligned} v_0(0) &\le [I-\Phi(\omega)]^{-1}\Big[\Phi(\omega-t_1)I_1(u(t_1)) \\ &\quad + \int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s)(f(s, v_0(s), Fv_0(s), Gv_0(s)) + Cv_0(s))ds\Big]. \end{aligned}$$

Continuing such a process interval by interval to $J_m$. On the other hand, from (10), we have

$$\begin{aligned} Q(v_0)(t) = \Phi(t)B\Big[\int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s, v_0(s), Fv_0(s), Gv_0(s)) \\ + Cv_0(s)]ds + \sum_{i=1}^{k}\Phi(\omega-t_i)I_i(v_0(t_i))\Big] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}\Phi(t-t_i)I_i(v_0(t_i))\\
&+\int_0^t (t-s)^{\alpha-1}\Psi(t-s)(f(s,v_0(s),Fv_0(s),Gv_0(s))\\
&\qquad + Cv_0(s))ds,\ t\in J.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&Q(v_0)(t)-v_0(t)\\
\geq{}& \Phi(t)\Big\{B\Big[\int_0^{\omega}(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,v_0(s),Fv_0(s),Gv_0(s))+Cv_0(s)]ds\\
&+\sum_{i=1}^{k}\Phi(\omega-t_i)I_i(v_0(t_i))\Big]-v_0(0)\Big\}\geq 0
\end{aligned}
$$

for all $t\in J$. It implies that $v_0\leq Qv_0$. Similarly, it can be show that $Qw_0\leq w_0$. So $Q:[v_0,w_0]\to[v_0,w_0]$ is a continuously increasing operator.

Next, we show that $Q:[v_0,w_0]\to[v_0,w_0]$ is completely continuous. Let

$$
\begin{aligned}
(Wu)(t)&=\int_0^t (t-s)^{\alpha-1}\Psi(t-s)(f(s,u(s),Fu(s),Gu(s))+Cu(s))ds,\\
(Vu)(t)&=\sum_{i=1}^{k}\Phi(t-t_i)I_i(u(t_i)),\ u\in[v_0,w_0].
\end{aligned}\tag{11}
$$

On the one hand, we prove that for any $0<t\leq\omega$, $Y(t)=\{(Wu)(t): u\in[v_0,w_0]\}$ is precompact in $E$. For $0<\epsilon<t$ and $u\in[v_0,w_0]$,

$$
\begin{aligned}
(W_\epsilon u)(t)={}&\int_0^{t-\epsilon}(t-s)^{\alpha-1}\Psi(t-s)(f(s,u(s),Fu(s),Gu(s))+Cu(s))ds\\
={}&\int_0^{t-\epsilon}(t-s)^{\alpha-1}S(\epsilon^\alpha\delta)\Big[\alpha\int_0^{t-\epsilon}\int_\delta^\infty \eta\theta_\alpha(\eta)S((t-s)^\alpha\eta-\epsilon^\alpha\delta)d\eta\\
&\times[f(s,u(s),Fu(s),Gu(s))+Cu(s)]ds\Big].
\end{aligned}\tag{12}
$$

For any $u\in[v_0,w_0]$, by assumption (H1), we have

$$
\begin{aligned}
f(t,v_0(t),Fv_0(t),Gv_0(t))+Cv_0(t)&\leq f(t,u(t),Fu(t),Gu(s))+Cu(t)\\
&\leq f(t,w_0(t),Fw_0(t),Gw_0(t))+Cw_0(t).
\end{aligned}
$$

By the normality of the cone $P$, there exists $\overline{M_1}>0$ such that

$$
\|f(t,u(t),Fu(t),Gu(t))+Cu(t)\|\leq\overline{M_1},\ u\in[v_0,w_0].
$$

By the compactness of $S(\epsilon)$, $Y_\epsilon(t)=\{(W_\epsilon u)(t): u\in[v_0,w_0]\}$ is precompact in $E$. Since

$$
\|(Wu)(t)-(W_\epsilon u)(t)\|\leq\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}\|\Psi(t-s)\|
$$

$$\cdot \|f(s,u(s),Fu(s),Gu(s))+Cu(s)\|ds$$
$$\leq \frac{M\overline{M_1}}{\Gamma(\alpha+1)}\epsilon^{\alpha},$$

the set $Y(t)$ is totally bounded in $E$. Furthermore, $Y(t)$ is precompact in $E$.

On the other hand, for any $0 \leq t_1 \leq t_2 \leq \omega$, we have

$$\begin{aligned}
&\|(Wu)(t_2)-(Wu)(t_1)\| \\
&= \Big\| \int_0^{t_1} (t_1-s)^{\alpha-1}\Big(\Psi(t_2-s)-\Psi(t_1-s)\Big) \\
&\quad [f(s,u(s),Fu(s),Gu(s))+Cu(s)]ds \\
&\quad + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\Psi(t_2-s)(f(s,u(s),Fu(s),Gu(s))+Cu(s))ds \\
&\quad + \int_0^{t_1} \Big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big)\Psi(t_2-s) \\
&\quad (f(s,u(s),Fu(s),Gu(s))+Cu(s))ds\Big\| \qquad (13) \\
&\leq \overline{M_1}\int_0^{t_1}|t_1-s|^{\alpha-1}\|\Psi(t_2-s)-\Psi(t_1-s)\|ds + \frac{M\overline{M_1}}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} \\
&\quad + \frac{M\overline{M_1}}{\Gamma(\alpha)}\int_0^{t_1}\Big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big)ds \\
&\leq \overline{M_1}\int_0^{\omega}\|\Psi(t_2-t_1+s)-\Psi(s)\|ds + \frac{M\overline{M_1}}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} \\
&\quad + \frac{M\overline{M_1}}{\Gamma(\alpha)}\int_0^{t_1}\Big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big)ds.
\end{aligned}$$

The right side of (13) depends on $t_2-t_1$, but is independent of $u$. As $S(\cdot)$ is compact, $\Psi(\cdot)$ is also compact and therefore $\Psi(t)$ is continuous in the uniform operator topology for $t>0$. So, the right side of (13) tends to zero as $t_2-t_1 \to 0$. Hence $W([v_0,w_0])$ is equicontinuous function of cluster in $Y$.

