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ON A GROUP CLOSELY RELATED WITH THE AUTOMORPHIC LANGLANDS GROUP

Kâzim İlhan İkeda

Abstract. Let L_K denote the hypothetical automorphic Langlands group of a number field K. In our recent study, we briefly introduced a certain unconditional non-commutative topological group $\mathscr{W}\mathscr{A}_K^{\varphi}$, called the Weil-Arthur idèle group of K, which, assuming the existence of L_K, comes equipped with a natural topological group homomorphism $\mathsf{NR}_K^{\varphi^{\mathrm{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K$ that we called the "Langlands form" of the global non-abelian norm-residue symbol of K. In this work, we present a detailed construction of $\mathscr{W}\mathscr{A}_K^{\varphi}$ and $\mathsf{NR}_K^{\varphi^{\mathrm{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K$, and discuss their basic properties.

1. Introduction

This work is a detailed and extended account of the first part of our short communication presented in the Seoul ICM 2014, which corresponds to Section 8 of [12].

Let K be a number field and L_K the hypothetical automorphic Langlands group, also called the absolute Langlands group, of K[1]. As pointed out by Arthur [1], the existence of this universal group L_K attached to the number field K has been conjectured by Langlands [19] in order to represent the functor

$$\mathcal{G}\mathrm{rp}_K^{\mathrm{red}} \rightsquigarrow \mathcal{S}\mathrm{et}$$

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[1] We can more generally assume that K is a global field and develop the ideas presented in this work under this assumption. However, we shall refrain to do so and only assume that K is a number field in this work. Regarding the automorphic Langlands group of a function field, look at 1.6.2 of [4].

from the category $\mathcal{G}\mathrm{rp}_K^{\mathrm{red}}$ of connected (quasisplit) reductive groups over K to the category $\mathcal{S}\mathrm{et}$ of sets defined by

$$\mathrm{G} \rightsquigarrow \mathcal{C}_{\mathrm{aut}}(\mathrm{G}(\mathbb{A}_K)),$$

where for a connected (quasisplit) reductive group G over K, the set $\mathcal{C}_{\mathrm{aut}}(\mathrm{G}(\mathbb{A}_K))$ denotes the collection whose elements are the equivalence classes of automorphic families $c^S = \{c_\upsilon \mid \upsilon \notin S\}$ of semisimple $\widehat{\mathrm{G}}(\mathbb{C})$-conjugacy classes in the L-group ${}^L\mathrm{G}(\mathbb{C}) = \widehat{\mathrm{G}}(\mathbb{C}) \rtimes W_K$ of the reductive group G over K in Weil form, where $\widehat{\mathrm{G}}$ denotes the dual of G. In other words, it is conjectured that, there must be a topological group L_K such that, for each connected (quasisplit) reductive group G over K, a bijective correspondence

$$\mathrm{Hom}(L_K, {}^L\mathrm{G}(\mathbb{C})) \rightleftarrows \mathcal{C}_{\mathrm{aut}}(\mathrm{G}(\mathbb{A}_K))$$

should exist, and the global Langlands reciprocity principle over K, in the most general sense, states that, this bijection satisfies the "naturality" conditions. Currently, the conjecture on the existence of L_K stated in [19] is one of the most important and central open problems in the Langlands Program.

In this direction, Arthur [1] proposed a candidate L_K^{Arthur} for the hypothetical group L_K. However, Arthur's construction is conditional; namely, his construction lives in the "ideal Langlands universe", where the local Langlands reciprocity principle, the global Langlands functoriality principle, and all related conjectures are valid.

On the other hand, in an attempt to develop global non-abelian class field theory [12], we have introduced [12, Definition 8.1] a certain concrete non-commutative topological group $\mathscr{W}\mathscr{A}_K^{\varphi}$ and called it the *Weil-Arthur idèle group of the number field K*. Moreover, without proving, we stated [12, Theorem 8.4] that, if the hypothetical locally compact group L_K exists, then the topological group $\mathscr{W}\mathscr{A}_K^{\varphi}$ comes equipped with a natural topological group homomorphism

$$\mathsf{NR}_K^{\varphi^{\mathrm{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K,$$

unique up to "local L_K-conjugation", which we have called the *"Langlands form" of the global non-abelian*[2] *norm-residue symbol* (*or map*) *of K*. Furthermore, we have conjectured [12, Conjecture 8.5] that this topological group homomorphism is open and surjective.

The first aim of this work is to provide a detailed construction of the topological group $\mathscr{W}\mathscr{A}_K^{\varphi}$ and a proof of Theorem 8.4 of [12], which appears as *Theorem 4.1* in the current text. The second aim of the paper is to list the basic properties of the global non-abelian norm-residue symbols in Langlands

[2]Instead of saying "non-abelian global/local ..." as in [12, 13] and in [15, 16], we shall say "global/local non-abelian ..." from now on, which seems to be a better terminology. In fact, we have a general theory—the non-abelian class field theory—which in the global fields case is constructed in [12, 13] and, which in the local fields case is developed in [15, 16]. Moreover, the global theory and the local theories are compatible and their abelianizations are the global class field theory and the local class field theories respectively.

form following the same lines of the Lahore paper [13]. We, however, find it appropriate to discuss the compatibility of our theory with Arthur's topological group L_K^{Arthur} in a separate work.

The plan of the paper is as follows: The next two sections are the background material and respectively summarize the parts of [12,13], [15,16], and [1,5,18] that are used in our work. More precisely, in the next section; that is, in Section 2, following the ideas and methods of [12] closely, we shall present a detailed construction of the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}_K^{\varphi}$ of the number field K. In order to do so, we shall briefly recall restricted free topological products of topological groups introduced in [12,13] and the local non-abelian class field theory in the sense of Koch [15,16] (also look at [21]). Along the way, we shall discuss the relationship between the topological groups $\mathscr{W}\mathscr{A}_K^{\varphi}$ and $\mathscr{J}_K^{\varphi}$ both attached to the number field K, where $\mathscr{J}_K^{\varphi}$ denotes the non-abelian idèle group of K studied in [12,13] in detail. Next, in Section 3, we shall review the formal properties of the hypothetical automorphic Langlands group L_K of K following [1,5,18]. Finally, in Sections 4 and 5, we shall first construct the global non-abelian norm-residue symbol

$$\mathsf{NR}_K^{\varphi^{\text{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K,$$

of the number field K in Langlands form in Theorem 4.1, and propose a surjectivity conjecture in Conjecture 4.5, and then list the basic properties of this topological homomorphism.

2. The Weil-Arthur idèle group $\mathscr{W}\mathscr{A}_K^{\varphi}$ of a number field K

In this section, we shall construct the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}_K^{\varphi}$ of a given number field K. In order to do so, we shall first review restricted free topological products of topological groups, and then recall the local non-abelian class field theory in the sense of Koch. Finally, we shall introduce and study the basic properties of the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}_K^{\varphi}$ of the number field K.

2.1. Restricted free topological products of topological groups

The idea of restricted free topological products of topological groups and their applications to number theory first appears in the works of Miyake [22] and Neukirch et al. [24], where the topological groups under consideration are profinite. In this subsection, we follow closely [12, Section 2] and [13, Subsection 2.1], and use this occasion to clarify the parts in the works [12,13] related to the direct limits of topological groups.

Let $\{G_i\}_{i\in I}$ be a collection of topological groups. Let $*_{i\in I}\, G_i$ denote the free topological product of the collection $\{G_i\}_{i\in I}$, which exists in the category of topological groups by Graev [11], also look at [23]. The group topology on $*_{i\in I}\, G_i$ introduced in [11] is called the Graev topology of $*_{i\in I}\, G_i$ in this work. The topological group $*_{i\in I} G_i$ comes equipped with a natural continuous embedding

$$\iota_{i_o} : G_{i_o} \hookrightarrow \mathop{*}_{i\in I} G_i$$

for each $i_o \in I$, and satisfies the following universal mapping property: Let H denote a topological group. If for every $i_o \in I$, $\phi_{i_o} : G_{i_o} \to H$ is a continuous homomorphism, then there exists a unique continuous homomorphism $\phi : *_{i\in I} G_i \to H$ such that $\phi \circ \iota_{i_o} = \phi_{i_o}$ for every $i_o \in I$. That is, the following triangle

$$\begin{tikzcd}
 & *_{i\in I} G_i \arrow[dd, "\exists!\phi"] \\
G_{i_o} \arrow[ur, "\iota_{i_o}"] \arrow[dr, "\phi_{i_o}"'] & \\
 & H
\end{tikzcd}$$

is commutative for every $i_o \in I$.

Assume that the index set I is countable. If, for each $i_o \in I$, the underlying topological space of G_{i_o} is furthermore a k_ω-space[3] [9], then by Ordman [25, Theorem 3.2], the free topological product $*_{i\in I} G_i$ is a k_ω-topological group as well, and furthermore $\iota_{i_o}(G_{i_o}) = G_{i_o}$ is a closed subgroup of $*_{i\in I} G_i$ for every $i_o \in I$. Fix now a finite subset I_∞ of the index set I. For each $i_o \in I - I_\infty$, fix furthermore an open subgroup O_{i_o} of G_{i_o}. As closed subspaces of a k_ω-space are also k_ω by [9, Fact 14], the open subgroup O_{i_o} is a k_ω-topological group for every $i_o \in I - I_\infty$. Now, for every finite subset S of I satisfying $I_\infty \subseteq S$, introduce a topological group $\mathscr{G}_S$ by the free topological product

$$\mathscr{G}_S := \left(\mathop{*}_{i\notin S} O_i\right) * \left(\mathop{*}_{i\in S} G_i\right)$$

of the topological groups O_i for $i \in I - S$ and of G_i for $i \in S$, equipped with the canonical continuous embeddings

$$\iota_{i_o}^{(S)} : \left\{ \begin{array}{ll} O_{i_o} & (i_o \notin S) \\ G_{i_o} & (i_o \in S) \end{array} \right\} \hookrightarrow \mathscr{G}_S,$$

defined for all $i_o \in I$. Observe that, by Ordman [25, Theorem 3.2], the topological group $\mathscr{G}_S$ is k_ω. For finite subsets S and T of I, satisfying $I_\infty \subseteq S \subseteq T$, there exists a unique continuous homomorphism

$$\tau_S^T : \mathscr{G}_S \to \mathscr{G}_T,$$

[3]A topological space X is called a k_ω-space with k_ω-decomposition $X = \bigcup_{i=1}^\infty X_i$, if the following properties hold: X is Hausdorff; X is covered by $\{X_i\}_{i=1}^\infty$; each X_i is a compact subset of X and $X_i \subseteq X_{i+1}$ for $i = 1, 2, \ldots$; and Y is closed in X if and only if $Y \cap X_i$ is closed in X for every $i = 1, 2, \ldots$. As pointed out in [9], k_ω-spaces are a generalization of countable CW-complexes.

which is the inclusion mapping $\mathscr{G}_S \hookrightarrow \mathscr{G}_T$, defined naturally by the universal mapping property of free topological products of topological groups. Moreover, the collection $\left\{\mathscr{G}_S; \tau_S^T : \mathscr{G}_S \to \mathscr{G}_T\right\}_{I_\infty \subseteq S \subseteq T}$, where S and T runs over all finite subsets of I satisfying $I_\infty \subseteq S \subseteq T$, forms a direct system of k_ω-topological groups, where the directed set $\{S\}$ is countable, as I is a countable set.

Notation 2.1. Let G be a (quasi-)topological group. The underlying abstract group of G is denoted by $\underline{G}$ and the underlying topology of G by $\mathcal{T}_G$.

The direct limit

$$\varinjlim_{S} \mathscr{G}_S =: \mathscr{G}$$

of the direct system $\left\{\mathscr{G}_S; \tau_S^T : \mathscr{G}_S \to \mathscr{G}_T\right\}_{I_\infty \subseteq S \subseteq T}$ has a natural quasi-topological group[4] structure equipped with the canonical continuous homomorphisms

$$c_S : \mathscr{G}_S \to \mathscr{G}$$

defined for all finite $S \subset I$ satisfying $I_\infty \subseteq S$, where the underlying topology $\mathcal{T}_{\mathscr{G}}$ of $\mathscr{G}$ is the direct limit topology $\varinjlim_{S} \mathcal{T}_{\mathscr{G}_S}$ on $\underline{\mathscr{G}} = \varinjlim_{S} \underline{\mathscr{G}}_S$, which is defined by declaring $X \subseteq \mathscr{G}$ to be open if $X \cap \mathscr{G}_S$ is open in $\mathscr{G}_S$ for every S. Moreover, as $\mathscr{G}_S$ is a k_ω-group for every S, by Glöckner et al. [10], the direct limit topology $\mathcal{T}_{\mathscr{G}}$ on $\underline{\mathscr{G}}$ coincides with the final topology on $\underline{\mathscr{G}}$ with respect to the family of mappings $\{c_S : \mathscr{G}_S \to \underline{\mathscr{G}}\}_S$, which is defined as the finest group topology on $\underline{\mathscr{G}}$ that makes the mappings $c_S : \mathscr{G}_S \to \underline{\mathscr{G}}$ continuous for all S. Thus, to sum up, $\mathscr{G} = \varinjlim_{S} \mathscr{G}_S$ endowed with the direct limit topology $\mathcal{T}_{\mathscr{G}} = \varinjlim_{S} \mathcal{T}_{\mathscr{G}_S}$ is a topological group equipped with the canonical continuous homomorphisms

$$c_S : \mathscr{G}_S \to \mathscr{G}$$

defined for all finite $S \subset I$ satisfying $I_\infty \subseteq S$. Furthermore, as $\mathscr{G}_S$ is a k_ω-group for every such S, the topological group $\mathscr{G}$ is Hausdorff as well.

The *restricted free topological product* of the collection $\{G_i\}_{i \in I}$ of k_ω-topological groups G_i where i runs over a countable index set I with respect to the collection $\{O_i\}_{i \in I - I_\infty}$ of open subgroups O_i of G_i for $i \in I - I_\infty$, which is denoted by $*'_{i \in I}(G_i : O_i)$, is defined by the direct limit

$$\mathop{*'}_{i \in I}(G_i : O_i) := \varinjlim_{S} \mathscr{G}_S = \mathscr{G}$$

of the direct system of k_ω-groups $\left\{\mathscr{G}_S; \tau_S^T : \mathscr{G}_S \to \mathscr{G}_T\right\}_{I_\infty \subseteq S \subseteq T}$, and it has a natural Hausdorff topological group structure. We maintain the notation and the assumptions introduced so far till the end of this subsection.

[4]A triple $(Q, \mu, \mathcal{T})$ consisting of a set Q together with a binary operation $\mu : Q \times Q \to Q$ and a topology $\mathcal{T}$ on Q is called a quasi-topological group, if Q is a group under μ; if the mapping $\mu : Q \times Q \to Q$ is separately continuous with respect to $\mathcal{T}$; and if the inversion map on Q with respect to μ defined by $x \mapsto x^{-1}$ for all $x \in Q$ is continuous with respect to $\mathcal{T}$.

The restricted free topological products of countable collections of k_ω-topological groups have the following universal mapping property.

Theorem 2.2 (Universal mapping property of restricted free topological products)**.** *Let H be a topological group (not necessarily k_ω). Assume that, for each $i_o \in I$, a continuous homomorphism*

$$\phi_{i_o} : G_{i_o} \to H$$

is given. Then, for each finite subset S of I satisfying $I_\infty \subseteq S$, there exists a unique continuous homomorphism

$$\phi_S : \mathscr{G}_S \to H$$

such that

$$\phi_S \circ \iota_{i_o}^{(S)} = \phi_{i_o}$$

for every $i_o \in I$, and a unique continuous homomorphism

$$\phi = \varinjlim_S \phi_S : \varinjlim_S \mathscr{G}_S = \mathscr{G} \to H$$

satisfying

$$\phi_S = \phi \circ c_S : \mathscr{G}_S \xrightarrow{c_S} \mathscr{G} \xrightarrow{\phi} H,$$

where $c_S : \mathscr{G}_S \to \mathscr{G}$ is the canonical continuous homomorphism defined for every finite subset S of I satisfying $I_\infty \subseteq S$.