The same idea can be used to prove the compactness of $V$. For $0 \leq t \leq \omega$, since $\{Qu(t) : u \in [v_0,w_0]\} = \Big\{\Phi(t)B\Big[\int_0^{\omega}(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,v_0(s),Fv_0(s), Gv_0(s))+Cv_0(s)]ds+\sum_{i=1}^{k}\Phi(\omega-t_i)I_i(v_0(t_i))\Big]+(Wu)(t)+(Vu)(t) : u \in [v_0,w_0]\Big\}$, and $Qu(0) = B\Big[\int_0^{\omega}(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,v_0(s),Fv_0(s),Gv_0(s))+Cv_0(s)]ds+\sum_{i=1}^{k}\Phi(\omega-t_i)I_i(v_0(t_i))\Big] = u(\omega)$ is precompact in $E$. Hence, $Q([v_0,w_0])$ is precompact in $PC(J,E)$ by the Arzela-Ascoli theorem. So $Q:[v_0,w_0]\to[v_0,w_0]$ is completely continuous. Hence, $Q$ has minimal and maximal fixed points $\underline{u}$ and $\overline{u}$ in $[v_0,w_0]$, and therefore, they are the minimal and maximal mild solutions of the PBVP (1) in $[v_0,w_0]$, respectively. $\square$

*Remark* 3.3. By Lemma 2.16 and Corollary 2.19, we can replace the assumption of $\{T(t)\}_{t\geq 0}$ being compact by $\|T(t)\| < 1$ for $t \in (0, \omega]$ or $\|\Phi(\omega)\| < 1$ directly. It is obvious that we have the following the result.

**Theorem 3.4.** *Let $E$ be an ordered Banach space, whose positive cone $P$ is normal, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive $C_0$-semigroup $T(t)(t \geq 0)$ in $E$ and $\|T(t)\| < 1$ for $t \in (0, \omega]$, $f \in C(J \times E \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$. If the PBVP* (1) *has a lower solution $v_0 \in PC(J, E) \cap C^\alpha(J, E) \cap C(J, E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^\alpha(J, E) \cap C(J, E_1)$ with $v_0 \leq w_0$, conditions* (H1) *and* (H2) *hold, and satisfy*

(H3) *There exist a constant $L \geq 0$ such that for all $t \in J$,*

$$\alpha(\{f(t, u_n, v_n, z_n)\}) \leq L(\alpha(\{u_n\}) + \alpha(\{v_n\}) + \alpha(\{z_n\})),$$

*and increasing or decreasing sequences $\{u_n\} \subset [v_0(t), w_0(t)]$, $\{v_n\} \subset [Fv_0(t), Fw_0(t)]$ and $\{z_n\} \subset [Gv_0(t), Gw_0(t)]$.*

(H4) *The following inequality*

$$\frac{2[(L + C) + 2\omega L(K_0 + H_0)]}{\Gamma(\alpha + 1)}\omega^\alpha < 1,$$

*hold, where $K_0 = \max_{(t,s)\in D} K(t, s), H_0 = \max_{(t,s)\in D_0} H(t, s)$.*

(H5) *The sequences $v_n(0)$ and $w_n(0)$ are convergent, where $v_n = Q(v_{n-1}, w_{n-1}), w_n = Q(w_{n-1}, v_{n-1}), n = 1, 2, \ldots$.*

*Then the PBVP* (1) *has minimal and maximal mild solutions between $v_0$ and $w_0$, which can be obtained by a monotone iterative procedure starting from $v_0$ and $w_0$ respectively.*

*Proof.* Let $C > \delta$, it is easy to see that $-(A + CI)$ generates an exponentially stable, positive $C_0$-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$. Also, it is compact. Let $\Phi(t) = \int_0^\infty \theta_\alpha(\sigma)S(t^\alpha\sigma)d\sigma$, $\Psi(t) = \alpha \int_0^\infty \sigma\theta_\alpha(\sigma)S(t^\alpha\sigma)d\sigma$, by Remark 2.18 and Lemma 2.16, the operators $\Phi(t)$ and $\Psi(t)$ are also positive and compact for all $t \geq 0$. By Lemma 2.16 and $\|T(t)\| < 1$, we have that

$$\|\Phi(t)\| < 1, \ \|\Psi(t)\| < \frac{1}{\Gamma(\alpha)}, \ t \geq 0.$$

Let $J_0 = [t_0, t_1] = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we define the mapping $Q : [v_0, w_0] \to PC(J, E)$ by (10). Clearly, $Q : [v_0, w_0] \to PC(J, E)$ is continuous. By Corollary 2.19, the mild solution of the PBVP (1) is equivalent to the fixed point of the operator $Q$.

From Theorem 3.2, we know that $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator. Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Qv_{n-1}, \quad w_n = Qw_{n-1}, \quad n = 1, 2, \ldots. \tag{14}$$

Then from the monotonicity of $Q$, it follows that

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0. \tag{15}$$

We prove that $\{v_n\}$ and $\{w_n\}$ are convergent in $J$.

For convenience, we denote $B = \{v_n : n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} : n \in \mathbb{N}\}$. Then $B = Q(B_0)$. From $B_0 = B \bigcup \{v_0\}$ it follows that $\alpha(B_0(t)) = \alpha(B(t))$ for $t \in J$. Let $\varphi(t) := \alpha(B(t))$, $t \in J$, going from $J_0$ to $J_m$ interval by interval we show that $\varphi(t) \equiv 0$ in $J$.