Proof. Look at the proof of Proposition 2.1 in [12]. □

For each $i_o \in I$, setting $\mathscr{G}_{i_o} = G_{i_o}$, there exists a natural continuous homomorphism

$$q_{i_o} : \mathscr{G}_{i_o} \to \mathscr{G} \tag{2.1}$$

defined explicitly via the commutative triangle

$$\begin{tikzcd} & \mathscr{G}_S \arrow[dd, "c_S"] \\ \mathscr{G}_{i_o} \arrow[ur, "\iota_{i_o}^{(S)}"] \arrow[dr, "q_{i_o}"'] & \\ & \mathscr{G} \end{tikzcd}$$

where S is a finite subset of I satisfying $I_\infty \subseteq S$ and $i_o \in S$. It turns out that the homomorphism $q_{i_o} : \mathscr{G}_{i_o} \to \mathscr{G}$ does not depend on the choice of S [12, Section 4]. Moreover, keeping the notation and the assumptions of Theorem 2.2, we have the following theorems:

Theorem 2.3. *The triangle*

$$
\begin{tikzcd}
 & \mathscr{G} \arrow[dd, "\phi"] \\
\mathscr{G}_{i_o} \arrow[ur, "q_{i_o}"] \arrow[dr, "\phi_{i_o}"'] & \\
 & H
\end{tikzcd}
\tag{2.2}
$$

is commutative.

Proof. In fact, for any finite subset S of I satisfying $I_\infty \subseteq S$ and $i_o \in S$, the identities

$$\phi \circ q_{i_o} = \phi \circ c_S \circ \iota_{i_o}^{(S)} = \phi_S \circ \iota_{i_o}^{(S)} = \phi_{i_o}$$

hold, proving the commutativity of the triangle (2.2). □

Theorem 2.4. *Let*

$$\phi' : \mathscr{G} \to H$$

be a continuous homomorphism such that

$$\phi' \circ q_{i_o} = \phi_{i_o}$$

for every $i_o \in I$. Then

$$\phi' = \phi.$$

Proof. Let $\phi' : \mathscr{G} = \varinjlim_S \mathscr{G}_S \to H$ be a continuous homomorphism. Then

$$\phi' = \varinjlim_S \phi'_S,$$

where $\phi'_S = \phi' \circ c_S : \mathscr{G}_S \xrightarrow{c_S} \varinjlim_S \mathscr{G}_S = \mathscr{G} \xrightarrow{\phi'} H$ for any finite subset S of I satisfying $I_\infty \subseteq S$. Moreover, for such an S,

$$\phi'_S \circ \iota_{i_o}^{(S)} = \phi' \circ c_S \circ \iota_{i_o}^{(S)} = \phi' \circ q_{i_o} = \phi_{i_o}$$

by assumption, and

$$\phi_S \circ \iota_{i_o}^{(S)} = \phi \circ c_S \circ \iota_{i_o}^{(S)} = \phi \circ q_{i_o} = \phi_{i_o}.$$

Therefore, by the universal mapping property of free topological products of topological groups, $\phi'_S = \phi_S$, and

$$\phi' = \varinjlim_S \phi'_S = \varinjlim_S \phi_S = \phi,$$

which completes the proof. □

Let G and H be two topological groups, and

$$\xi : G \to H$$

be a continuous homomorphism from G to H. Recall that, a continuous homomorphism

$$\xi' : G \to H$$

from G to H is said to be H-conjugate to $\xi : G \to H$, denoted by $\xi' \sim_H \xi$, if there exists $h \in H$ such that

$$\xi' = \iota_h \circ \xi,$$

where $\iota_h : H \xrightarrow{\sim} H$ is the inner-automorphism of H defined by the h-conjugation as $\iota_h(x) = h^{-1}xh$ for every $x \in H$. Clearly, if $\xi' \sim_H \xi$, then $\ker(\xi') = \ker(\xi)$.

For the next definition, which will be used from Section 3 on, and for the theorem following it, we maintain the notation and the assumptions of Theorem 2.2.

Definition 2.5. Let

$$\phi : \mathscr{G} \to H$$

be a continuous homomorphism from $\mathscr{G}$ to H. A continuous homomorphism

$$\phi' : \mathscr{G} \to H$$

is said to be locally H-conjugate to $\phi : \mathscr{G} \to H$ if

$$\phi'_{i_o} = \phi' \circ q_{i_o} \sim_H \phi \circ q_{i_o} = \phi_{i_o}$$

for every $i_o \in I$.

Theorem 2.6. *Let*

$$\phi, \phi' : \mathscr{G} \to H$$

be two continuous homomorphisms from $\mathscr{G}$ to H. If ϕ and ϕ' are locally H-conjugate, then

$$\ker(\phi) = \ker(\phi').$$

Proof. As ϕ and ϕ' are locally H-conjugate, $\phi \circ q_{i_o} \sim_H \phi' \circ q_{i_o}$ for every $i_o \in I$. Therefore, $\ker(\phi \circ q_{i_o}) = \ker(\phi' \circ q_{i_o})$. On the other hand, for any finite subset S of I satisfying $I_\infty \subseteq S$,

$$\begin{aligned}(\phi_S)_{i_o} = \phi_S \circ \iota_{i_o}^{(S)} = \phi \circ c_S \circ \iota_{i_o}^{(S)} = \phi \circ q_{i_o} \\ \sim_H \phi' \circ q_{i_o} = \phi' \circ c_S \circ \iota_{i_o}^{(S)} = \phi'_S \circ \iota_{i_o}^{(S)} = (\phi'_S)_{i_o}\end{aligned}$$

for every $i_o \in I$. Thus, $\ker((\phi_S)_{i_o}) = \ker((\phi'_S)_{i_o})$, which proves that

$$U := \bigcup_{i \in I} \ker((\phi_S)_i) = \bigcup_{i \in I} \ker((\phi'_S)_i) =: U'$$

and that $\ker(\phi_S) = \ker(\phi'_S)$ as $\ker(\phi_S)$ is the topological closure of the normal closure in $\mathscr{G}_S$ of U and $\ker(\phi'_S)$ is the topological closure of the normal closure in $\mathscr{G}_S$ of U'. Hence, the following equalities

$$\begin{aligned}\ker(\phi) &= \ker(\varinjlim_S \phi_S)\\ &= \varinjlim_S(\ker(\phi_S))\\ &= \varinjlim_S(\ker(\phi'_S))\\ &= \ker(\varinjlim_S \phi'_S) = \ker(\phi')\end{aligned}$$

hold, proving $\ker(\phi) = \ker(\phi')$. □

The following theorem, which generalizes Theorem 2.5 of [12], is about the abelianization, that is the maximal abelian Hausdorff quotient, of a restricted free topological product of a countable collection of k_ω-topological groups.

Theorem 2.7. *The maximal abelian Hausdorff quotient* $\left(*'_{i\in I}(G_i : O_i)\right)^{\mathrm{ab}}$ *of the restricted free topological product* $*'_{i\in I}(G_i : O_i)$ *of the collection* $\{G_i\}_{i\in I}$ *of* k_ω*-topological groups* G_i *where* i *runs over a countable index set* I *with respect to the collection* $\{O_i\}_{i\in I-I_\infty}$ *of open subgroups* O_i *of* G_i *for* $i \in I - I_\infty$ *is isomorphic to* $\prod'_{i\in I}(G_i^{\mathrm{ab}} : O_i^{\mathrm{ab}})$ *as topological groups, where* $\prod'_{i\in I}(G_i^{\mathrm{ab}} : O_i^{\mathrm{ab}})$ *denotes the restricted direct topological product of the collection* $\{G_i^{\mathrm{ab}}\}_{i\in I}$ *with respect to the collection* $\{O_i^{\mathrm{ab}}\}_{i\in I-I_\infty}$.

Proof. We first recall that the abelianization; that is passing to the maximal abelian Hausdorff quotient commutes with the direct limit as the abelianization functor $\mathrm{ab} : \mathcal{G}\mathrm{rp}^{\mathrm{top}} \rightsquigarrow \mathcal{A}\mathrm{b}^{\mathrm{top}}$ from the category of topological groups $\mathcal{G}\mathrm{rp}^{\mathrm{top}}$ to the category of abelian topological groups $\mathcal{A}\mathrm{b}^{\mathrm{top}}$ is left adjoint to the inclusion (=forgetful) functor $\mathrm{inc} : \mathcal{A}\mathrm{b}^{\mathrm{top}} \rightsquigarrow \mathcal{G}\mathrm{rp}^{\mathrm{top}}$ from $\mathcal{A}\mathrm{b}^{\mathrm{top}}$ to $\mathcal{G}\mathrm{rp}^{\mathrm{top}}$ [2, Introduction]. Therefore, there exists an isomorphism of topological groups

$$\left(*'_{i\in I}(G_i : O_i)\right)^{\mathrm{ab}} = (\varinjlim_S \mathscr{G}_S)^{\mathrm{ab}} \simeq \varinjlim_S \mathscr{G}_S^{\mathrm{ab}},$$

where S runs over all finite subsets of I containing I_∞. Here, the direct limit $\varinjlim_S \mathscr{G}_S^{\mathrm{ab}}$ is defined with respect to the connecting morphisms

$$(\tau_S^T)^{\mathrm{ab}} : \mathscr{G}_S^{\mathrm{ab}} \to \mathscr{G}_T^{\mathrm{ab}}$$

defined by

$$(\tau_S^T)^{\mathrm{ab}} : x\mathscr{G}_S^{\mathrm{c}} \mapsto \tau_S^T(x)\mathscr{G}_T^{\mathrm{c}}$$

for every $x \in \mathscr{G}_S$, where S and T are finite subsets of I satisfying $I_\infty \subseteq S \subseteq T$, and has a Hausdorff topological group structure with respect to the direct limit topology $\varinjlim_S \mathcal{T}_{\mathscr{G}_S^{\mathrm{ab}}}$, as $\mathscr{G}_S^{\mathrm{ab}}$ is a k_ω-group with respect to the quotient topology $\mathcal{T}_{\mathscr{G}_S^{\mathrm{ab}}}$ induced from the Graev topology $\mathcal{T}_{\mathscr{G}_S}$ of $\mathscr{G}_S$, by Morita's result [9, Fact 11], for every such S.

Next, we recall that [3, Proposition 21, p. 234], if G_o is a dense subgroup of a topological group G and $H_o \trianglelefteq G_o$, then $\overline{G_o/H_o} = G/\overline{H}_o$. Thus, returning to our discussion, the abelianization $\mathscr{G}_S^{\mathrm{ab}} = \mathscr{G}_S/\mathscr{G}_S^{\mathrm{c}}$ of $\mathscr{G}_S$, where $\mathscr{G}_S^{\mathrm{c}}$ denotes the closure $\overline{\mathscr{G}'_S}$ of the 1st commutator subgroup $\mathscr{G}'_S$ of $\mathscr{G}_S$, is the closure $\overline{\mathscr{G}_S/\mathscr{G}'_S}$ of the quotient group $\mathscr{G}_S/\mathscr{G}'_S$ which is the weak direct topological product $\prod^{w}_{i\notin S} O_i^{\mathrm{ab}} \times \prod^{w}_{i\in S} G_i^{\mathrm{ab}}$ of the collection $\{O_i^{\mathrm{ab}}\}_{i\notin S} \cup \{G_i^{\mathrm{ab}}\}_{i\in S}$. That is,

$$\mathscr{G}_S^{\mathrm{ab}} = \left((\mathop{\ast}_{i\notin S} O_i) \ast (\mathop{\ast}_{i\in S} G_i) \right)^{\mathrm{ab}} = \overline{\prod_{i\notin S}^{w} O_i^{\mathrm{ab}} \times \prod_{i\in S}^{w} G_i^{\mathrm{ab}}},$$

proving that

$$\mathscr{G}_S^{\mathrm{ab}} = \prod_{i\notin S} O_i^{\mathrm{ab}} \times \prod_{i\in S} G_i^{\mathrm{ab}},$$

as the weak direct topological product of a family of topological groups is dense in the direct topological product of the same family, which completes the proof. $\square$

Remark 2.8 (On abelianization–Part I). There seems to be two types of abelianization mappings

$$\mathscr{G} \to \prod_{i\in I}{}' (G_i^{\mathrm{ab}} : O_i^{\mathrm{ab}}).$$

The first one

$$\mathsf{s} : \mathscr{G} \xrightarrow{\mathrm{ab}} \mathscr{G}^{\mathrm{ab}} \xrightarrow[\text{(Thm.2.7)}]{\sim} \prod_{i\in I}{}' (G_i^{\mathrm{ab}} : O_i^{\mathrm{ab}})$$

via computing the maximal abelian Hausdorff quotient $\mathscr{G}^{\mathrm{ab}}$ of $\mathscr{G}$, and the second one as the unique continuous homomorphism

$$\mathsf{a} : \mathscr{G} \to \prod_{i\in I}{}' (G_i^{\mathrm{ab}} : O_i^{\mathrm{ab}})$$

satisfying

$$\mathsf{a} \circ q_{i_o} = \mathsf{a}_{i_o},$$

where

$$\mathsf{a}_{i_o} : G_{i_o} \xrightarrow{\mathrm{ab}} G_{i_o}^{\mathrm{ab}} \xrightarrow{\varepsilon_{i_o} = q_{i_o}^{\mathrm{ab}}} \prod_{i\in I}{}' (G_i^{\mathrm{ab}} : O_i^{\mathrm{ab}})$$

for every $i_o \in I$, whose existence and uniqueness is guaranteed by the universal mapping property of restricted free topological products and by Theorem 2.4. Now, we claim that

$$\mathsf{s} = \mathsf{a},$$

which also answers a question asked in [13, footnote 5]. By Theorem 2.4, it suffices to check that

$$\mathsf{s}_{i_o} = \mathsf{a}_{i_o}, \ {}^{\forall} i_o \in I.$$

Let $i_o \in I$, and S_{i_o} a finite subset of I such that $I_\infty \subseteq S_{i_o}$ and $i_o \in S_{i_o}$. For any $x_{i_o} \in G_{i_o}$,

$$\mathsf{s}_{i_o}(x_{i_o}) = \mathsf{s} \circ q_{i_o}(x_{i_o}) = \mathsf{s} \circ c_{S_{i_o}} \circ \iota_{i_o}^{(S_{i_o})}(x_{i_o}) = \mathsf{s}([\iota_{i_o}^{(S_{i_o})}(x_{i_o})]) = [\iota_{i_o}^{(S_{i_o})}(x_{i_o})\mathscr{G}_{S_{i_o}}^{\mathrm{c}}],$$

where $\iota_{i_o}^{(S_{i_o})}(x_{i_o})\mathscr{G}_{S_{i_o}}^{\mathrm{c}} \in \mathscr{G}_{S_{i_o}}^{\mathrm{ab}}$, and

$$\mathsf{a} \circ q_{i_o}(x_{i_o}) = \mathsf{a}_{i_o}(x_{i_o}) = \varepsilon_{i_o}(x_{i_o}G_{i_o}^c) = q_{i_o}^{\mathrm{ab}}(x_{i_o}G_{i_o}^c) = [(\iota_{i_o}^{(S_{i_o})})^{\mathrm{ab}}(x_{i_o}G_{i_o}^c)],$$

where $(\iota_{i_o}^{(S_{i_o})})^{\mathrm{ab}}(x_{i_o}G_{i_o}^c) = \iota_{i_o}^{(S_{i_o})}(x_{i_o})\mathscr{G}_{S_{i_o}}^{c}$. Thus,

$$\mathsf{s}_{i_o}(x_{i_o}) = \mathsf{a}_{i_o}(x_{i_o}), \ {}^\forall x_{i_o} \in G_{i_o},$$

proving that $\mathsf{s} = \mathsf{a}$.

2.2. Examples of k_ω-groups (Part I)

Let $\mathbb{h}_K$ and $\mathbb{o}_K$ denote the sets of finite (henselian) and infinite (archimedean) places of the number field K respectively. For $\upsilon \in \mathbb{h}_K \cup \mathbb{o}_K$, the completion of K at the place υ is denoted by K_υ as usual. In case $\upsilon \in \mathbb{h}_K$, denote the ring of integers of the henselian local field K_υ by O_{K_υ} and its unique prime ideal by $\mathfrak{p}_{K_\upsilon}$. The residue class field of K_υ is denoted by κ_{K_υ}, which is a finite field with q_{K_υ} elements. For $\upsilon \in \mathbb{h}_K \cup \mathbb{o}_K$, let G_{K_υ} and W_{K_υ} denote the absolute Galois group and the absolute Weil group of the local field K_υ respectively. Recall that, W_{K_υ} is a locally compact Hausdorff group endowed with a continuous homomorphism

$$\beta_{K_\upsilon} : W_{K_\upsilon} \to G_{K_\upsilon}$$

having dense image. Moreover, the Weil group topology $\mathcal{T}_{K_\upsilon}^{\mathrm{Weil}}$ of W_{K_υ} is not the same as the topology on $\underline{W}_{K_\upsilon}$ induced from the Krull topology $\mathcal{T}_{K_\upsilon}^{\mathrm{Krull}}$ of G_{K_υ}.

Recall that [26], in case $\upsilon \in \mathbb{h}_K$, there exists a canonical surjective and continuous homomorphism

$$\varrho_{K_\upsilon} : G_{K_\upsilon} \twoheadrightarrow G_{\kappa_{K_\upsilon}} = \widehat{\langle \Phi_{\kappa_{K_\upsilon}} \rangle} (\simeq \widehat{\mathbb{Z}})$$

from the absolute Galois group G_{K_υ} of K_υ to the absolute Galois group $G_{\kappa_{K_\upsilon}}$ of κ_{K_υ}, where $\Phi_{\kappa_{K_\upsilon}} \in G_{\kappa_{K_\upsilon}}$ denotes the Frobenius automorphism of $\overline{\kappa}_{K_\upsilon}$, which is a topological generator of $G_{\kappa_{K_\upsilon}}$ $(\simeq \widehat{\mathbb{Z}})$. The the topology $\mathcal{T}_{K_\upsilon}^{\mathrm{Weil}}$ of W_{K_υ} is defined as the weakest topology on $\underline{W}_{K_\upsilon} = \varrho_{K_\upsilon}^{-1}\left(\underline{\langle \Phi_{\kappa_{K_\upsilon}} \rangle}\right)$ that makes the sequence

$$1 \to I_{K_\upsilon} \to W_{K_\upsilon} \xrightarrow{\varrho_{K_\upsilon}} \langle \Phi_{\kappa_{K_\upsilon}} \rangle \to 1 \tag{2.3}$$

topological short exact, where the inertia group I_{K_υ} of K_υ is equipped with the profinite topology induced from the Krull topology $\mathcal{T}_{K_\upsilon}^{\mathrm{Krull}}$ of G_{K_υ} and $\langle \Phi_{\kappa_{K_\upsilon}} \rangle$ $(\simeq \mathbb{Z})$ equipped with the discrete topology. Thus, there exists a homeomorphism

$$I_{K_\upsilon} \times \mathbb{Z} \xrightarrow{\approx} W_{K_\upsilon}$$

defined by

$$(\iota, n) \mapsto \iota \varrho_{K_v}^{-1}(\Phi_{\kappa_{K_v}})^n$$

for every $\iota \in I_{K_v}$ and $n \in \mathbb{Z}$. Recall that, the discrete space $\mathbb{Z}$ is k_ω, as any locally compact Hausdorff space X is k_ω if and only if X is σ-compact [9, Fact 10]. Moreover, the continuous projection map

$$I_{K_v} \times \mathbb{Z} \to \mathbb{Z}$$

is a k-mapping[5] from the Hausdorff space $I_{K_v} \times \mathbb{Z}$ onto the k_ω-space $\mathbb{Z}$. Therefore, $I_{K_v} \times \mathbb{Z}$ is a k_ω-space. As W_{K_v} is homeomorphic to $I_{K_v} \times \mathbb{Z}$, it follows that the absolute Weil group W_{K_v} of K_v is a k_ω group.

Now, if $v \in \mathbb{a}_K$, we have two cases. In case $v \in \mathbb{a}_{K,\mathbb{C}}$, that is v is a complex archimedean prime of K, then $K_v = \mathbb{C}$ and the absolute Weil group $W_\mathbb{C}$ of $\mathbb{C}$ is defined as the topological group $W_\mathbb{C} = \mathbb{C}^\times$ equipped with the trivial map $\beta_\mathbb{C} : W_\mathbb{C} \to G_\mathbb{C}$. In the remaining case $v \in \mathbb{a}_{K,\mathbb{R}}$, that is v is a real archimedean prime of K, then $K_v = \mathbb{R}$ and the absolute Weil group $W_\mathbb{R}$ of $\mathbb{R}$ is defined as the topological group $W_\mathbb{R} = \mathbb{C}^\times \cup j\mathbb{C}^\times$, where $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for every $z \in \mathbb{C}^\times$, equipped with the map $\beta_\mathbb{R} : W_\mathbb{R} \to G_\mathbb{R}$ defined by $\beta_\mathbb{R}(\mathbb{C}^\times) = \mathrm{id}_\mathbb{C}$ and $\beta_\mathbb{R}(j\mathbb{C}^\times) =$ "complex conjugation" map on $\mathbb{C}$. Note that, in both cases, $W_\mathbb{C}$ and $W_\mathbb{R}$ are locally compact and Hausdorff groups, which are furthermore σ-compact, proving that they are k_ω-groups [9, Fact 10].

Now, let G be a locally compact Hausdorff and σ-compact group. Then, G is a k_ω-group [9, Fact 10]. Therefore, the direct product $W_{K_v} \times \mathrm{G}$ is a k_ω-group for every $v \in \mathbb{h}_K \cup \mathbb{a}_K$, and as I_{K_v} is an open subgroup of W_{K_v}, the direct product $I_{K_v} \times \mathrm{G}$ is a k_ω-group for every $v \in \mathbb{h}_K$ [9, Fact 4]. Thus, the following theorem follows.

Theorem 2.9. *Let* G *be a locally compact Hausdorff and σ-compact group. Then*:

- *The direct product $W_{K_v} \times \mathrm{G}$ is a k_ω-group for every $v \in \mathbb{h}_K \cup \mathbb{a}_K$;*
- *The open subgroup $I_{K_v} \times \mathrm{G}$ of $W_{K_v} \times \mathrm{G}$ is a k_ω-group for every $v \in \mathbb{h}_K$.*

2.3. Local non-abelian reciprocity map of K_v for $v \in \mathbb{h}_K$

In this subsection, we follow closely [15, 16]. For every $v \in \mathbb{h}_K$, fix an extension φ_{K_v} of the Frobenius automorphism Fr_{K_v} of K_v^{nr} to K_v^{sep}; namely, fix a *Lubin-Tate splitting* φ_{K_v} over K_v [17]. Generalizing [6] and [17], the "Galois form" of the local non-abelian reciprocity law of the non-archimedean local field K_v in the sense of Koch is constructed [15], which is an algebraic and topological isomorphism

$$\Phi_{K_v}^{(\varphi_{K_v})^{\mathrm{Galois}}} : G_{K_v} \xrightarrow{\sim} \nabla_{K_v}^{(\varphi_{K_v})}$$

[5] Recall that, a map $f : X \to Y$ from a topological space X to a topological space Y is called a k-mapping, if the preimage $f^{-1}(C)$ of any compact subset C of the target space Y is compact in the source space X.

between the absolute Galois group G_{K_v} of the local field K_v and a certain topological group $\nabla_{K_v}^{(\varphi_{K_v})}$ depending only on K_v and on the Lubin-Tate splitting φ_{K_v} over K_v.

In what follows, we shall however consider the "Weil form" of the local non-abelian reciprocity law of K_v, which is a topological group isomorphism

$$\Phi_{K_v}^{(\varphi_{K_v})^{\mathrm{Weil}}} : W_{K_v} \xrightarrow{\sim} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$$

between the absolute Weil group W_{K_v} of K_v and a certain dense subgroup ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ of $\nabla_{K_v}^{(\varphi_{K_v})}$. The constructions in [15] of the topological group $\nabla_{K_v}^{(\varphi_{K_v})}$ and its dense subgroup ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ use the theory of APF-extensions and the fields of norms introduced by Fontaine and Wintenberger [7,8].

Moreover, there exists a subgroup ${}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{o}}$ of ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ such that, the "Weil form" of the local non-abelian reciprocity law $\Phi_{K_v}^{(\varphi_{K_v})^{\mathrm{Weil}}} : W_{K_v} \xrightarrow{\sim} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ of K_v induces an isomorphism

$$\Phi_{K_v}^{(\varphi_{K_v})^{\mathrm{Weil}}} : W_{K_v}^{\underline{o}} \xrightarrow{\sim} {}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{o}}$$

of topological groups [16]. Here, $W_{K_v}^{\underline{o}}$ is the 0^{th} ramification subgroup of W_{K_v} in upper numbering. Recall that, the upper ramification filtration $\{W_{K_v}^e\}_{e\in\mathbb{R}_{\geq -1}}$ of W_{K_v} is defined by $W_{K_v}^e = W_{K_v} \cap G_{K_v}^e$, where $G_{K_v}^e$ is the e^{th} ramification subgroup of G_{K_v} in upper numbering for $e \in \mathbb{R}_{\geq -1}$. Thus, $W_{K_v}^{\underline{o}} = I_{K_v}$, the inertia subgroup of W_{K_v}.

The reason why the topological isomorphism $\Phi_{K_v}^{(\varphi_{K_v})^{\mathrm{Galois}}} : G_{K_v} \xrightarrow{\sim} \nabla_{K_v}^{(\varphi_{K_v})}$, or its Weil form $\Phi_{K_v}^{(\varphi_{K_v})^{\mathrm{Weil}}} : W_{K_v} \xrightarrow{\sim} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$, deserves to be called the *local non-abelian reciprocity law of* K_v is that, it is "natural" in the sense that the non-abelian analogues of the local abelian class field theoretic properties, such as "existence", "functoriality", and a certain "ramification theoretic" property, are all satisfied [15,16].

Remark 2.10 (On abelianization–Part II). Recall that [15], for every $v \in h_K$, the topological group $\nabla_{K_v}^{(\varphi_{K_v})}$ is defined as a projective limit

$$\nabla_{K_v}^{(\varphi_{K_v})} = \varprojlim_{L} \nabla_{L/K_v}^{(\varphi_{K_v})}$$

over certain type (look at Section 6 of [15] or Subsection 3.2 of [16]) of infinite APF Galois extensions L of K_v satisfying $K_v \subset L \subset K_v^{\varphi_{K_v}^d}$ with $f(L/K_v) = [\kappa_L : \kappa_{K_v}] = d$ and $L_o = L \cap K_v^{\mathrm{nr}}$, where

$$\nabla_{L/K_v}^{(\varphi_{K_v})} = K_v^\times / \mathrm{N}_{L_o/K_v} L_o^\times \times U_{\widetilde{\mathbb{X}}(L/K_v)}^{\diamond} / Y_{L/L_o},$$

relative to the connecting morphisms

$$(e_{L_o/M_o}^{\mathrm{CFT}}, \widetilde{\mathscr{N}}_{L/M}^{\mathrm{Coleman}}) : \nabla_{L/K_v}^{(\varphi_{K_v})} \to \nabla_{M/K_v}^{(\varphi_{K_v})},$$

where M is an infinite APF Galois extension of K_v of the same type as L such that $K_v \subset M \subset K_v^{\varphi_{K_v}^{d'}}$ with $f(M/K_v) = [\kappa_M : \kappa_{K_v}] = d'$ and $M_o = M \cap K_v^{\mathrm{nr}}$, satisfying $M \subseteq L$ and $d' \mid d$. Here, the arrow

$$e^{\mathrm{CFT}}_{L_o/M_o} : K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times \to K_v^\times/\mathrm{N}_{M_o/K_v}M_o^\times$$

is the natural morphism constructed via the existence theorem of local class field theory, and the arrow

$$\widetilde{\mathscr{N}}^{\mathrm{Coleman}}_{L/M} : U^\diamond_{\widetilde{\mathbb{X}}(L/K_v)}/Y_{L/L_o} \to U^\diamond_{\widetilde{\mathbb{X}}(M/K_v)}/Y_{M/M_o}$$

is the Coleman norm map from L to M introduced in Lemma 2.21, and the equations (2.47) and (2.48) of [14].

Observe that, by [6, Remark 4], the kernel of the continuous surjective homomorphism

$$\mathrm{id}_{K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times} \times \mathrm{Pr}_{\widetilde{L}_o} : \nabla^{(\varphi_{K_v})}_{L/K_v} \to K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times \times U_{L_o}/\mathrm{N}_{L/L_o}U_L \tag{2.4}$$

is the commutator subgroup $\left(\nabla^{(\varphi_{K_v})}_{L/K_v}\right)^c$ of $\nabla^{(\varphi_{K_v})}_{L/K_v}$. Therefore, the continuous surjective homomorphism (2.4) induces an isomorphism of topological groups (2.5)

$$(\mathrm{id}_{K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times} \times \mathrm{Pr}_{\widetilde{L}_o})^* : \left(\nabla^{(\varphi_{K_v})}_{L/K_v}\right)^{\mathrm{ab}} \xrightarrow{\sim} K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times \times U_{L_o}/\mathrm{N}_{L/L_o}U_L$$

as the groups under discussion are compact and Hausdorff. Moreover, the diagram

$$\begin{tikzcd}
\nabla^{(\varphi_{K_v})}_{L/K_v} \arrow[r, "\mathrm{ab}_{L/K_v}"'] \arrow[rr, bend left, "\mathrm{id}_{K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times}\times\mathrm{Pr}_{\widetilde{L}_o}(\widetilde{L}_o=\widetilde{K}_v)"] \arrow[d, "(e^{\mathrm{CFT}}_{L_o/M_o}{,}\widetilde{\mathscr{N}}^{\mathrm{Coleman}}_{L/M})"'] & \left(\nabla^{(\varphi_{K_v})}_{L/K_v}\right)^{\mathrm{ab}} \arrow[r, "\sim", "(\mathrm{id}\times\mathrm{Pr}_{\widetilde{L}_o})^*"'] \arrow[d, "(e^{\mathrm{CFT}}_{L_o/M_o}{,}\widetilde{\mathscr{N}}^{\mathrm{Coleman}}_{L/M})^{\mathrm{ab}}"] & K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times \times U_{L_o}/\mathrm{N}_{L/L_o}U_L \arrow[d, "(e^{\mathrm{CFT}}_{L_o/M_o}{,}\mathrm{N}_{L_o/M_o})"] \\
\nabla^{(\varphi_{K_v})}_{M/K_v} \arrow[r, "\mathrm{ab}_{M/K_v}"'] \arrow[rr, bend right, "\mathrm{id}_{K_v^\times/\mathrm{N}_{M_o/K_v}M_o^\times}\times\mathrm{Pr}_{\widetilde{M}_o}(\widetilde{M}_o=\widetilde{K}_v)"'] & \left(\nabla^{(\varphi_{K_v})}_{M/K_v}\right)^{\mathrm{ab}} \arrow[r, "\sim", "(\mathrm{id}\times\mathrm{Pr}_{\widetilde{M}_o})^*"'] & K_v^\times/\mathrm{N}_{M_o/K_v}M_o^\times \times U_{M_o}/\mathrm{N}_{M/M_o}U_M
\end{tikzcd}$$

is commutative [14, p. 130, (ii)]. Thus, passing to the projective limits, there exists a continuous surjective homomorphism

$$\mathrm{id}_{\widehat{\mathbb{Z}}} \times \varprojlim_L \mathrm{Pr}_{\widetilde{L}_o} : \underbrace{\varprojlim_L \nabla^{(\varphi_{K_v})}_{L/K_v}}_{\nabla^{(\varphi_{K_v})}_{K_v}} \to \underbrace{\varprojlim_L \left(K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times \times U_{L_o}/\mathrm{N}_{L/L_o}U_L\right)}_{\widehat{\mathbb{Z}}\times U_{K_v}},$$

which factors as

$$\begin{tikzcd}[column sep=large]
\nabla^{(\varphi_{K_v})}_{K_v} \arrow[r, "\varprojlim_L \mathrm{ab}_{L/K_v}"'] \arrow[rr, bend left, "\mathrm{id}_{\widehat{\mathbb{Z}}}\times\varprojlim_L \mathrm{Pr}_{\widetilde{L}_o}"] & \varprojlim_L \left(\nabla^{(\varphi_{K_v})}_{L/K_v}\right)^{\mathrm{ab}} \arrow[r, "\sim", "\varprojlim_L (\mathrm{id}_{K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times}\times\mathrm{Pr}_{\widetilde{L}_o})^*"'] & \widehat{\mathbb{Z}}\times U_{K_v}\,.
\end{tikzcd}$$