For $t \in J$, there exists a $J_{k-1}$ such that $t \in J_{k-1}$. By Lemma 2.8, we have

$$\begin{aligned}
\alpha(F(B_0)(t)) &= \alpha\Big(\Big\{ \int_0^t K(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big) \\
&\le \sum_{j=1}^{k-1} \alpha\Big(\Big\{ \int_{t_{j-1}}^{t_j} K(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big) \\
&\quad + \alpha\Big(\Big\{ \int_{t_{k-1}}^{t} K(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big) \\
&\le 2K_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \alpha(B_0(s))ds + 2K_0 \int_{t_{k-1}}^{t} \alpha(B_0(s))ds \\
&= 2K_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \varphi(s)ds + 2K_0 \int_{t_{k-1}}^{t} \varphi(s)ds \\
&= 2K_0 \int_0^t \varphi(s)ds,
\end{aligned}$$

and therefore,

$$\int_0^t \alpha(F(B_0)(s))ds \le 2\omega K_0 \int_0^t \varphi(s)ds, \tag{16}$$

and

$$\begin{aligned}
\alpha(G(B_0)(t)) &= \alpha\Big(\Big\{ \int_0^\omega H(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big) \\
&\le \sum_{j=1}^{k-1} \alpha\Big(\Big\{ \int_{t_{j-1}}^{t_j} H(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big) \\
&\quad + \alpha\Big(\Big\{ \int_{t_{k-1}}^{\omega} H(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big) \\
&\le 2H_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \alpha(B_0(s))ds + 2H_0 \int_{t_{k-1}}^{\omega} \alpha(B_0(s))ds \\
&= 2H_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \varphi(s)ds + 2H_0 \int_{t_{k-1}}^{\omega} \varphi(s)ds
\end{aligned}$$

$$= 2H_0 \int_0^\omega \varphi(s)ds,$$

and therefore,

$$\int_0^\omega \alpha(G(B_0)(s))ds \leq 2\omega H_0 \int_0^\omega \varphi(s)ds. \tag{17}$$

For $t \in J_0$, by (9), Lemma 2.8 and the positivity of operator $\Phi(t), \Psi(t)$, and assumption (H3), we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(Q(B_0)(t)) \\
&= \alpha\Big(\Big\{\Phi(t)B\Big[\int_0^\omega (\omega - s)^{\alpha-1}\Psi(\omega - s) \\
&\quad\ [f(s, v_{n-1}(s), Fv_{n-1}(s), Gv_{n-1}(s)) + Cv_{n-1}(s)]ds\Big] \\
&\quad + \int_0^t (t-s)^{\alpha-1}\Psi(t-s) \\
&\quad\ (f(s, v_{n-1}(s), Fv_{n-1}(s), Gv_{n-1}(s)) + Cv_{n-1}(s))ds\Big\}\Big) \\
&\leq \alpha\Big(\Big\{\Phi(t)v_n(0)\Big\}\Big) + \frac{2}{\Gamma(\alpha)}\int_0^t \alpha(\{(t-s)^{\alpha-1} \\
&\quad\ (f(s, v_{n-1}(s), Fv_{n-1}(s), Gv_{n-1}(s)) + Cv_{n-1}(s))\})ds \\
&\leq \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad\ \Big(L(\alpha(B_0(s)) + \alpha(F(B_0)(s)) + \alpha(G(B_0)(s))) + C\alpha(B_0(s))\Big)ds \\
&\leq \frac{2}{\Gamma(\alpha)}(L+C)\int_0^t (t-s)^{\alpha-1}\varphi(s)ds + \frac{4}{\Gamma(\alpha)}\omega L K_0 \int_0^t (t-s)^{\alpha-1}\varphi(s)ds \\
&\quad + \frac{4}{\Gamma(\alpha)}\omega L H_0 \int_0^\omega (t-s)^{\alpha-1}\varphi(s)ds.
\end{aligned}$$

Hence by (H4) and Lemma 2.9, $\varphi(t) \equiv 0$ in $J_0$. In particular, $\alpha(B(t_1)) = \alpha(B_0(t_1)) = \varphi(t_1) = 0$, this implies that $B(t_1)$ and $B_0(t_1)$ are precompact in $E$. Thus $I_1(B_0(t_1))$ is precompact in $E$, and $\alpha(I_1(B_0(t_1))) = 0$.

Now, for $t \in J_1$, by the above argument for $t \in J_0$, we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(Q(B_0)(t)) \\
&= \alpha\Big(\Big\{\Phi(t)B\Big[\int_0^\omega (\omega - s)^{\alpha-1}\Psi(\omega - s) \\
&\quad\ [f(s, v_{n-1}(s), Fv_{n-1}(s), Gv_{n-1}(s)) + Cv_{n-1}(s)]ds \\
&\quad + \Phi(\omega - t_1)I_1(v_{n-1}(t_1))\Big] + \int_0^t (t-s)^{\alpha-1}\Psi(t-s) \\
&\quad\ (f(s, v_{n-1}(s), Fv_{n-1}(s), Gv_{n-1}(s)) + Cv_{n-1}(s))ds
\end{aligned}$$

$$
\begin{aligned}
&+\Phi(t-t_1)I_1(v_{n-1}(t_1))\Big\}\Big)\\
&\le \alpha\Big(\Big\{\Phi(t)v_n(0)\Big\}\Big)+\frac{2}{\Gamma(\alpha)}(L+C)\int_0^t (t-s)^{\alpha-1}\varphi(s)ds\\
&+\frac{4}{\Gamma(\alpha)}\omega LK_0\int_0^t (t-s)^{\alpha-1}\varphi(s)ds\\
&+\frac{4}{\Gamma(\alpha)}\omega LH_0\int_0^\omega (t-s)^{\alpha-1}\varphi(s)ds.
\end{aligned}
$$

Again by (H4) and Lemma 2.9, $\varphi(t)\equiv 0$ in $J_1$, from which we obtain that $\alpha(B_0(t_2))=0$ and $\alpha(I_2(B_0(t_2)))=0$.

Continuing such a process interval by interval up to $J_m$, we can prove that $\varphi(t)\equiv 0$ in every $J_k$, $k=0,1,2,\dots,m$. Hence, for any $t\in J, \{v_n(t)\}$ is precompact, and $\{v_n(t)\}$ has a convergent subsequence. Combing this with the monotonicity (15), we easily prove that $\{v_n(t)\}$ itself is convergent, i.e., $\lim_{n\to\infty} v_n(t)=\underline{u}(t)$, $t\in J$. Similarly, $\lim_{n\to\infty} w_n(t)=\overline{u}(t)$, $t\in J$.