Thus, the following diagram

$$\begin{tikzcd}[column sep=large, row sep=huge]
G_{K_v} \arrow[r, "\mathrm{ab}"] \arrow[d, "\Phi^{(\varphi_{K_v})^{\mathrm{Galois}}}_{K_v}"', "\wr"] & G^{\mathrm{ab}}_{K_v} \arrow[r, "\mathsf{Art}_{K_v}", "\sim"'] \arrow[d, "(\Phi^{(\varphi_{K_v})^{\mathrm{Galois}}}_{L/K_v})^{\mathrm{ab}}"', "\wr"] & \widehat{K_v^\times} \arrow[d, "\wr"', "\varphi_{K_v}\text{ determines }\pi_{K_v}"] \\
\nabla^{(\varphi_{K_v})}_{K_v} \arrow[r, "\mathrm{ab}"'] & (\nabla^{(\varphi_{K_v})}_{K_v})^{\mathrm{ab}} \arrow[r, "\sim", "\varprojlim_L (\mathrm{id}_{K_v^\times/\mathrm{N}_{L_o/K_v}L_o^\times}\times\mathrm{Pr}_{\widetilde{L}_o})^*"'] & \widehat{\mathbb{Z}}\times U_{K_v}
\end{tikzcd}$$

is commutative, where the commutativity follows from the construction of the local non-abelian reciprocity law $\Phi^{(\varphi_{K_v})^{\mathrm{Galois}}}_{K_v} : G_{K_v} \xrightarrow{\sim} \nabla^{(\varphi_{K_v})}_{K_v}$ of K_v in Galois form and from [6, Remark 4].

Thus, via Remark 2.10, the "local non-abelian class field theory" can be summarized in terms of the following tables:

Local non-abelian C.F.T. (φ_{K_v} fixed)	
G_{K_v}	$\nabla^{(\varphi_{K_v})}_{K_v}$
W_{K_v}	${}_{\mathbb{Z}}\nabla^{(\varphi_{K_v})}_{K_v}$
$W^0_{K_v}$	${}_1\nabla^{(\varphi_{K_v})\underline{0}}_{K_v}$
$W^\delta_{K_v},\ \delta\in(i-1,i]$	${}_1\nabla^{(\varphi_{K_v})\underline{i}}_{K_v}$

$\xrightarrow{\mathrm{ab}}$

Local abelian class field theory	
$G^{ab}_{K_v}$	$\widehat{\mathbb{Z}}\times U_{K_v}$
$W^{ab}_{K_v}$	$\mathbb{Z}\times U_{K_v}$
$W^{ab\ 0}_{K_v}$	U_{K_v}
$W^{ab\ \delta}_{K_v},\ \delta\in(i-1,i]$	$U^i_{K_v}$

Following [12], denote the inverse $\left(\Phi^{(\varphi_{K_v})^{\mathrm{Galois}}}_{K_v}\right)^{-1} : \nabla^{(\varphi_{K_v})}_{K_v} \xrightarrow{\sim} G_{K_v}$ of the local non-abelian reciprocity law $\Phi^{(\varphi_{K_v})^{\mathrm{Galois}}}_{K_v} : G_{K_v} \xrightarrow{\sim} \nabla^{(\varphi_{K_v})}_{K_v}$ of K_v by

$$\{\bullet, K_v\}^{\mathrm{Galois}}_{\varphi_{K_v}} : \nabla^{(\varphi_{K_v})}_{K_v} \xrightarrow{\sim} G_{K_v},$$

and call it the *"Galois form" of the local non-abelian norm-residue symbol of* K_v. Likewise, the "Weil-form" of the local non-abelian norm-residue symbol

$$\{\bullet, K_v\}^{\mathrm{Weil}}_{\varphi_{K_v}} : {}_{\mathbb{Z}}\nabla^{(\varphi_{K_v})}_{K_v} \xrightarrow{\sim} W_{K_v}$$

of K_v is defined by the inverse $\left(\Phi^{(\varphi_{K_v})^{\mathrm{Weil}}}_{K_v}\right)^{-1} : {}_{\mathbb{Z}}\nabla^{(\varphi_{K_v})}_{K_v} \xrightarrow{\sim} W_{K_v}$ of $\Phi^{(\varphi_{K_v})^{\mathrm{Weil}}}_{K_v} : W_{K_v} \xrightarrow{\sim} {}_{\mathbb{Z}}\nabla^{(\varphi_{K_v})}_{K_v}$.

2.4. Examples of k_ω-groups (Part II)

Recall that, Theorem 2.9 of Subsection 2.2 states that $W_{K_v} \times \mathrm{G}$ is a k_ω-group for every $v \in \mathbb{h}_K \cup \mathbb{a}_K$, and $I_{K_v} \times \mathrm{G}$ is a k_ω-group for every $v \in \mathbb{h}_K$, where G denotes a locally compact Hausdorff and σ-compact group. In case $v \in \mathbb{h}_K$, the "Weil form" of the local non-abelian reciprocity law

$$\Phi_{K_v}^{(\varphi_{K_v})^{\mathrm{Weil}}} : W_{K_v} \xrightarrow{\sim} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$$

of K_v, reviewed in Subsection 2.3, naturally defines an isomorphism

$$\Phi_{K_v}^{(\varphi_{K_v})^{\mathrm{Weil}}} \times \mathrm{id}_{\mathrm{G}} : W_{K_v} \times \mathrm{G} \xrightarrow{\sim} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{G}$$

of topological groups, which induces a topological group isomorphism

$$\Phi_{K_v}^{(\varphi_{K_v})^{\mathrm{Weil}}} \times \mathrm{id}_{\mathrm{G}} : W_{K_v}^{\underline{o}} \times \mathrm{G} \xrightarrow{\sim} {}_{1}\nabla_{K_v}^{(\varphi_{K_v})\underline{o}} \times \mathrm{G},$$

where $W_{K_v}^{\underline{o}} = I_{K_v}$. Therefore, ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{G}$ and ${}_{1}\nabla_{K_v}^{(\varphi_{K_v})\underline{o}} \times \mathrm{G}$ are k_ω-groups. So we have the following theorem.

Theorem 2.11. *Let* G *be a locally compact Hausdorff and σ-compact group. Then* ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{G}$ *and* ${}_{1}\nabla_{K_v}^{(\varphi_{K_v})\underline{o}} \times \mathrm{G}$ *are k_ω-groups, where $v \in \mathbb{h}_K$.*

2.5. Weil-Arhur idèle group $\mathscr{W}\mathscr{A}_K^{\varphi}$ of K

For every $v \in \mathbb{h}_K \cup \mathbb{a}_K$, the absolute Langlands group L_{K_v} of the local field K_v is defined as the absolute Weil-Arthur group WA_{K_v} of K_v, where:

- $WA_{K_v} = W_{K_v} \times \mathrm{SU}(2)$ if $v \in \mathbb{h}_K$;
- $WA_{K_v} = W_{K_v}$ if $v \in \mathbb{a}_K$.

Remark 2.12. Note that, instead of the traditional absolute Weil-Deligne group WD_{K_v} of the local field K_v, in this work we use, following Langlands [20], the *absolute Weil-Arthur group* WA_{K_v} of K_v defined as above.

Observe that, by Theorem 2.9, for every $v \in \mathbb{h}_K \cup \mathbb{a}_K$, the absolute Langlands group L_{K_v} of the local field K_v is a k_ω-group, and for every $v \in \mathbb{h}_K$, the open subgroup $I_{K_v} \times \mathrm{SU}(2)$ of L_{K_v} is k_ω.

Hypothetically, there is a global counterpart of the collection of topological groups $\{L_{K_v}\}_{v \in \mathbb{h}_K \cup \mathbb{a}_K}$ called the automorphic Langlands group L_K of the number field K (the absolute Langlands group L_K of K). The expected formal properties of this conjectural topological group L_K attached to K are listed in Section 3. In this subsection, we shall introduce an unconditional topological group $\mathscr{W}\mathscr{A}_K^{\varphi}$ attached to K closely related with this hypothetical topological group L_K.

For each $v \in \mathbb{h}_K$, fix a Lubin-Tate splitting φ_{K_v} over K_v. The "Weil-form" of the local non-abelian norm-residue symbol

$$\{\bullet, K_v\}_{\varphi_{K_v}}^{\mathrm{Weil}} : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \xrightarrow{\sim} W_{K_v}$$

of K_v induces an isomorphism

$$\{\bullet, K_v\}^{\text{Langlands}}_{\varphi_{K_v}} := \{\bullet, K_v\}^{\text{Weil}}_{\varphi_{K_v}} \times \mathrm{id}_{\mathrm{SU}(2)} : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SU}(2) \xrightarrow{\sim} L_{K_v},$$

the "Langlands-form" of the local non-abelian norm-residue symbol of K_v.

Note that, by Theorem 2.11, the topological groups ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ and ${}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}}$ as well as ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SU}(2)$ and ${}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}} \times \mathrm{SU}(2)$ are all k_ω. Now, the "thickened" version $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ of the non-abelian idèle group $\mathscr{J}_K^{\underline{\varphi}}$ of the number field K, where $\mathscr{J}_K^{\underline{\varphi}}$ is the Hausdorff topological group defined as the restricted free product

$$\mathscr{J}_K^{\underline{\varphi}} := \mathop{\ast'}_{v \in \mathbb{h}_K} \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} : {}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}} \right) \ast W_{\mathbb{R}}^{\ast r_1} \ast W_{\mathbb{C}}^{\ast r_2}$$

of the collection $\{{}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}\}_{v \in \mathbb{h}_K} \cup \{W_{K_v}\}_{v \in \mathbb{a}_K}$ with respect to the collection $\{{}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}}\}_{v \in \mathbb{h}_K}$ introduced in [12], is defined as follows:

Definition 2.13. For each $v \in \mathbb{h}_K$, fix a Lubin-Tate splitting φ_{K_v} and let $\underline{\varphi} = \{\varphi_{K_v}\}_{v \in \mathbb{h}_K}$.

- The Hausdorff topological group $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ defined by the restricted free product

$$\mathscr{W}\mathscr{A}_K^{\underline{\varphi}} := \mathop{\ast'}_{v \in \mathbb{h}_K} \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SU}(2) : {}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}} \times \mathrm{SU}(2) \right) \ast W_{\mathbb{R}}^{\ast r_1} \ast W_{\mathbb{C}}^{\ast r_2}$$

  of the collection $\{{}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SU}(2)\}_{v \in \mathbb{h}_K} \cup \{W_{K_v}\}_{v \in \mathbb{a}_K}$ with respect to the collection $\{{}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}} \times \mathrm{SU}(2)\}_{v \in \mathbb{h}_K}$ is called the *Weil-Arthur idèle group of the number field* K;
- The finite (=henselian) part $\mathscr{W}\mathscr{A}_{K,\mathbb{h}}^{\underline{\varphi}}$ of $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ is defined by

$$\mathscr{W}\mathscr{A}_{K,\mathbb{h}}^{\underline{\varphi}} := \mathop{\ast'}_{v \in \mathbb{h}_K} \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SU}(2) : {}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}} \times \mathrm{SU}(2) \right),$$

  and;
- The infinite (=archimedean) part $\mathscr{W}\mathscr{A}_{K,\mathbb{a}}^{\underline{\varphi}}$ of $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ is defined by

$$\mathscr{W}\mathscr{A}_{K,\mathbb{a}}^{\underline{\varphi}} := W_{\mathbb{R}}^{\ast r_1} \ast W_{\mathbb{C}}^{\ast r_2}.$$

Here, as usual r_1 and r_2 denote the number of real and the number of conjugate pairs of complex embeddings of the number field K in $\mathbb{C}$, respectively.

As in Subsection 2.1, for every finite subset S of $\mathbb{h}_K \cup \mathbb{a}_K$ satisfying $\mathbb{a}_K \subseteq S$, the k_ω-topological group $(\mathscr{W}\mathscr{A}_K^{\underline{\varphi}})_S$ is defined by

$$(\mathscr{W}\mathscr{A}_K^{\underline{\varphi}})_S := \mathop{\ast}_{v \notin S} ({}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}} \times \mathrm{SU}(2)) \ast \left(\mathop{\ast}_{v \in S - \mathbb{a}_K} ({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SU}(2)) \right) \ast \mathscr{W}\mathscr{A}_{K,\mathbb{a}}^{\underline{\varphi}},$$

and following Theorem 2.2,

$$c_S : (\mathscr{W}\mathscr{A}_K^{\underline{\varphi}})_S \to \mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$$

denotes the canonical homomorphism.

Note that, the topological group $\mathscr{W}\mathscr{A}_K^{\varphi}$ is an "extremely big" group, but its definition *depends only on the number field* K.

2.6. Basic properties of the Weil-Arhur idèle group $\mathscr{W}\mathscr{A}_K^{\varphi}$ of K

The following theorem describes the abelianization ${\mathscr{W}\mathscr{A}_K^{\varphi}}^{ab}$ of the topological group $\mathscr{W}\mathscr{A}_K^{\varphi}$.

Theorem 2.14. *The abelianization* ${\mathscr{W}\mathscr{A}_K^{\varphi}}^{ab}$ *of the topological group* $\mathscr{W}\mathscr{A}_K^{\varphi}$ *is the idèle group* $\mathbb{J}_K$ *of* K*.*

Proof. Follows directly from Theorem 2.7. □

The proof of the next theorem depends on the following "open mapping theorem" from the general theory of topological groups.

Lemma 2.15. *Let* G *and* H *be two topological groups such that, there exist a continuous and surjective homomorphism*

$$\phi : G \to H,$$

and a continuous homomorphism

$$s : H \to G$$

such that

$$\phi \circ s = \mathrm{id}_H.$$

Then $\phi : G \to H$ *is an open mapping.*

Proof. Let U be an open subset of G. First observe that, as $s : H \to G$ is a continuous mapping, $s^{-1}(U)$ is an open subset of H. Moreover, the condition $\phi \circ s = \mathrm{id}_H$ implies that $s^{-1}(U) \subseteq \phi(U)$. Next, observe that, for any $\varepsilon \in \ker(\phi)$, the inclusion $s^{-1}(U\varepsilon^{-1}) \subseteq \phi(U)$ holds. Finally, in case $s^{-1}(U) \subsetneq \phi(U)$, for any $y \in \phi(U) - s^{-1}(U)$, there exists $\varepsilon_y \in \ker(\phi)$ such that $y \in s^{-1}(U\varepsilon_y^{-1})$. In fact, there exists $x \in U$ such that $\phi(x) = y$ and $y \notin s^{-1}(U)$; that is, $s(y) \notin U$. Therefore, $x \neq s(y)$, while $\phi(x) = \phi(s(y)) = y$. So, there exists $1 \neq \varepsilon_y \in \ker(\phi)$, such that $s(y)\varepsilon_y = x \in U$, which shows that $y \in s^{-1}(U\varepsilon_y^{-1})$. Thus, $\phi(U)$ is an open subset of H, as $s^{-1}(U\varepsilon_y^{-1})$ is an open subset of H, for every $y \in \phi(U) - s^{-1}(U)$. □

Theorem 2.16. – *There exists a surjective, open and continuous homomorphism*

$$f_K^{\varphi} : \mathscr{W}\mathscr{A}_K^{\varphi} \to \mathscr{J}_K^{\varphi},$$

defined uniquely by the continuous homomorphisms

$$(f_K^{\varphi})_v : (\mathscr{W}\mathscr{A}_K^{\varphi})_v \to \mathscr{J}_K^{\varphi}$$

given by

$$(f_K^{\varphi})_v = \begin{cases} q_v \circ \mathrm{Pr}_{\mathbb{Z}\nabla_{K_v}^{(\varphi_{K_v})}}, & v \in \mathbb{h}_K; \\ q_v \circ \mathrm{id}_{W_{K_v}}, & v \in \mathbb{a}_K; \end{cases}$$

– *There exists a continuous homomorphism*

$$\sigma_K^{\varphi} : \mathscr{J}_K^{\varphi} \to \mathscr{W}\mathscr{A}_K^{\varphi}$$

satisfying

$$f_K^{\varphi} \circ \sigma_K^{\varphi} = \mathrm{id}_{\mathscr{J}_K^{\varphi}}.$$

Proof. For each $v \in \mathbb{h}_K \cup \mathbb{a}_K$, there exists a natural topological group homomorphism defined by

$$q_v \circ \mathrm{Pr}_{\mathbb{Z}\nabla_{K_v}^{(\varphi_{K_v})}} : (\mathscr{W}\mathscr{A}_K^{\varphi})_v = {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SU}(2) \xrightarrow{\mathrm{Pr}_{\mathbb{Z}\nabla_{K_v}^{(\varphi_{K_v})}}} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} = (\mathscr{J}_K^{\varphi})_v \xrightarrow{q_v} \mathscr{J}_K^{\varphi},$$

if $v \in \mathbb{h}_K$, and by

$$q_v \circ \mathrm{id}_{W_{K_v}} : (\mathscr{W}\mathscr{A}_K^{\varphi})_v = W_{K_v} \xrightarrow{\mathrm{id}_{W_{K_v}}} W_{K_v} = (\mathscr{J}_K^{\varphi})_v \xrightarrow{q_v} \mathscr{J}_K^{\varphi},$$

if $v \in \mathbb{a}_K$. Therefore, by Theorem 2.2 on the universal mapping property of restricted free products and by Theorem 2.3, there exists a continuous homomorphism

$$f_K^{\varphi} : \mathscr{W}\mathscr{A}_K^{\varphi} \to \mathscr{J}_K^{\varphi}$$

defined uniquely by the continuous homomorphisms

$$(f_K^{\varphi})_v : (\mathscr{W}\mathscr{A}_K^{\varphi})_v \to \mathscr{J}_K^{\varphi}$$

given by

$$(f_K^{\varphi})_v = \begin{cases} q_v \circ \mathrm{Pr}_{\mathbb{Z}\nabla_{K_v}^{(\varphi_{K_v})}}, & v \in \mathbb{h}_K; \\ q_v \circ \mathrm{id}_{W_{K_v}}, & v \in \mathbb{a}_K. \end{cases}$$

To prove the surjectivity of this morphism, it suffices to observe the existence of a continuous map

$$\sigma_K^{\varphi} : \mathscr{J}_K^{\varphi} \to \mathscr{W}\mathscr{A}_K^{\varphi}$$

satisfying

$$f_K^{\varphi} \circ \sigma_K^{\varphi} = \mathrm{id}_{\mathscr{J}_K^{\varphi}}.$$

In fact, again by Theorems 2.2 and 2.3, there exists a unique continuous homomorphism

$$\sigma_K^{\varphi} : \mathscr{J}_K^{\varphi} \to \mathscr{W}\mathscr{A}_K^{\varphi}$$

defined by the continuous homomorphisms

$$(\sigma_K^{\varphi})_v : (\mathscr{J}_K^{\varphi})_v \to \mathscr{W}\mathscr{A}_K^{\varphi}$$

given by

$$(\sigma_K^{\varphi})_v = \begin{cases} q_v \circ (\mathrm{id}_{\mathbb{Z}\nabla_{K_v}^{(\varphi_{K_v})}}(\bullet) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}), & v \in \mathbb{h}_K; \\ q_v \circ \mathrm{id}_{W_{K_v}}, & v \in \mathbb{a}_K. \end{cases}$$

Now, for $\upsilon \in \mathbb{h}_K \cup \mathbb{a}_K$,

$$\begin{aligned}(f_K^{\varphi} \circ \sigma_K^{\varphi})_{\upsilon} &= (f_K^{\varphi} \circ \sigma_K^{\varphi}) \circ q_{\upsilon} \\ &= f_K^{\varphi} \circ (\sigma_K^{\varphi} \circ q_{\upsilon}) \\ &= f_K^{\varphi} \circ (\sigma_K^{\varphi})_{\upsilon}.\end{aligned}$$

Thus, if $\upsilon \in \mathbb{h}_K$, then

$$\begin{aligned}(f_K^{\varphi} \circ \sigma_K^{\varphi})_{\upsilon} &= f_K^{\varphi} \circ (\sigma_K^{\varphi})_{\upsilon} \\ &= f_K^{\varphi} \circ q_{\upsilon} \circ (\mathrm{id}_{\mathbb{Z}\nabla_{K_{\upsilon}}^{(\varphi K_{\upsilon})}}(\bullet) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \\ &= (f_K^{\varphi})_{\upsilon} \circ (\mathrm{id}_{\mathbb{Z}\nabla_{K_{\upsilon}}^{(\varphi K_{\upsilon})}}(\bullet) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \\ &= q_{\upsilon} \circ \mathrm{Pr}_{\mathbb{Z}\nabla_{K_{\upsilon}}^{(\varphi K_{\upsilon})}} \circ (\mathrm{id}_{\mathbb{Z}\nabla_{K_{\upsilon}}^{(\varphi K_{\upsilon})}}(\bullet) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \\ &= q_{\upsilon} \circ \mathrm{id}_{\mathbb{Z}\nabla_{K_{\upsilon}}^{(\varphi K_{\upsilon})}} \\ &= q_{\upsilon},\end{aligned}$$

and if $\upsilon \in \mathbb{a}_K$, then

$$\begin{aligned}(f_K^{\varphi} \circ \sigma_K^{\varphi})_{\upsilon} &= f_K^{\varphi} \circ (\sigma_K^{\varphi})_{\upsilon} \\ &= f_K^{\varphi} \circ q_{\upsilon} \circ \mathrm{id}_{W_{K_{\upsilon}}} \\ &= (f_K^{\varphi})_{\upsilon} \circ \mathrm{id}_{W_{K_{\upsilon}}} \\ &= q_{\upsilon} \circ \mathrm{id}_{W_{K_{\upsilon}}} \circ \mathrm{id}_{W_{K_{\upsilon}}} \\ &= q_{\upsilon} \circ \mathrm{id}_{W_{K_{\upsilon}}} \\ &= q_{\upsilon}.\end{aligned}$$

Therefore, by the universal mapping property of free products of topological groups and by the construction of the natural homomorphism (2.1),

$$(f_K^{\varphi} \circ \sigma_K^{\varphi})_S = c_S,$$

where S is a finite subset of $\mathbb{h}_K \cup \mathbb{a}_K$ such that $\mathbb{a}_K \subseteq S$ and $\upsilon \in S$. Now, passing to the direct limits, the identity

$$f_K^{\varphi} \circ \sigma_K^{\varphi} = \varinjlim_{S} c_S = \mathrm{id}_{\mathscr{I}_K^{\varphi}}$$

follows. Finally, the openness of the continuous homomorphism $f_K^{\varphi} : \mathscr{W}\mathscr{A}_K^{\varphi} \to \mathscr{I}_K^{\varphi}$ follows from the open mapping theorem stated as Lemma 2.15, which completes the proof. □

Proposition 2.17. *The kernel* $\mathscr{S}_K^{\varphi} := \ker(f_K^{\varphi})$ *of the surjective and continuous homomorphism*

$$f_K^{\varphi} : \mathscr{W}\mathscr{A}_K^{\varphi} \to \mathscr{I}_K^{\varphi}$$

is a closed normal subgroup of $\mathscr{W}\mathscr{A}_K^{\varphi}$.

Proof. Note that, $\mathscr{J}_K^{\varphi}$ is a Hausdorff topological group. Therefore, $\left\langle 1_{\mathscr{J}_K^{\varphi}} \right\rangle$ is a closed subgroup of $\mathscr{J}_K^{\varphi}$. As $f_K^{\varphi} : \mathscr{W}\mathscr{A}_K^{\varphi} \to \mathscr{J}_K^{\varphi}$ is a continuous homomorphism, $\mathscr{S}_K^{\varphi}$ is a closed normal subgroup of $\mathscr{W}\mathscr{A}_K^{\varphi}$ by Theorem 2.16. □

So, maintaining the notation introduced in Theorem 2.16 and Proposition 2.17, and their proof, the following corollary follows naturally:

Corollary 2.18. *The topological group* $\mathscr{W}\mathscr{A}_K^{\varphi}$ *sits in the split topological short exact sequence*

$$1 \longrightarrow \mathscr{S}_K^{\varphi} \xrightarrow{\text{inc.}} \mathscr{W}\mathscr{A}_K^{\varphi} \underset{\sigma_K^{\varphi}}{\overset{f_K^{\varphi}}{\rightleftarrows}} \mathscr{J}_K^{\varphi} \longrightarrow 1$$

of topological groups, where $\mathscr{S}_K^{\varphi} = \ker(f_K^{\varphi})$, *and* $\sigma_K^{\varphi} : \mathscr{J}_K^{\varphi} \to \mathscr{W}\mathscr{A}_K^{\varphi}$ *is a continuous section of* $f_K^{\varphi} : \mathscr{W}\mathscr{A}_K^{\varphi} \to \mathscr{J}_K^{\varphi}$ *which is furthermore a topological group homomorphism.*

Proof. The assertion follows immediately from the proof of Theorem 2.16 and from Proposition 2.17. □

Thus, Corollary 2.18 combined with [3, pages 240–241], implies that the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}_K^{\varphi}$ of K is isomorphic as a topological group to the external topological semi-direct product $\mathscr{S}_K^{\varphi} \rtimes_{\sigma_K^{\varphi}} \mathscr{J}_K^{\varphi}$ of $\mathscr{J}_K^{\varphi}$ by $\mathscr{S}_K^{\varphi}$ relative to the continuous section σ_K^{φ} of f_K^{φ}.

3. Automorphic Langlands group L_K of a number field K

There is a conjectural topological group L_K, called the automorphic Langlands group of the number field K (the absolute Langlands group of K), which is closely related with the absolute Weil group W_K of K. Recall that, W_K is a locally compact topological group which comes equipped with a continuous homomorphism $\beta_K : W_K \to G_K$, where G_K is the absolute Galois group of K [13, 26].

The expected properties of this conjectural topological group L_K attached to the number field K are listed below.

3.1. Formal properties of the hypothetical group L_K (Part I)

Following Arthur [1], Fan [5], and Kottwitz [18], L_K is a certain hypothetical locally compact group such that:

(1) There exists a surjective topological group homomorphism

$$L_K \xrightarrow{f_K} W_K;$$

(2) The kernel of the topological group homomorphism

$$L_K \xrightarrow{f_K} W_K \xrightarrow{\beta_K} G_K$$

is connected;

(3) The abelianization L_K^{ab} of L_K is isomorphic to the abelianization W_K^{ab} of W_K via the isomorphism

$$L_K^{\mathrm{ab}} \xrightarrow[\sim]{f_K^{\mathrm{ab}}} W_K^{\mathrm{ab}};$$

(4) The kernel of the topological group homomorphism

$$L_K \xrightarrow{\mathrm{ab}} L_K^{\mathrm{ab}} \xrightarrow[\sim]{f_K^{\mathrm{ab}}} W_K^{\mathrm{ab}} \xrightarrow[\sim]{\mathsf{Art}_K} C_K \xrightarrow{\log|\bullet|} \mathbb{R}$$

is compact, where C_K denotes the idèle class group of K, and as usual $\mathsf{Art}_K : W_K^{\mathrm{ab}} \xrightarrow{\sim} C_K$ is the global Artin reciprocity law of K;

(5) For every valuation v of K, the absolute Langlands group L_{K_v} of the local field K_v, introduced in Subsection 2.5, which is the local analogue of L_K, comes equipped with a surjective continuous homomorphism

$$L_{K_v} \xrightarrow{f_{K_v}} W_{K_v},$$

which is defined by:
- $f_{K_v} = \mathrm{Pr}_{W_{K_v}}$, if v is a henselian valuation of K;
- $f_{K_v} = \mathrm{id}_{W_{K_v}}$, if v is an archimedean valuation of K;

(6) Every embedding $e_v : K^{\mathrm{sep}} \hookrightarrow K_v^{\mathrm{sep}}$ determines a continuous embedding $e_v^{\mathrm{Langlands}} : L_{K_v} \to L_K$, which is unique up to L_K-conjugacy, and extends the conjugacy classes of embeddings $e_v^{\mathrm{Weil}} : W_{K_v} \to W_K$ and $e_v^{\mathrm{Galois}} : G_{K_v} \to G_K$ again determined by $e_v : K^{\mathrm{sep}} \hookrightarrow K_v^{\mathrm{sep}}$ via the commutativity of the following diagram

$$\begin{tikzcd} L_{K_v} \arrow[r, "f_{K_v}"] \arrow[d, "e_v^{\mathrm{Langlands}}"'] & W_{K_v} \arrow[r, "\beta_{K_v}"] \arrow[d, "e_v^{\mathrm{Weil}}"'] & G_{K_v} \arrow[d, "e_v^{\mathrm{Galois}}"] \\ L_K \arrow[r, "f_K"] & W_K \arrow[r, "\beta_K"] & G_K \end{tikzcd} \tag{3.1}$$

of continuous homomorphisms.

In this work, the properties listed above together with the one listed in Subsection 5.3 are called the *formal properties* of the locally compact group L_K. There is an extra property called the *universality* of L_K, namely the *global Langlands reciprocity principle* over K, whose discussion is postponed to a separate work.

Assumption 3.1. From now on we shall assume the existence of the hypothetical automorphic Langlands group of a number field with the desired properties listed above.

4. Global non-abelian norm-residue symbol in Langlands form

For $v \in \mathbb{h}_K \cup \mathbb{a}_K$, choose and fix an embedding

$$e_v : K^{sep} \hookrightarrow K_v^{sep}.$$

Recall from the 6th basic property of the automorphic Langlands group L_K of the number field K given in Subsection 3.1 that, this embedding determines a continuous homomorphism

$$e_v^{\text{Langlands}} : L_{K_v} \to L_K,$$

which is unique up to L_K-conjugation. Therefore, there exists a continuous homomorphism

$$\mathsf{NR}_{K_v}^{(\varphi_{K_v})^{\text{Langlands}}} : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SU}(2) \xrightarrow[\sim]{\{\bullet, K_v\}_{\varphi_{K_v}}^{\text{Langlands}}} L_{K_v} \xrightarrow{e_v^{\text{Langlands}}} L_K$$

defined up to L_K-conjugation, for every $v \in \mathbb{h}_K$, and a continuous homomorphism

$$\mathsf{NR}_{K_v}^{(\varphi_{K_v})^{\text{Langlands}}} = e_v^{\text{Langlands}} : W_{K_v} \xrightarrow{e_v^{\text{Langlands}}} L_K,$$

defined up to L_K-conjugation, for every $v \in \mathbb{a}_K$. So, by Theorem 2.2 on the universal mapping property of restricted free products, we have the following theorem.