Evidently $\{v_n(t)\}\in PC(J,E)$, so $\underline{u}(t)$ is bounded integrable in every $J_k$, $k=0,1,2,\dots,m$. Since for any $t\in J_k$, we have

$$
v_n(t)=\begin{cases}
\Phi(t)B\Big[\int_0^\omega(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,v_{n-1}(s),Fv_{n-1}(s),Gv_{n-1}(s))\\
+Cv_{n-1}(s)]ds\Big]\\
+\int_0^t(t-s)^{\alpha-1}\Psi(t-s)(f(s,v_{n-1}(s),Fv_{n-1}(s),Gv_{n-1}(s))\\
+Cv_{n-1}(s))ds,\ t\in[0,t_1],\\
\Phi(t)B\Big[\int_0^\omega(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,v_{n-1}(s),Fv_{n-1}(s),Gv_{n-1}(s))\\
+Cv_{n-1}(s)]ds\\
+\Phi(\omega-t_1)I_1(v_{n-1}(t_1))\Big]+\Phi(t-t_1)I_1(v_{n-1}(t_1))\\
+\int_0^t(t-s)^{\alpha-1}\Psi(t-s)(f(s,v_{n-1}(s),Fv_{n-1}(s),Gv_{n-1}(s))\\
+Cv_{n-1}(s))ds,\ t\in(t_1,t_2],\\
\vdots\\
\Phi(t)B\Big[\int_0^\omega(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,v_{n-1}(s),Fv_{n-1}(s),Gv_{n-1}(s))\\
+Cv_{n-1}(s)]ds\\
+\sum_{i=1}^{k}\Phi(\omega-t_i)I_i(u(t_i))\Big]+\sum_{i=1}^{k}\Phi(t-t_i)I_i(v_{n-1}(t_i))\\
+\int_0^t(t-s)^{\alpha-1}\Psi(t-s)(f(s,v_{n-1}(s),Fv_{n-1}(s),Gv_{n-1}(s))\\
+Cv_{n-1}(s))ds,\ t\in(t_m,\omega],
\end{cases}
$$

letting $n\to\infty$, by the Lebesgue dominated convergence theorem, for all $t\in J_k$, $k=0,1,2,\dots,m$, we get

$$
\underline{u}(t)=\Phi(t)B\Big[\int_0^\omega(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,\underline{u}(s),F\underline{u}(s),G\underline{u}(s))+C\underline{u}(s)]ds
$$

$$+\sum_{i=1}^{k}\Phi(\omega-t_i)I_i(\underline{u}(t_i))\Big]$$
$$+\int_0^t (t-s)^{\alpha-1}\Psi(t-s)(f(s,\underline{u}(s),F\underline{u}(s),G\underline{u}(s))+C\underline{u}(s))ds$$
$$+\sum_{i=1}^{m}\Phi(t-t_i)I_i(\underline{u}(t_i)),$$

and $\underline{u}(t)\in PC(J_k,E)$, $k=0,1,2,\ldots,m$. So, for $t\in J$, we have

$$\underline{u}(t)=\begin{cases}\Phi(t)B\Big[\int_0^\omega(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,\underline{u}(s),F\underline{u}(s),G\underline{u}(s))+C\underline{u}(s)]ds\Big]\\ +\int_0^t(t-s)^{\alpha-1}\Psi(t-s)(f(s,\underline{u}(s),F\underline{u}(s),G\underline{u}(s))+C\underline{u}(s))ds,\ t\in[0,t_1],\\ \Phi(t)B\Big[\int_0^\omega(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,\underline{u}(s),F\underline{u}(s),G\underline{u}(s))+C\underline{u}(s)]ds\\ +\Phi(\omega-t_1)I_1(\underline{u}(t_1))\Big]+\Phi(t-t_1)I_1(\underline{u}(t_1))\\ +\int_0^t(t-s)^{\alpha-1}\Psi(t-s)(f(s,\underline{u}(s),F\underline{u}(s),G\underline{u}(s))+C\underline{u}(s))ds,\ t\in(t_1,t_2],\\ \vdots\\ \Phi(t)B\Big[\int_0^\omega(\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,\underline{u}(s),F\underline{u}(s),G\underline{u}(s))+C\underline{u}(s)]ds\\ +\sum_{i=1}^{k}\Phi(\omega-t_i)I_i(\underline{u}(t_i))\Big]+\sum_{i=1}^{k}\Phi(t-t_i)I_i(\underline{u}(t_i))\\ +\int_0^t(t-s)^{\alpha-1}\Psi(t-s)(f(s,\underline{u}(s),F\underline{u}(s),G\underline{u}(s))+C\underline{u}(s))ds,\ t\in(t_m,\omega].\end{cases}$$

Therefore, $\underline{u}(t)\in PC(J,E)$, and $\underline{u}=Q\underline{u}$. Similarly, $\overline{u}(t)\in PC(J,E)$, and $\overline{u}=Q\overline{u}$. Combing this with monotonicity (15), we see that $v_0\le\underline{u}\le\overline{u}\le w_0$. By the monotonicity of $Q$, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of $Q$ in $[v_0,w_0]$. Therefore, $\underline{u}$ and $\overline{u}$ are the minimal and maximal mild solutions of the PBVP (1) in $[v_0,w_0]$, respectively. ☐

**Corollary 3.5.** *Let $E$ be an ordered Banach space, whose positive cone $P$ is regular, $A:D(A)\subset E\to E$ be a closed linear operator and $-A$ generates a positive $C_0$-semigroup $T(t)(t\ge 0)$ in $E$ and $\|T(t)\|<1$ for $t\in(0,\omega]$, $f\in C(J\times E\times E\times E,E)$ and $I_k\in C(E,E)$, $k=1,2,\ldots,m$. If the PBVP* (1) *has a lower solution $v_0\in PC(J,E)\cap C^\alpha(J,E)\cap C(J,E_1)$ and an upper solution $w_0\in PC(J,E)\cap C^\alpha(J,E)\cap C(J,E_1)$ with $v_0\le w_0$, and conditions* (H1), (H2), (H4) *and* (H5) *are satisfied, then the PBVP* (1) *has minimal and maximal mild solutions between $v_0$ and $w_0$, which can be obtained by a monotone iterative procedure starting from $v_0$ and $w_0$ respectively.*

*Proof.* Since $P$ is regular, any ordered monotonic and ordered bounded sequence in $E$ is convergent. For $t\in J$, let $\{x_n\}$ be an increasing or decreasing sequence in $[v_0(t),w_0(t)]$. By (H1), $\{f(t,x_n,y_n,z_n)+Cx_n\}$ is an ordered monotonic and ordered bounded sequence in $E$. Then, $\alpha(\{f(t,x_n,y_n,z_n)+Cx_n\})=\alpha(\{x_n\})=0$. By the properties of measure of noncompactness, we have

$$\alpha(\{f(t,x_n,y_n,z_n)\})\le\alpha(\{f(t,x_n,y_n,z_n)+Cx_n\})+C\alpha(\{x_n\})=0.$$

So, (H3) holds. Then, by Theorem 3.2, the proof is complete. □

**Theorem 3.6.** *Let $E$ be an ordered Banach space, whose positive cone $P$ is normal, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive and equicontinuous $C_0$-semigroup $T(t)(t \geq 0)$ in $E$ and $\|T(t)\| < 1$ for $t \in (0, \omega]$, $f \in C(J \times E \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$. If the PBVP* (1) *has a lower solution $v_0 \in PC(J, E) \cap C^{\alpha}(J, E) \cap C(J, E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^{\alpha}(J, E) \cap C(J, E_1)$ with $v_0 \leq w_0$, conditions* (H1), (H2) *hold, and satisfy*

(H6) *There exist a constant $0 < L_1 < \frac{\Gamma(\alpha+1)[1-4\sum_{k=1}^{m} M_k]}{4(1+2\omega K_0+2\omega H_0)\omega^{\alpha}}$ such that*

$$\alpha(\{f(t, u_n, v_n, z_n) + Cu_n\}) \leq L_1(\alpha(\{u_n\}) + \alpha(\{v_n\}) + \alpha(\{z_n\})),$$

*for all $t \in J$, and equicontinuous countable subsets $\{x_n\}, \{y_n\}, \{z_n\} \subset [v_0(t), w_0(t)]$.*

(H7) *There exists $M_k > 0, k = 1, 2, \ldots, m$ with $\sum_{k=1}^{m} M_k < \frac{1}{4}$ such that*

$$\alpha(\{I_k(x_n(t_k))\}) \leq M_k \alpha(x_n(t_k))$$

*for any equicontinuous countable subsets $\{x_n\} \subset [v_0, w_0]$.*

*Then the PBVP* (1) *has a minimal mild solutions $\underline{u}$ and a maximal mild solution $\overline{u}$ in $[v_0, w_0]$; moreover*

$$v_n(t) \to \underline{u}(t),\ w_n(t) \to \overline{u},\ (n \to \infty) \text{ uniformly for } t \in J,$$

*where $v_n(t) = Qv_{n-1}(t), w_n(t) = Qw_{n-1}(t)$ which satisfy*

$$\begin{aligned} v_0(t) &\leq v_1(t) \leq \cdots \leq v_n(t) \leq \cdots \\ &\leq \underline{u}(t) \leq \overline{u}(t) \leq \cdots \\ &\leq w_n(t) \leq \cdots \leq w_1(t) \leq w_0(t),\ t \in J. \end{aligned}$$

*Proof.* Let $C > \delta$, it is easy to see that $-(A + CI)$ generates an exponentially stable, positive $C_0$-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$. Also, it is compact. Let $\Phi(t) = \int_0^{\infty} \theta_{\alpha}(\sigma) S(t^{\alpha}\sigma) d\sigma$, $\Psi(t) = \alpha \int_0^{\infty} \sigma \theta_{\alpha}(\sigma) S(t^{\alpha}\sigma) d\sigma$, by Remark 2.18 and Lemma 2.16, the operators $\Phi(t)$ and $\Psi(t)$ are also positive and compact for all $t \geq 0$. By Lemma 2.8 and $\|T(t)\| < 1$, we have that

$$\|\Phi(t)\| < 1,\ \|\Psi(t)\| < \frac{1}{\Gamma(\alpha)},\ t \geq 0.$$

Let $J_0 = [t_0, t_1] = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we define the mapping $Q : [v_0, w_0] \to PC(J, E)$ by (10). Clearly, $Q : [v_0, w_0] \to PC(J, E)$ is continuous. By Corollary 2.19, the mild solution of the PBVP(1) is equivalent to the fixed point of the operator $Q$.

From the proof of Theorem 3.1, clearly, $Q : [v_0, w_0] \to [v_0, w_0]$ is continuous. Since $T(t)(t \geq 0)$ is a equicontinuous $C_0$-semigroup, $S(t)(t \geq 0)$ is also a equicontinuous $C_0$-semigroup, we also know that $Q : [v_0, w_0] \to [v_0, w_0]$ is a equicontinuous operator. For any $D \subset [v_0, w_0]$, $Q(D)$ is bounded and equicontinuous. So, by Lemma 2.6, there exists a countable set $D_0 = \{x_n\}$ such that

$$\alpha(Q(D)) \leq 2\alpha(Q(D_0)).$$

For $t \in J_0 = [0, t_1]$, by assumptions (H6), (H7) and Lemma 2.6, we have

$$\begin{aligned}
\alpha(Q(D_0(t))) &= \alpha\Big(\Big\{\Phi(t)B\Big[\int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,x_n(s),Fx_n(s),G_n(s)) \\
&\qquad + Cx_n(s)]ds\Big] + \int_0^t (t-s)^{\alpha-1}\Psi(t-s) \\
&\qquad (f(s,x_n(s),Fx_n(s),Gx_n(s)) + Cx_n(s))ds\Big\}\Big) \\
&\leq \alpha\Big(\Big\{\Phi(t)B\Big[\int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s) \\
&\qquad [f(s,x_n(s),Fx_n(s),Gx_n(s)) + Cx_n(s)]ds\Big]\Big) \\
&\quad + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(\Big\{(f(s,x_n(s),Fx_n(s),Gx_n(s)) \\
&\qquad + Cx_n(s))\Big\}\Big)ds \\
&\leq \frac{2M^*}{\Gamma(\alpha)}\int_0^\omega (\omega-s)^{\alpha-1}\Big(L_1\Big[\alpha(D_0(s)) + \alpha(F(D_0)(s)) \\
&\qquad + \alpha(G(D_0)(s))\Big]\Big)ds \\
&\quad + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(L_1\Big[\alpha(D_0(s)) + \alpha(F(D_0)(s)) \\
&\qquad + \alpha(G(D_0)(s))\Big]\Big)ds \\
&\leq \frac{2M^*L_1}{\Gamma(\alpha)}\int_0^\omega (\omega-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + \frac{4M^*}{\Gamma(\alpha)}\omega L_1 K_0 \int_0^\omega (\omega-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + \frac{4M^*}{\Gamma(\alpha)}\omega L_1 H_0 \int_0^\omega (\omega-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + \frac{2L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + \frac{4}{\Gamma(\alpha)}\omega L_1 K_0 \int_0^t (t-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + \frac{4}{\Gamma(\alpha)}\omega L_1 H_0 \int_0^\omega (t-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds
\end{aligned}$$