Theorem 4.1 ("Langlands form" of the global non-abelian norm-residue symbol)**.** *There exists a continuous homomorphism*

$$\mathsf{NR}_K^{\varphi^{\text{Langlands}}} : \mathscr{W}\!\mathscr{A}_K^{\varphi} \to L_K, \tag{4.1}$$

which is unique up to local L_K-conjugacy introduced in Definition 2.5.

The continuous homomorphism

$$\mathsf{NR}_K^{\varphi^{\text{Langlands}}} : \mathscr{W}\!\mathscr{A}_K^{\varphi} \to L_K,$$

which is unique up to local L_K-conjugation, should be considered as the *"ultimate form" of the global non-abelian norm-residue symbol of K*, because of the following theorem:

Theorem 4.2. *The diagram*

$$\begin{tikzcd} \mathscr{W}\!\mathscr{A}_K^{\varphi} \arrow[r, "\mathsf{NR}_K^{\varphi^{\text{Langlands}}}"] \arrow[d, "f_K^{\varphi}"'] & L_K \arrow[d, "f_K"] \\ \mathscr{J}_K^{\varphi} \arrow[r, "\mathsf{NR}_K^{\varphi^{\text{Weil}}}"] & W_K \end{tikzcd} \tag{4.2}$$

is commutative, where the continuous homomorphism

$$\mathsf{NR}_K^{\varphi^{\text{Weil}}} : \mathscr{J}_K^{\varphi} \to W_K,$$

which is unique up to local W_K-conjugation, is the "Weil form" of the global non-abelian norm-residue symbol of K introduced in [12] *and studied in detail in* [13].

Proof. If $\upsilon \in \mathbb{h}_K$, the following diagram

$$\begin{tikzcd}
{}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})} \times \mathrm{SU}(2) \arrow[r, "\mathrm{Pr}_{{}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}}"] \arrow[d, "\{\bullet, K_\upsilon\}_{\varphi_{K_\upsilon}}^{\mathrm{Langlands}}"'] & {}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})} \arrow[d, "\{\bullet, K_\upsilon\}_{\varphi_{K_\upsilon}}^{\mathrm{Weil}}"] \\
L_{K_\upsilon} \arrow[r, "\mathrm{Pr}_{W_{K_\upsilon}}"] & W_{K_\upsilon}
\end{tikzcd} \tag{4.3}$$

is commutative. Moreover, by Theorem 2.16

$$\mathsf{NR}_K^{\varphi^{\mathrm{Weil}}} \circ f_K^{\varphi} \circ q_\upsilon : (\mathscr{W}\mathscr{A}_K^{\varphi})_\upsilon \xrightarrow{q_\upsilon} \mathscr{W}\mathscr{A}_K^{\varphi} \xrightarrow{f_K^{\varphi}} \mathscr{J}_K^{\varphi} \xrightarrow{\mathsf{NR}_K^{\varphi^{\mathrm{Weil}}}} W_K$$

is given by

$$\begin{aligned}
\mathsf{NR}_K^{\varphi^{\mathrm{Weil}}} \circ (f_K^{\varphi} \circ q_\upsilon) &= \mathsf{NR}_K^{\varphi^{\mathrm{Weil}}} \circ (q_\upsilon \circ \mathrm{Pr}_{{}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}}) \\
&= (\mathsf{NR}_K^{\varphi^{\mathrm{Weil}}} \circ q_\upsilon) \circ \mathrm{Pr}_{{}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}} \\
&= (e_\upsilon^{\mathrm{Weil}} \circ \{\bullet, K_\upsilon\}_{\varphi_{K_\upsilon}}^{\mathrm{Weil}}) \circ \mathrm{Pr}_{{}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}} \\
&= e_\upsilon^{\mathrm{Weil}} \circ (\{\bullet, K_\upsilon\}_{\varphi_{K_\upsilon}}^{\mathrm{Weil}} \circ \mathrm{Pr}_{{}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}}).
\end{aligned}$$

On the other hand, by the commutative diagram (3.1) and by the formal property (5) of the topological group L_K,

$$f_K \circ \mathsf{NR}_K^{\varphi^{\mathrm{Langlands}}} \circ q_\upsilon : (\mathscr{W}\mathscr{A}_K^{\varphi})_\upsilon \xrightarrow{q_\upsilon} \mathscr{W}\mathscr{A}_K^{\varphi} \xrightarrow{\mathsf{NR}_K^{\varphi^{\mathrm{Langlands}}}} L_K \xrightarrow{f_K} W_K$$

is given by

$$\begin{aligned}
f_K \circ (\mathsf{NR}_K^{\varphi^{\mathrm{Langlands}}} \circ q_\upsilon) &= f_K \circ (\mathsf{NR}_K^{\varphi^{\mathrm{Langlands}}})_\upsilon \\
&= f_K \circ (e_\upsilon^{\mathrm{Langlands}} \circ \{\bullet, K_\upsilon\}_{\varphi_{K_\upsilon}}^{\mathrm{Langlands}}) \\
&= (f_K \circ e_\upsilon^{\mathrm{Langlands}}) \circ \{\bullet, K_\upsilon\}_{\varphi_{K_\upsilon}}^{\mathrm{Langlands}} \\
&= (e_\upsilon^{\mathrm{Weil}} \circ f_{K_\upsilon}) \circ \{\bullet, K_\upsilon\}_{\varphi_{K_\upsilon}}^{\mathrm{Langlands}} \\
&= (e_\upsilon^{\mathrm{Weil}} \circ \mathrm{Pr}_{W_{K_\upsilon}}) \circ \{\bullet, K_\upsilon\}_{\varphi_{K_\upsilon}}^{\mathrm{Langlands}} \\
&= e_\upsilon^{\mathrm{Weil}} \circ (\mathrm{Pr}_{W_{K_\upsilon}} \circ \{\bullet, K_\upsilon\}_{\varphi_{K_\upsilon}}^{\mathrm{Langlands}}).
\end{aligned}$$

Therefore, the identity

$$\mathsf{NR}_K^{\varphi^{\mathrm{Weil}}} \circ f_K^{\varphi} \circ q_\upsilon = f_K \circ \mathsf{NR}_K^{\varphi^{\mathrm{Langlands}}} \circ q_\upsilon$$

follows from the commutativity of the diagram (4.3).

If $v \in \mathbb{a}_K$, again by Theorem 2.16,

$$\mathsf{NR}_K^{\varphi\,\mathrm{Weil}} \circ f_K^{\varphi} \circ q_v : (\mathscr{W}\mathscr{A}_K^{\varphi})_v \xrightarrow{q_v} \mathscr{W}\mathscr{A}_K^{\varphi} \xrightarrow{f_K^{\varphi}} \mathscr{J}_K^{\varphi} \xrightarrow{\mathsf{NR}_K^{\varphi\,\mathrm{Weil}}} W_K$$

is given by

$$\begin{aligned}\mathsf{NR}_K^{\varphi\,\mathrm{Weil}} \circ (f_K^{\varphi} \circ q_v) &= \mathsf{NR}_K^{\varphi\,\mathrm{Weil}} \circ (q_v \circ \mathrm{id}_{W_{K_v}}) \\ &= (\mathsf{NR}_K^{\varphi\,\mathrm{Weil}} \circ q_v) \circ \mathrm{id}_{W_{K_v}} \\ &= e_v^{\mathrm{Weil}} \circ \mathrm{id}_{W_{K_v}},\end{aligned}$$

and again by the commutative diagram (3.1),

$$f_K \circ \mathsf{NR}_K^{\varphi\,\mathrm{Langlands}} \circ q_v : (\mathscr{W}\mathscr{A}_K^{\varphi})_v \xrightarrow{q_v} \mathscr{W}\mathscr{A}_K^{\varphi} \xrightarrow{\mathsf{NR}_K^{\varphi\,\mathrm{Langlands}}} L_K \xrightarrow{f_K} W_K$$

is given by

$$\begin{aligned}f_K \circ (\mathsf{NR}_K^{\varphi\,\mathrm{Langlands}} \circ q_v) &= f_K \circ (\mathsf{NR}_K^{\varphi\,\mathrm{Langlands}})_v \\ &= f_K \circ e_v^{\mathrm{Langlands}} \\ &= e_v^{\mathrm{Weil}} \circ f_{K_v}.\end{aligned}$$

Now, the formal property (5) of L_K proves the identity

$$\mathsf{NR}_K^{\varphi\,\mathrm{Weil}} \circ f_K^{\varphi} \circ q_v = f_K \circ \mathsf{NR}_K^{\varphi\,\mathrm{Langlands}} \circ q_v.$$

So, for every $v \in \mathbb{h}_K \cup \mathbb{a}_K$, the identity

$$\left(\mathsf{NR}_K^{\varphi\,\mathrm{Weil}} \circ f_K^{\varphi}\right)_v = \left(f_K \circ \mathsf{NR}_K^{\varphi\,\mathrm{Langlands}}\right)_v$$

follows. Therefore, by Theorems 2.2, 2.3, and 2.4,

$$\mathsf{NR}_K^{\varphi\,\mathrm{Weil}} \circ f_K^{\varphi} = f_K \circ \mathsf{NR}_K^{\varphi\,\mathrm{Langlands}},$$

which proves that the diagram (4.2) is commutative. □

The following definition singles out an extremely important closed normal subgroup of $\mathscr{W}\mathscr{A}_K^{\varphi_K}$.

Definition 4.3. The kernel $\ker(\mathsf{NR}_K^{\varphi_K\,\mathrm{Langlands}})$ of the global non-abelian norm-residue symbol

$$\mathsf{NR}_K^{\varphi_K\,\mathrm{Langlands}} : \mathscr{W}\mathscr{A}_K^{\varphi_K} \to L_K$$

of K in Langlands form is denoted by $\mathscr{N}_K^{\varphi_K\,\mathrm{Langlands}}$ and called the automorphic kernel of $\mathscr{W}\mathscr{A}_K^{\varphi_K}$.

Remark 4.4. Although the global non-abelian norm-residue symbol

$$\mathsf{NR}_K^{\varphi\,\mathrm{Langlands}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K$$

of K in Langlands form is defined up to local L_K-conjugation, by Theorem 2.6, the automorphic kernel ${\mathscr{N}_K^{\varphi_K}}^{\mathrm{Langlands}}$ of $\mathscr{W}\mathscr{A}_K^{\varphi_K}$ is well-defined in the sense that it depends only on the local L_K-conjugacy class.

Moreover, in [12], we proposed the following "meta-conjecture":

Conjecture 4.5. *The continuous homomorphism*

$$\mathsf{NR}_K^{\varphi\,\mathrm{Langlands}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K$$

is open[6] *and surjective.*

This completes the detailed discussion of the first part of our Seoul communication and Section 8 of [12]. We close this section with a remark:

Remark 4.6. Let G be a connected (quasisplit) reductive group over K. In the remaining part of our Seoul communication, we introduced certain homomorphisms

$$\phi : \mathscr{W}\mathscr{A}_K^{\varphi} \to {}^L\mathrm{G}(\mathbb{C}),$$

whose equivalence classes are called the "WA-parameters" for G over K. We plan to discuss the possible applications of "WA-parameters" for G over K to the Langlands reciprocity and functoriality principles for G over K in a separate work.

5. Basic properties of the global non-abelian norm-residue symbol in Langlands form

In this section, we shall list the basic properties of the global non-abelian norm-residue symbol

$$\mathsf{NR}_K^{\varphi\,\mathrm{Langlands}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K$$

of K in Langlands form which is defined up to L_K-conjugation. We shall skip the proofs of these properties, because all of them follows from the proofs of the Weil form counterparts of the basic properties presented in the Lahore paper [13, Section 4] with very minor modifications.

[6]So that, the induced map $\mathscr{W}\mathscr{A}_K^{\varphi}/{\mathscr{N}_K^{\varphi_K}}^{\mathrm{Langlands}} \xrightarrow{\sim} L_K$ is a topological group isomorphism.

5.1. Local-global compatibility of the non-abelian norm-residue symbols in Langlands form

The *"local-global compatibility"* of $\{\bullet, K_v\}^{\mathrm{Langlands}}_{\varphi_{K_v}}$ and $\mathsf{NR}^{\varphi^{\mathrm{Langlands}}}_{K}$ states the commutativity of the square

$$\begin{tikzcd} {}_{\mathbb{Z}}\nabla^{(\varphi_{K_v})}_{K_v} \times \mathrm{SU}(2) \arrow[r, "q_v"] \arrow[d, "\{\bullet, K_v\}^{\mathrm{Langlands}}_{\varphi_{K_v}}"'] & \mathscr{W}\mathscr{A}^{\varphi}_{K} \arrow[d, "\mathsf{NR}^{\varphi}_{K}{}^{\mathrm{Langlands}}"] \\ L_{K_v} \arrow[r, "e_v^{\mathrm{Langlands}}"'] & L_K \end{tikzcd}$$

for $v \in \mathbb{h}_K$. Note that both $e_v^{\mathrm{Langlands}} : L_{K_v} \to L_K$ and $\mathsf{NR}^{\varphi^{\mathrm{Langlands}}}_{K} : \mathscr{W}\mathscr{A}^{\varphi}_{K} \to L_K$ are defined up to L_K-conjugation for $v \in \mathbb{h}_K$. Look at [12, Theorem 4.1].

5.2. Relationship with the global abelian norm-residue symbol

The global non-abelian norm-residue symbol

$$\mathsf{NR}^{\varphi^{\mathrm{Langlands}}}_{K} : \mathscr{W}\mathscr{A}^{\varphi}_{K} \to L_K$$

of K in Langlands form defined up to L_K-conjugation sits in the following commutative diagram

$$\begin{tikzcd} \mathscr{W}\mathscr{A}^{\varphi}_{K} \arrow[r, "\mathsf{NR}^{\varphi}_{K}{}^{\mathrm{Langlands}}"] \arrow[d, "\mathrm{ab}"] \arrow[dd, bend right=60, "(\text{Remark 2.8})\ \mathsf{a}_K = \mathsf{s}_K"'] & L_K \arrow[d, "\mathrm{ab}"] \\ \mathscr{W}\mathscr{A}^{\varphi^{\mathrm{ab}}}_{K} \arrow[d, "\wr"', "(\text{Thm. 2.14})"] & L_K^{\mathrm{ab}} \arrow[d, "\wr"', "f_K^{\mathrm{ab}} (\text{Formal Prop. 3})"] \\ \mathbb{J}_K \arrow[r, "\text{Global abelian CFT}", "(\bullet, K)"'] & W_K^{\mathrm{ab}}. \end{tikzcd}$$

The proof is similar to the proof of Theorem 2 of [13].

5.3. Formal properties of the hypothetical group L_K (Part II)

We follow closely Tate [26] in this subsection.

The absolute Weil group W_K of the number field K has the following property. Let E/K be a finite extension. The absolute Weil group W_E of E is the open subgroup of W_K defined by

$$W_E = \beta_K^{-1}(G_E),$$

where the absolute Weil group W_K of the number field K comes equipped with a continuous homomorphism $\beta_K : W_K \to G_K$ with dense image. Moreover, the open subgroup W_E of W_K is equipped with a continuous homomorphism $\beta_E : W_E \to G_E$ which sits in the commutative square

$$\begin{tikzcd} W_E \arrow[r, "\beta_E"] \arrow[d, "\gamma_{E/K}"'] & G_E \arrow[d, "\text{inc.}"] \\ W_K \arrow[r, "\beta_K"'] & G_K \end{tikzcd} \tag{5.1}$$

and with dense image, where the left vertical arrow

$$\gamma_{E/K} : W_E \hookrightarrow W_K$$

is the natural embedding; that is, the identity map defined by the inclusion mapping $W_E := \beta_K^{-1}(G_E) \subset W_K$.