$$\leq (M^*+1)\frac{2L_1(1+2\omega K_0+2\omega H_0)\omega^\alpha}{\Gamma(\alpha+1)}\alpha(D),$$

where $M^* = \|B\|$. For $t \in J_k = (t_k, t_{k+1}], k = 1, 2, \ldots, m$, by assumptions (H6), (H7), (3.16), (3.17) and Lemma 2.6, we have

$$\begin{aligned}
\alpha(Q(D_0(t))) &= \alpha\Big(\Big\{\Phi(t)B\Big[\int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s)[f(s,x_n(s),Fx_n(s),Gx_n(s)) \\
&\qquad + Cx_n(s)]ds + \sum_{i=1}^{k}\Phi(\omega-t_i)I_i(x_n(t_i))\Big] \\
&\quad + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)(f(s,x_n(s),Fx_n(s),Gx_n(s)) \\
&\quad + Cx_n(s))ds + \sum_{i=1}^{m}\Phi(t-t_i)I_i(x_n(t_i))\Big\}\Big) \\
&\leq \frac{2M^*}{\Gamma(\alpha)}\int_0^\omega (\omega-s)^{\alpha-1}\alpha\Big(\Big\{f(s,x_n(s),Fx_n(s),Gx_n(s)) \\
&\qquad + Cx_n(s)\Big\}\Big)ds + 2M^*\sum_{i=1}^{m}M_i\alpha(I_i(x_n(t_i))) \\
&\quad + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(\Big\{f(s,x_n(s),Fx_n(s),Gx_n(s)) \\
&\qquad + Cx_n(s)\Big\}\Big)ds + 2\sum_{i=1}^{m}M_i\alpha(I_i(x_n(t_i))) \\
&\leq \frac{2M^*L_1}{\Gamma(\alpha)}\int_0^\omega (\omega-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + \frac{4M^*}{\Gamma(\alpha)}\omega L_1 K_0\int_0^\omega (\omega-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + \frac{4M^*}{\Gamma(\alpha)}\omega L_1 H_0\int_0^\omega (\omega-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + 2M^*\sum_{k=1}^{m}M_k\alpha(D_0(t_k)) \\
&\quad + \frac{2L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + \frac{4}{\Gamma(\alpha)}\omega L_1 K_0\int_0^t (t-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds \\
&\quad + \frac{4}{\Gamma(\alpha)}\omega L_1 H_0\int_0^\omega (t-s)^{\alpha-1}\alpha\Big(D_0(s)\Big)ds
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{k=1}^{m} M_k \alpha(D_0(t_k))\\
&\leq (M^*+1)\Big(\frac{2L_1(1+2\omega K_0+2\omega H_0)\omega^\alpha}{\Gamma(\alpha+1)}+2\sum_{k=1}^{m} M_k\Big)\alpha(D),
\end{aligned}
$$

where $M^* = \|B\|$. Hence for any $t \in J$, we have

$$
\alpha(Q(D_0(t))) \leq (M^*+1)\Big(\frac{2L_1(1+2\omega K_0+2\omega H_0)\omega^\alpha}{\Gamma(\alpha+1)}+2\sum_{k=1}^{m} M_k\Big)\alpha(D).
$$

Since $Q(D_0)$ is bounded and equicontinuous, by Lemma 2.8, we have

$$
\begin{aligned}
\alpha(Q(D)) &\leq 2\alpha(Q(D_0)) = 2\max_{t\in J}\alpha(Q(D_0(t)))\\
&\leq (M^*+1)\Big(\frac{4L_1(1+2\omega K_0+2\omega H_0)\omega^\alpha}{\Gamma(\alpha+1)}+4\sum_{k=1}^{m} M_k\Big)\alpha(D)\\
&\leq \gamma\alpha(D),
\end{aligned}
$$

where $\gamma = (M^*+1)\Big(\frac{4L_1(1+2\omega K_0+2\omega H_0)\omega^\alpha}{\Gamma(\alpha+1)}+4\sum_{k=1}^{m} M_k\Big)$.

By (H6) and (H7), we known that $\gamma < 1$. Therefore, the $Q : [v_0, w_0], \to [v_0, w_0]$ is a strict set contraction operator. Hence, our conclusion follows from Lemma 2.10. □

*Remark* 3.7. Analytic semigroup and differentiable semigroup are equicontinuous semigroup [21]. In applications of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroup is an analytic semigroup. So Theorem 3.3 has extensive applicability.

Now we discuss the uniqueness of the mild solution to the PBVP (1) in $[v_0, w_0]$. If we further assume that the following conditions hold:

(H1)* There exists a positive constant $C$ with $C < \frac{\Gamma(\alpha+1)}{2\omega^\alpha N(M^*+1)}$ such that

$$
f(t, u_2, v_2, z_2) - f(t, u_1, v_1, z_1) \geq -C(u_2 - u_1),
$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $Fv_0(t) \leq v_1 \leq v_2 \leq Fw_0(t)$, $Gv_0(t) \leq z_1 \leq z_2 \leq Gw_0(t)$.

(H8) There exist positive constants $\overline{C}$, $\overline{L}, \overline{N}$ with $\overline{C} + \omega\overline{L}K_0 + \omega\overline{N}H_0 < \frac{\Gamma(\alpha+1)}{2\omega^\alpha N(M^*+1)}$ such that

$$
f(t, u_2, v_2, z_2) - f(t, u_1, v_1, z_1) \leq \overline{C}(u_2 - u_1) + \overline{L}(v_2 - v_1) + \overline{N}(z_2 - z_1),
$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $Fv_0(t) \leq v_1 \leq v_2 \leq Fw_0(t)$, $Gv_0(t) \leq z_1 \leq z_2 \leq Gw_0(t)$.