The local version of the property of W_K just stated now is as follows. Let E/K be a finite extension. Let $\upsilon \in \mathbb{h}_K \cup \mathbb{c}_K$ and $\mu \in \mathbb{h}_E \cup \mathbb{c}_E$ such that $\mu \mid \upsilon$. Then, E_μ is a finite extension of K_υ. The absolute Weil group W_{E_μ} of E_μ is the open subgroup of W_{K_υ} defined by

$$W_{E_\mu} = \beta_{K_\upsilon}^{-1}(G_{E_\mu}),$$

where the absolute Weil group W_{K_υ} of the local field K_υ comes equipped with a continuous homomorphism $\beta_{K_\upsilon} : W_{K_\upsilon} \to G_{K_\upsilon}$ with dense image. Likewise, the open subgroup W_{E_μ} of W_{K_υ} is equipped with a continuous homomorphism $\beta_{E_\mu} : W_{E_\mu} \to G_{E_\mu}$ with dense image and the square

$$\begin{tikzcd} W_{E_\mu} \arrow[r, "\beta_{E_\mu}"] \arrow[d, "\gamma_{E_\mu/K_\upsilon}"'] & G_{E_\mu} \arrow[d, "\text{inc.}"] \\ W_{K_\upsilon} \arrow[r, "\beta_{K_\upsilon}"'] & G_{K_\upsilon} \end{tikzcd} \tag{5.2}$$

commutes, where the left vertical arrow

$$\gamma_{E_\mu/K_\upsilon} : W_{E_\mu} \hookrightarrow W_{K_\upsilon}$$

is the natural embedding; namely, the identity mapping defined by the inclusion $W_{E_\mu} := \beta_{K_\upsilon}^{-1}(G_{E_\mu}) \subset W_{K_\upsilon}$.

The commutative diagrams (5.1) and (5.2) are related with each other as well. In fact, any embedding $e_\upsilon : K^{\text{sep}} \hookrightarrow K_\upsilon^{\text{sep}}$ defines conjugacy classes of embeddings e_μ^{Galois} and e_μ^{Weil} as well as $e_\upsilon^{\text{Galois}}$ and e_υ^{Weil}, since $e_\upsilon : K^{\text{sep}} \hookrightarrow K_\upsilon^{\text{sep}}$ naturally and uniquely defines an embedding $e_\mu : E^{\text{sep}} \hookrightarrow E_\mu^{\text{sep}}$, and the following diagram

$$\begin{tikzcd}
 & W_E \arrow[rr, "\beta_E"] \arrow[dd, "\gamma_{E/K}"] & & G_E \arrow[dd, "\text{inc.}"] \\
W_{E_\mu} \arrow[ur, "e_\mu^{\text{Weil}}"] \arrow[rr, "\beta_{E_\mu}"] \arrow[dd, "\gamma_{E_\mu/K_\upsilon}"'] & & G_{E_\mu} \arrow[ur, "e_\mu^{\text{Galois}}"] \arrow[dd, "\text{inc.}"] & \\
 & W_K \arrow[rr, "\beta_K"] & & G_K \\
W_{K_\upsilon} \arrow[ur, "e_\upsilon^{\text{Weil}}"] \arrow[rr, "\beta_{K_\upsilon}"] & & G_{K_\upsilon} \arrow[ur, "e_\upsilon^{\text{Galois}}"] &
\end{tikzcd}$$

is commutative.

So, it is natural to expect that the hypothetical locally compact group L_K has the following property:

(7) If E/K is a finite extension, then L_E is an open subgroup of L_K defined by
$$L_E = f_K^{-1}(\gamma_{E/K}(W_E)),$$
where the absolute Langlands group L_K of the number field K comes equipped with a surjective topological group homomorphism $f_K : L_K \to W_K$. Moreover, the open subgroup L_E of L_K is equipped with a surjective topological group homomorphism $f_E : L_E \to W_E$ which sits in the commutative diagram

$$\begin{tikzcd}
L_E \arrow[r, "f_E"] \arrow[d, "\omega_{E/K}"'] & W_E \arrow[r, "\beta_E"] \arrow[d, "\gamma_{E/K}"'] & G_E \arrow[d, "\text{inc.}"] \\
L_K \arrow[r, "f_K"'] & W_K \arrow[r, "\beta_K"'] & G_K
\end{tikzcd} \tag{5.3}$$

where the left vertical arrow
$$\omega_{E/K} : L_E \hookrightarrow L_K$$
is the natural embedding; that is, the identity map defined by the inclusion $L_E := f_K^{-1}(\gamma_{E/K}(W_E)) \subset L_K$.

The expected property of the hypothetical locally compact group L_K just stated above is listed as the formal property (7) of L_K. There is also the local analogue of the formal property (7) of L_K, which is no longer an expectation, but a fact. Let again E/K be a finite extension. Let $\upsilon \in \mathbb{h}_K \cup \mathbb{0}_K$ and $\mu \in \mathbb{h}_E \cup \mathbb{0}_E$ such that $\mu \mid \upsilon$. Then, E_μ is a finite extension of K_υ. The formal property (8) of L_K states that:

(8) L_{E_μ} is the open subgroup of L_{K_υ} satisfying
$$L_{E_\mu} = f_{K_\upsilon}^{-1}(\gamma_{E_\mu/K_\upsilon}(W_{E_\mu})),$$

where the absolute Langlands group L_{K_υ} of the local field K_υ comes equipped with a surjective continuous homomorphism $f_{K_\upsilon} : L_{K_\upsilon} \to W_{K_\upsilon}$. Likewise, the open subgroup L_{E_μ} of L_{K_υ} is equipped with a surjective continuous homomorphism $f_{E_\mu} : L_{E_\mu} \to W_{E_\mu}$, and the rectangle

$$\begin{tikzcd}
L_{E_\mu} \arrow[r, "f_{E_\mu}"] \arrow[d, "\omega_{E_\mu/K_\upsilon}"'] & W_{E_\mu} \arrow[r, "\beta_{E_\mu}"] \arrow[d, "\gamma_{E_\mu/K_\upsilon}"'] & G_{E_\mu} \arrow[d, "\text{inc.}"] \\
L_{K_\upsilon} \arrow[r, "f_{K_\upsilon}"'] & W_{K_\upsilon} \arrow[r, "\beta_{K_\upsilon}"'] & G_{K_\upsilon}
\end{tikzcd} \tag{5.4}$$

is commutative, where the left vertical arrow

$$\omega_{E_\mu/K_\upsilon} : L_{E_\mu} \hookrightarrow L_{K_\upsilon}$$

is the natural embedding; that is, the identity map defined by the inclusion $L_{E_\mu} := f_{K_\upsilon}^{-1}(\gamma_{E_\mu/K_\upsilon}(W_{E_\mu})) \subset L_{K_\upsilon}$.

The commutative diagrams (5.3) and (5.4) are related with each other as follows: any embedding $e_\upsilon : K^{\text{sep}} \hookrightarrow K_\upsilon^{\text{sep}}$ defines conjugacy classes of embeddings e_μ^{Galois}, e_μ^{Weil} and $e_\mu^{\text{Langlands}}$ as well as $e_\upsilon^{\text{Galois}}$, e_υ^{Weil} and $e_\upsilon^{\text{Langlands}}$, since $e_\upsilon : K^{\text{sep}} \hookrightarrow K_\upsilon^{\text{sep}}$ naturally and uniquely defines an embedding $e_\mu : E^{\text{sep}} \hookrightarrow E_\mu^{\text{sep}}$, and the following diagram

$$\begin{tikzcd}
 & L_E \arrow[rr, "f_E"] \arrow[dd, "\omega_{E/K}"'] & & W_E \arrow[rr, "\beta_E"] \arrow[dd, "\gamma_{E/K}"'] & & G_E \arrow[dd, "\text{inc.}"] \\
L_{E_\mu} \arrow[ur, "e_\mu^{\text{Langlands}}"] \arrow[rr, "f_{E_\mu}"] \arrow[dd, "\omega_{E_\mu/K_\upsilon}"'] & & W_{E_\mu} \arrow[ur, "e_\mu^{\text{Weil}}"] \arrow[rr, "\beta_{E_\mu}"] \arrow[dd, "\gamma_{E_\mu/K_\upsilon}"'] & & G_{E_\mu} \arrow[ur, "e_\mu^{\text{Galois}}"] \arrow[dd, "\text{inc.}"'] & \\
 & L_K \arrow[rr, "f_K"] & & W_K \arrow[rr, "\beta_K"] & & G_K \\
L_{K_\upsilon} \arrow[ur, "e_\upsilon^{\text{Langlands}}"] \arrow[rr, "f_{K_\upsilon}"] & & W_{K_\upsilon} \arrow[ur, "e_\upsilon^{\text{Weil}}"] \arrow[rr, "\beta_{K_\upsilon}"] & & G_{K_\upsilon} \arrow[ur, "e_\upsilon^{\text{Galois}}"] &
\end{tikzcd}$$

is commutative.

5.4. Weil-Arthur idèles in field extensions

Let E be a finite extension of the number field K. In this subsection, we shall discuss the relationship between the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}_K}$ of K and the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}_E^{\underline{\varphi}_E}$ of E.

Remark 5.1. Let E be a finite extension of the number field K. Fixing $\underline{\varphi}_K = \{\varphi_{K_\upsilon}\}_{\upsilon \in \mathbb{h}_K}$ uniquely determines $\underline{\varphi}_E = \{\varphi_{E_\mu}\}_{\mu \in \mathbb{h}_E}$ via Koch-de Shalit process applied to compatible extensions of K_υ for each $\upsilon \in \mathbb{h}_K$. For details, look at [15, Section 7] and [17, p. 89].

Now, for any $\upsilon \in \mathbb{h}_K$ and for any $\mu \in \mathbb{h}_E$ satisfying $\mu \mid \upsilon$, define a continuous homomorphism

$$\mathscr{N}^{\infty}_{E_\mu/K_\upsilon} \times \mathrm{id}_{\mathrm{SU}(2)} : {}_{\mathbb{Z}}\nabla^{(\varphi_{E_\mu})}_{E_\mu} \times \mathrm{SU}(2) \to {}_{\mathbb{Z}}\nabla^{(\varphi_{K_\upsilon})}_{K_\upsilon} \times \mathrm{SU}(2)$$

via the commutative diagram

$$\begin{tikzcd}
{}_{\mathbb{Z}}\nabla^{(\varphi_{E_\mu})}_{E_\mu} \arrow[r, "\{\bullet, E_\mu\}^{\mathrm{Weil}}_{\varphi_{E_\mu}}", "\sim"'] \arrow[d, "\mathscr{N}^{\infty}_{E_\mu/K_\upsilon}"'] & W_{E_\mu} \arrow[d, "\gamma_{E_\mu/K_\upsilon}"] \\
{}_{\mathbb{Z}}\nabla^{(\varphi_{K_\upsilon})}_{K_\upsilon} & W_{K_\upsilon} \arrow[l, "\sim"', "\mathbf{\Phi}^{(\varphi_{K_\upsilon})\mathrm{Weil}}_{K_\upsilon}"]
\end{tikzcd}$$

introduced in [15, p. 39]. Then, clearly the following square

$$\begin{tikzcd}
{}_{\mathbb{Z}}\nabla^{(\varphi_{E_\mu})}_{E_\mu} \times \mathrm{SU}(2) \arrow[r, "\{\bullet, E_\mu\}^{\mathrm{Langlands}}_{\varphi_{E_\mu}}", "\sim"'] \arrow[d, "\mathscr{N}^{\infty}_{E_\mu/K_\upsilon} \times \mathrm{id}_{\mathrm{SU}(2)}"'] & L_{E_\mu} \arrow[d, "\gamma_{E_\mu/K_\upsilon} \times \mathrm{id}_{\mathrm{SU}(2)} = \omega_{E_\mu/K_\upsilon}"] \\
{}_{\mathbb{Z}}\nabla^{(\varphi_{K_\upsilon})}_{K_\upsilon} \times \mathrm{SU}(2) & L_{K_\upsilon} \arrow[l, "\sim"', "\mathbf{\Phi}^{(\varphi_{K_\upsilon})\mathrm{Langlands}}_{K_\upsilon}"]
\end{tikzcd} \tag{5.5}$$

is commutative.

Thus, for every $\mu \in \mathbb{h}_E$, there exists a continuous homomorphism

$${}_{\mathbb{Z}}\nabla^{(\varphi_{E_\mu})}_{E_\mu} \times \mathrm{SU}(2) \to \mathscr{W}\mathscr{A}^{\varphi_K}_K$$

defined by the composition

$$q_\upsilon \circ \mathscr{N}^{\infty}_{E_\mu/K_\upsilon} \times \mathrm{id}_{\mathrm{SU}(2)} : {}_{\mathbb{Z}}\nabla^{(\varphi_{E_\mu})}_{E_\mu} \times \mathrm{SU}(2) \xrightarrow{\mathscr{N}^{\infty}_{E_\mu/K_\upsilon} \times \mathrm{id}_{\mathrm{SU}(2)}} {}_{\mathbb{Z}}\nabla^{(\varphi_{K_\upsilon})}_{K_\upsilon} \times \mathrm{SU}(2) \xrightarrow{q_\upsilon} \mathscr{W}\mathscr{A}^{\varphi_K}_K,$$

and for every $\mu \in \mathbb{o}_E$, a continuous homomorphism

$$W_{E_\mu} \to \mathscr{W}\mathscr{A}^{\varphi_K}_K$$

defined by the composition

$$q_\upsilon \circ \gamma_{E_\mu/K_\upsilon} : W_{E_\mu} \underset{\gamma_{E_\mu/K_\upsilon}}{\hookrightarrow} W_{K_\upsilon} \xrightarrow{q_\upsilon} \mathscr{W}\mathscr{A}^{\varphi_K}_K.$$

So, by Theorem 2.2, there exists a unique continuous homomorphism

$$\mathscr{N}^{\infty}_{E/K} : \mathscr{W}\mathscr{A}^{\varphi_E}_E \to \mathscr{W}\mathscr{A}^{\varphi_K}_K,$$

called the norm homomorphism from the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}^{\varphi_E}_E$ of E to the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}^{\varphi_K}_K$ of K.

Modifying the proof of Proposition 3 of [13], we see that the norm homomorphism is transitive; namely

$$\mathscr{N}^{\infty}_{F/K} = \mathscr{N}^{\infty}_{E/K} \circ \mathscr{N}^{\infty}_{F/E}$$

for any tower of finite extensions $K \subseteq E \subseteq F$ of the number field K.

The relationship between the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}_K^{\varphi_K}$ of K and the Weil-Arthur idèle group $\mathscr{W}\mathscr{A}_E^{\varphi_E}$ of E is encoded in the following commutative square

$$\begin{tikzcd} \mathscr{W}\mathscr{A}_E^{\varphi_E} \arrow[r, "\mathsf{NR}_E^{\varphi_E}{}^{\text{Langlands}}"] \arrow[d, "\mathscr{N}_{E/K}^{\infty}"'] & L_E \arrow[d, "\omega_{E/K}"] \\ \mathscr{W}\mathscr{A}_K^{\varphi_K} \arrow[r, "\mathsf{NR}_K^{\varphi_K}{}^{\text{Langlands}}"'] & L_K, \end{tikzcd}$$

where the commutativity of the diagram follows by an argument similar to the one given in the proof of Theorem 3 of [13].