(H9) There exist positive constants $\tau_k (k = 1, 2, \ldots, m)$ with
$$\sum_{k=1}^{m} \tau_k < \frac{\Gamma(\alpha+1) - N(M^*+1)(C + \overline{C} + \omega\overline{L}K_0 + \omega\overline{N}H_0)\omega^\alpha}{\Gamma(\alpha+1)N(M^*+1)}$$
such that
$$I_k(u_2) - I_k(u_1) \le \tau_k(u_2 - u_1),\ k = 1, 2, \ldots, m$$
for any $t \in J$, $v_0(t) \le u_1 \le u_2 \le w_0(t)$.

Then we have the following existence and uniqueness results in general ordered Banach space.

**Theorem 3.8.** *Let $E$ be an ordered Banach space, whose positive cone $P$ is normal, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive $C_0$-semigroup $T(t)(t \ge 0)$ in $E$ and $\|T(t)\| < 1$ for $t \in (0, \omega]$, $f \in C(J \times E \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$. If the PBVP* (1) *has a lower solution $v_0 \in PC(J, E) \cap C^\alpha(J, E) \cap C(J, E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^\alpha(J, E) \cap C(J, E_1)$ with $v_0 \le w_0$, such that conditions* (H1)*, (H2), (H8), (H9) *hold, then the PBVP* (1) *has a unique mild solution $u^*$ in $[v_0, w_0]$.*

*Proof.* From the proof of Theorem 3.2, when the conditions (H1)* and (H2) are satisfied, the iterative sequences $\{v_n\}$ and $\{w_n\}$ defined by (14) satisfy (15). Next, we show that there exists a unique $u^* \in PC(J, E)$ such that $u^* = Qu^*$. Since $\|T(t)\| < 1$, so $\|\Phi(t)\| < 1, \|\Psi(t)\| < \frac{1}{\Gamma(\alpha)}, t \in J$. For any $t \in J$, from (H8), (H9), (10), (14) and (15), we have
$$\begin{aligned}
\theta &\le w_n(t) - v_n(t) = Qw_{n-1}(t) - Qv_{n-1}(t) \\
&= \Phi(t)B\Big[\int_0^\omega (\omega - s)^{\alpha-1}\Psi(\omega - s)[f(s, w_n(s), Fw_n(s), Gw_n(s)) + Cw_n(s) \\
&\quad - f(s, v_n(s), Fv_n(s), Gv_n(s)) - Cv_n(s)]ds \\
&\quad + \sum_{0<t_k<t} \Phi(\omega - t_k)[I_k(w_n(t_k)) - I_k(v_n(t_k))]\Big] \\
&\quad + \sum_{0<t_k<t} \Phi(t - t_k)[I_k(w_n(t_k)) - I_k(v_n(t_k))] \\
&\quad + \int_0^t (t - s)^{\alpha-1}\Psi(t - s)[f(s, w_n(s), Fw_n(s), Gw_n(s)) + Cw_n(s) \\
&\quad - f(s, v_n(s), Fv_n(s), Gv_n(s)) - Cv_n(s)]ds \\
&\le \Phi(t)B\Big[(C + \overline{C} + \omega\overline{L}K_0 + \omega\overline{N}H_0) \\
&\quad \int_0^\omega (\omega - s)^{\alpha-1}\Psi(\omega - s)(w_{n-1}(s) - v_{n-1}(s))ds \\
&\quad + \sum_{0<t_k<t} \Phi(\omega - t_k)\tau_k(w_{n-1}(t_k) - v_{n-1}(t_k))\Big]
\end{aligned}$$

$$+ (C + \overline{C} + \omega\overline{L}K_0 + \omega\overline{N}H_0)$$
$$\int_0^t (\omega - s)^{\alpha-1}\Psi(\omega - s)(w_{n-1}(s) - v_{n-1}(s))ds$$
$$+ \sum_{0<t_k<t} \Phi(t - t_k)\tau_k(w_{n-1}(t_k) - v_{n-1}(t_k)).$$

By the normality of cone $P$ it follows that

$$\begin{aligned} &\|w_n(t) - v_n(t)\| \\ \le\ & NM^*\Big[(C + \overline{C} + \omega\overline{L}K_0 + \omega\overline{N}H_0)\frac{\omega^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^m \tau_k\Big]\|w_{n-1} - v_{n-1}\| \\ &+ N\Big[(C + \overline{C} + \omega\overline{L}K_0 + \omega\overline{N}H_0)\frac{\omega^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^m \tau_k\Big]\|w_{n-1} - v_{n-1}\|. \end{aligned}$$

Therefore

$$\begin{aligned} &\|w_n - v_n\| \\ \le\ & N(M^* + 1)\Big(\frac{(C + \overline{C} + \omega\overline{L}K_0 + \omega\overline{N}H_0)\omega^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^m \tau_k\Big)\|w_{n-1} - v_{n-1}\|. \end{aligned}$$

Repeat using the above inequality, we can obtain that

$$\begin{aligned} &\|w_n - v_n\| \\ \le\ & \Big[N(M^* + 1)\Big(\frac{(C + \overline{C} + \omega\overline{L}K_0 + \omega\overline{N}H_0)\omega^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^m \tau_k\Big)\Big]^n\|w_{n-1} - v_{n-1}\| \to 0 \end{aligned}$$

as $n \to \infty$. Then there exists a unique $u^* \in PC(J, E)$ such that $\lim_{n\to\infty} w_n = \lim_{n\to\infty} v_n = u^*$. Therefore, let $n \to \infty$ in (14), from the continuity of operator $Q$, we know that $u^* = Qu^*$, which means that $u^*$ is a unique mild solution of the problem PBVP(1). □

## 4. Examples

In this section, we give two examples to demonstrate how to utilize our results.

**Example 4.1.** We consider the impulsive fractional parabolic partial differential equation

$$\begin{cases} \partial_t^\alpha u(x,t) + A(x,D)u(x,t) \\ = f(x,t,u(x,t),Fu(x,t),Gu(x,t)),\ x \in \Omega,\ t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(x,t_k)),\ x \in \Omega,\ k = 1,2,\ldots,m, \\ u|_{\partial\Omega} = 0, \\ u(x,0) = u(x,\omega),\ x \in \Omega, \end{cases} \tag{18}$$

where $\partial_t^\alpha$ is the Caputo fractional partial derivative of order $0<\alpha<1$, $J=[0,\omega]$, $0<t_1<t_2<\cdots<t_m<\omega$, integer $N\geq 1$, let $\Omega\subset\mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$,

$$A(x,D)=-\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial y_j}+\sum_{i=1}^{N}a_i(x)\frac{\partial}{\partial x_i}+a_0(x)$$

is a strongly elliptic operator of second order, coefficient functions $a_{ij}(x),a_i(x)$ and $a_0(x)$ are Hölder continuous in $\Omega$, $f:\overline{\Omega}\times J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, $I_k:\mathbb{R}\to\mathbb{R}$ are also continuous, $k=1,2,\dots,m$.

Let $E=L^p(\Omega)$ with $p>N+2$, $P=\{u\in L^p(\Omega):u(x)\geq 0, a.e.\ x\in\Omega\}$, and define the operator $A$ as follows:

$$D(A)=\{u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega):u|_{\partial\Omega}=0\},\quad Au=A(x,D)u.$$

Then $E$ is a Banach space, $P$ is a regular cone of $E$, and $-A$ generates a positive and analytic $C_0$-semi-group $T(t)(t\geq 0)$ in $E$. So, the problem (18) can be transformed into the PBVP (1). For solving the problem (18), the following assumptions are needed.

(a) Let $f(x,t,0,0,0)\geq 0$, $I_k(0)\geq 0$, $u(x,\omega)\geq 0$, $x\in\Omega$, and there exists a function $w=w(x,t)\in PC(J,E)\cap C^\alpha(J,E)$, such that

$$\begin{cases}\partial_t^\alpha w+A(x,D)w\geq f(x,t,w,Fw,Gw),\ (x,t)\in\Omega\times J,\ t\neq t_k,\\ \Delta w|_{t=t_k}\geq I_k(w(x,t_k)),\ x\in\Omega,\ k=1,2,\dots,m,\\ u|_{\partial\Omega}=0,\\ w(x,0)\geq w(x,\omega),\ x\in\Omega.\end{cases}$$

(b) There exists a constant $M>0$ such that

$$f(x,t,x_2,y_2,z_2)-f(x,t,x_1,y_1,z_1)\geq -M(x_2-x_1),$$

for any $t\in J$, and $0\leq x_1\leq x_2\leq w(x,t)$, $0\leq y_1\leq y_2\leq Fw(x,t)$, $0\leq z_1\leq z_2\leq Gw(x,t)$.

(c) For any $u_1,u_2\in[0,w(x,t)]$ with $u_1\leq u_2$, we have

$$I_k(u_1(x,t_k))\leq I_k(u_2(x,t_k)),\quad x\in\Omega,\ k=1,2,\dots,m.$$

Assumption (a) implies that $v_0\equiv 0$ and $w_0\equiv w(x,t)$ are lower and upper solutions of the PBVP (18) respectively, and from (b) and (c), it is easy to verify that all conditions of Theorem 3.1 are satisfied, so the PBVP (18) has minimal and maximal mild solutions between 0 and $w(x,t)$, which can be obtained by a monotone iterative procedure starting from 0 and $w(x,t)$ respectively. □

**Example 4.2.** Consider the impulsive fractional differential equation of the form

$$(19)\quad \begin{cases}{}^cD_{0+}^\alpha u(t,y)+\frac{\partial^2}{\partial y^2}u(t,y)=f(t,y,u(t,y),Fu(t,y),Gu(t,y)),\ t\neq t_k,\\ \Delta u|_{t=t_k}=I_k(u(t_k,y)),\ y\in[0,\pi],\ k=1,2,\dots,m,\\ u(t,0)=u(t,\pi)=0,\ t\in[0,\omega]\\ u(0,y)=u(\omega,y),\ (t,y)\in[0,\omega]\times[0,\pi].\end{cases}$$

where ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, $(t,y) \in [0,\omega] \times [0,\pi]$.

Let $E = L^2([0,\pi])$. Define $Au = \frac{\partial^2}{\partial y^2}u$ for $u \in D(A)$, where

$$D(A) = \Big\{u \in E : \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2} \in E, u(0) = u(\pi) = 0\Big\}.$$

Then $-A$ generates a positive $C_0$-semigroup $T(t)(t \geq 0)$ in $E$, which is equicontinuous and $M = 1$.

Let $0 \leq w \in PC(J,E)$ satisfy the following conditions:

(i) $0 \leq I_k(w(t_k,y))$ and $I_k(w(t_k,y)) \leq \Delta w|_{t=t_k}$, $k = 1,2,\ldots,m,\ y \in [0,\pi]$;

(ii) $0 \leq w(\omega,y)$ and $w(\omega,y) \leq w(0,y),\ (t,y) \in [0,1] \times (0,\pi)$;

(iii) $Lw(t,y) \leq f(t,y,w(t,y),0,0)$ and $f(t,y,w(t,y),0,0) \leq^c D_{0+}^{\alpha}w(t,y) + (A - LI)w(t,y), (t,y) \in [0,\omega] \times [0,\pi], t \neq t_k$.

Then 0 and $w$ are lower and upper mild solutions of the problem (19). Therefore, if the functions $f$ and $I_k(k = 1,2,\ldots,m)$ satisfy the conditions (H1)-(H4) on the interval $[0,w]$, then the problem (19) has minimal and maximal mild solutions between 0 and $w$.

If the functions $f$ and $I_k(k = 1,2,\ldots,m)$ satisfy the conditions (H1), (H2), (H6) and (H7) on the interval $[0,w]$, then the problem (19) has at least mild solutions on $[0,w]$.

If the functions $f$ and $I_k(k = 1,2,\ldots,m)$ satisfy the conditions (H1)*, (H2), (H8) and (H9) on the interval $[0,w]$, then the problem (19) has a unique mild solution on$[0,w]$.

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