5.5. Relative global non-abelian norm-residue symbols in Langlands form

For each finite extension E/K, observe that, there exists a well-defined topological bijection

$$f_{E/K} : L_K/L_E \to W_K/\gamma_{E/K}(W_E)$$

defined by

$$f_{E/K} : xL_E \mapsto f_K(x)\gamma_{E/K}(W_E)$$

for every $x \in L_K$. Therefore, there exist the following topological bijections of homogenous spaces:

$$L_K/L_E \xrightarrow{f_{E/K}} W_K/\gamma_{E/K}(W_E) \xrightarrow{\beta_{E/K}} G_K/G_E.$$

In particular, if E/K is furthermore Galois, the topological bijection $f_{E/K} : L_K/L_E \to W_K/\gamma_{E/K}(W_E)$ is furthermore an isomorphism of topological groups.

Therefore, for any finite Galois extension E/K, there exists a continuous homomorphism

$$\mathsf{NR}_{E/K}^{\varphi_K}{}^{\text{Langlands}} : \mathscr{W}\mathscr{A}_K^{\varphi_K} \to \mathrm{Gal}(E/K)$$

defined up to $\mathrm{Gal}(E/K)$-conjugation, called the global non-abelian norm-residue symbol in Langlands form relative to the extension E/K, which is defined as the top horizontal arrow that makes the diagram

$$\begin{tikzcd} \mathscr{W}\mathscr{A}_K^{\varphi_K} \arrow[rrr, dotted, "\mathsf{NR}_{E/K}^{\varphi_K}{}^{\text{Langlands}}"] \arrow[d, "\mathsf{NR}_K^{\varphi_K}{}^{\text{Langlands}}"'] & & & \mathrm{Gal}(E/K) \\ L_K \arrow[r, two heads, "\mathrm{mod}_{L_E}"'] & L_K/L_E \arrow[r, "\sim", "f_{E/K}"'] & W_K/\gamma_{E/K}(W_E) \arrow[r, "\sim", "\beta_{E/K}"'] & G_K/G_E \arrow[u, "\wr"', "\mathrm{res}_E^*"] \end{tikzcd}$$

commutative. Recall that the arrow $\mathsf{NR}_K^{\varphi_K^{\mathrm{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi_K} \to L_K$ is defined up to L_K-conjugation.

Observe that, if Conjecture 4.5 holds, then obviously, the continuous homomorphism

$$\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi_K} \to \mathrm{Gal}(E/K) \tag{5.6}$$

is surjective. Moreover, an argument similar to the proof of Theorem 4 of [13] shows that this arrow has open kernel

$$\ker(\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Langlands}}}) = \mathscr{N}_{E/K}^{\infty}(\mathscr{W}\mathscr{A}_E^{\varphi_E})\mathscr{N}_K^{\varphi_K}{}^{\mathrm{Langlands}}$$

inducing a topological group isomorphism

$$\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Langlands}^*}} : \mathscr{W}\mathscr{A}_K^{\varphi_K}/\mathscr{N}_{E/K}^{\infty}(\mathscr{W}\mathscr{A}_E^{\varphi_E})\mathscr{N}_K^{\varphi_K}{}^{\mathrm{Langlands}} \xrightarrow{\sim} \mathrm{Gal}(E/K).$$

5.6. Relationship with the relative global abelian norm-residue symbol

Let E/K be a finite Galois extension. Let $(E/K)^{\mathrm{ab}}$ denote the maximal abelian extension of K inside E.

The global non-abelian norm-residue symbol in Langlands form

$$\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to \mathrm{Gal}(E/K)$$

relative to the extension E/K, which is defined up to $\mathrm{Gal}(E/K)$-conjugation, sits in the following commutative diagram

$$\begin{tikzcd}
\mathscr{W}\mathscr{A}_K^{\varphi_K} \arrow[rr, "\mathsf{NR}_{E/K}^{\varphi_K^{\mathrm{Langlands}}}"] \arrow[d, "\mathrm{ab}"] \arrow[dd, bend right, "(\text{Remark 2.8})\ \mathsf{a}_K=\mathsf{s}_K"'] & & \mathrm{Gal}(E/K) \arrow[d, "\mathrm{ab}"] \\
\mathscr{W}\mathscr{A}_K^{\varphi_K^{\mathrm{ab}}} \arrow[d, "\wr", "(\text{Thm. 2.14})"'] & & \mathrm{Gal}(E/K)^{\mathrm{ab}} \arrow[d, "\wr"] \\
\mathbb{J}_K \arrow[rr, "\text{Global abelian CFT}", "(\bullet{,}E/K)"'] & & \mathrm{Gal}((E/K)^{\mathrm{ab}}/K).
\end{tikzcd}$$

The proof of the commutativity of the above diagram is similar to the proof of Theorem 5 of [13].

5.7. Decomposition and inertia groups

Maintain the assumptions and the notation introduced in Subsection 5.5. Let E/K be a finite Galois extension. For $\upsilon \in \mathbb{h}_K \cup \mathbb{o}_K$, let $D_\upsilon := \mathrm{D}_\upsilon(E/K)$ and $I_\upsilon := \mathrm{I}_\upsilon(E/K)$ denote respectively the decomposition and the inertia groups of υ in $\mathrm{Gal}(E/K)$ determined by the continuous homomorphism $e_\upsilon^{\mathrm{Langlands}}$:

$L_{K_v} \to L_K$ defined up to L_K-conjugation. That is, the subgroups D_v and I_v of $\mathrm{Gal}(E/K)$ are defined up to $\mathrm{Gal}(E/K)$-conjugation by

$$D_v = \mathrm{res}_E^* \circ \beta_{E/K} \circ f_{E/K} \circ \mathrm{mod}_{L_E} \circ e_v^{\mathrm{Langlands}}(L_{K_v})$$

and

$$I_v = \mathrm{res}_E^* \circ \beta_{E/K} \circ f_{E/K} \circ \mathrm{mod}_{L_E} \circ e_v^{\mathrm{Langlands}}(L_{K_v}^0),$$

where for $v \in \mathbb{h}_K$, the group $L_{K_v}^0$ is defined by $W_{K_v}^0 \times \mathrm{SU}(2)$ and for $v \in \mathbb{a}_K$, the group $W_{K_v}^0$ is defined by $W_{K_v}^0 = W_{K_v}$.

For every prime $v \in \mathbb{h}_K \cup \mathbb{a}_K$,

- The image of the continuous homomorphism defined by the composition

$$\left(\mathscr{W}\mathscr{A}_K^{\varphi_K}\right)_v = \left\{\begin{matrix} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SU}(2), & v \in \mathbb{h}_K \\ W_{\mathbb{R}}, & v \in \mathbb{a}_{K,\mathbb{R}} \\ W_{\mathbb{C}}, & v \in \mathbb{a}_{K,\mathbb{C}} \end{matrix}\right\} \xrightarrow{q_v} \mathscr{W}\mathscr{A}_K^{\varphi_K} \xrightarrow{\mathsf{NR}_{E/K}^{\varphi_K \,\mathrm{Langlands}}} \mathrm{Gal}(E/K)$$

  is the decomposition group D_v of v in $\mathrm{Gal}(E/K)$, which is unique up to $\mathrm{Gal}(E/K)$-conjugation, determined by the continuous homomorphism $e_v^{\mathrm{Langlands}} : L_{K_v} \to L_K$, defined up to L_K-conjugation;
- The image of the continuous homomorphism defined by the composition

$$\left(\mathscr{W}\mathscr{A}_K^{\varphi_K}\right)_v^0 := \left\{\begin{matrix} {}_{1}\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}} \times \mathrm{SU}(2), & v \in \mathbb{h}_K \\ W_{\mathbb{R}}^0, & v \in \mathbb{a}_{K,\mathbb{R}} \\ W_{\mathbb{C}}^0, & v \in \mathbb{a}_{K,\mathbb{C}} \end{matrix}\right\} \xrightarrow{q_v} \mathscr{W}\mathscr{A}_K^{\varphi_K} \xrightarrow{\mathsf{NR}_{E/K}^{\varphi_K \,\mathrm{Langlands}}} \mathrm{Gal}(E/K)$$

  is the inertia group I_v of v in $\mathrm{Gal}(E/K)$, unique up to $\mathrm{Gal}(E/K)$-conjugation, determined by the continuous homomorphism $e_v^{\mathrm{Langlands}} : L_{K_v} \to L_K$, defined up to L_K-conjugation.

The proof is a straightforward modification of Theorem 6 of [13].

5.8. Basic functorial properties

Let $K \subseteq E \subseteq F$ be a tower of finite Galois extensions of the number field K. Then,

- The triangle

$$\begin{tikzcd} \mathscr{W}\mathscr{A}_K^{\varphi_K} \arrow[r, "\mathsf{NR}_{F/K}^{\varphi_K \,\mathrm{Langlands}}"] \arrow[dr, "\mathsf{NR}_{E/K}^{\varphi_K \,\mathrm{Langlands}}"'] & \mathrm{Gal}(F/K) \arrow[d, "\mathrm{res}_E"] \\ & \mathrm{Gal}(E/K) \end{tikzcd}$$

  where the right vertical arrow is the restriction map to E, is commutative;

– The square

$$\begin{tikzcd} \mathscr{W}\mathscr{A}_{E}^{\varphi_E} \arrow[r, "\mathsf{NR}_{F/E}^{\varphi_E \, \mathrm{Langlands}}"] \arrow[d, "\mathscr{N}_{E/K}^{\infty}"'] & \mathrm{Gal}(F/E) \arrow[d, "\mathrm{inc.}"] \\ \mathscr{W}\mathscr{A}_{K}^{\varphi_K} \arrow[r, "\mathsf{NR}_{F/K}^{\varphi_K \, \mathrm{Langlands}}"'] & \mathrm{Gal}(F/K) \end{tikzcd}$$

is commutative.

The proofs of the commutativity of the above two diagrams are similar to the proofs of Theorems 7 and 8 of [13].

5.9. Global non-abelian existence theorem in "Langlands form"

In this subsection, we assume that Conjecture 4.5 holds.

The non-abelian generalization of the existence theorem of global abelian class field theory, in "Langlands form", states the existence of an inclusion-reversing bijective correspondence

$$\left\{\begin{matrix}\text{Finite Galois exten–}\\ \text{sions of } K \text{ inside } K^{sep}\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{Open normal subgroups of finite index}\\ \text{in } \mathscr{W}\mathscr{A}_{K}^{\varphi_K} \text{ containing } \mathscr{N}_{K}^{\varphi_K \, \mathrm{Langlands}}\end{matrix}\right\},$$

which is defined by

$$E \mapsto \mathscr{N}_{E/K}^{\infty}(\mathscr{W}\mathscr{A}_{E}^{\varphi_E})\mathscr{N}_{K}^{\varphi_K \, \mathrm{Langlands}}$$

for every finite Galois extension E of K inside K^{sep}, where $\mathscr{N}_{K}^{\varphi_K \, \mathrm{Langlands}} = \ker(\mathsf{NR}_{K}^{\varphi_K \, \mathrm{Langlands}})$ is the automorphic kernel of $\mathscr{W}\mathscr{A}_{K}^{\varphi_K}$ introduced in Definition 4.3. Note that, the proof is similar to the one presented in Subsection 4.8 of [13].

5.10. Non-abelian ray class groups and non-abelian ray class fields

Let $S = \mathbb{O}_{K,\mathbb{C}}$ and $\mathfrak{m}$ be an S-cycle (=S-modulus) of the number field K. Note that, an S-cycle $\mathfrak{m} = \prod_{v \in \mathbb{h}_K \cup \mathbb{O}_K} v^{e_v}$ of K canonically defines a subgroup $\mathscr{U}_{\mathfrak{m}}^{\varphi_K}$ of $\mathscr{W}\mathscr{A}_{K}^{\varphi_K}$ by the free product

$$\mathscr{U}_{\mathfrak{m}}^{\varphi_K} = *_v \mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})},$$

where the local groups $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})}$ for $v \in \mathbb{h}_K \cup \mathbb{O}_K$ are defined as follows:

- $v \in \mathbb{h}_K$ and $e_v = 0$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = {}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}} \times \mathrm{SU}(2)$;
- $v \in \mathbb{h}_K$ and $e_v > 0$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = {}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{e_v}}} \times \mathrm{SU}(2)$;
- $v \in \mathbb{O}_{K,\mathbb{R}}$ and $e_v = 0$: $\mathscr{U}_{\mathfrak{m},v}^{(\varphi_{K_v})} = W_{\mathbb{R}}$;

– $\upsilon \in \mathbb{O}_{K,\mathbb{R}}$ and $e_\upsilon = 1$: $\mathscr{U}_{\mathfrak{m},\upsilon}^{(\varphi_{K_\upsilon})} = W_{\mathbb{R},>0}$, where $W_{\mathbb{R},>0}$ is the subgroup of $W_{\mathbb{R}}$ which is defined as the pre-image of the subgroup $\mathbb{R}_{>0}$ of $\mathbb{R}^\times$ under the natural abelianization homomorphism $W_{\mathbb{R}} \to W_{\mathbb{R}}^{ab} \xrightarrow{\sim} \mathbb{R}^\times$ of $W_{\mathbb{R}}$;
– $\upsilon \in \mathbb{O}_{K,\mathbb{C}}$: So $e_\upsilon = 0$ and we set $\mathscr{U}_{\mathfrak{m},\upsilon}^{(\varphi_{K_\upsilon})} = W_{\mathbb{C}}$.

For the definition and the basic properties of the subgroup ${}_1\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})^{\underline{i}}}$ of ${}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}$, where $\underline{i}$ is an "increasing net" in $\mathbb{R}_{\geq -1}$, look at [16].

Now, assume that Conjecture 4.5 holds in this subsection.
For any S-cycle $\mathfrak{m} = \prod_{\upsilon \in \mathbb{h}_K \cup \mathbb{O}_K} \upsilon^{e_\upsilon}$ of K, let $\mathscr{V}_{\mathfrak{m}}^{\varphi_K}$ denote the smallest open normal subgroup of finite index in $\mathscr{W}\!\mathscr{A}_K^{\varphi_K}$ containing ${\mathscr{N}_K^{\varphi_K}}^{\text{Langlands}}\mathscr{U}_{\mathfrak{m}}^{\varphi_K}$. By Subsection 5.9 on the global non-abelian existence theorem in Langlands form, there exists a finite Galois extension $R_{\mathfrak{m}}$ of K inside K^{sep}, called the *S-ray class field of* $\mathfrak{m}$, satisfying

$$\mathsf{NR}_{R_{\mathfrak{m}}/K}^{\varphi_K}{}^{\text{Langlands}^*} : \mathscr{W}\!\mathscr{A}_K^{\varphi_K}/\mathscr{V}_{\mathfrak{m}}^{\varphi_K} \xrightarrow{\sim} \text{Gal}(R_{\mathfrak{m}}/K).$$

The group $\mathscr{W}\!\mathscr{A}_K^{\varphi_K}/\mathscr{V}_{\mathfrak{m}}^{\varphi_K}$ is called the *S-ray class group of* $\mathfrak{m}$. Moreover,

– The Galois extension $R_{\mathfrak{m}}$ over K is unramified at all υ with $e_\upsilon = 0$.

Now, let E/K be any finite Galois extension. By the global non-abelian existence theorem in Langlands form stated in Subsection 5.9,

$${\mathscr{N}_K^{\varphi_K}}^{\text{Langlands}}\mathscr{N}_{E/K}^{\infty}(\mathscr{W}\!\mathscr{A}_E^{\varphi_E})$$

is the open normal subgroup of finite index in $\mathscr{W}\!\mathscr{A}_K^{\varphi_K}$ containing ${\mathscr{N}_K^{\varphi_K}}^{\text{Langlands}}$ and corresponding to E. Therefore, the subgroup

$$\mathsf{s}_K\left({\mathscr{N}_K^{\varphi_K}}^{\text{Langlands}}\mathscr{N}_{E/K}^{\infty}(\mathscr{W}\!\mathscr{A}_E^{\varphi_E})\right)$$

of $\mathbb{J}_K$ is open and finite index in $\mathbb{J}_K$. So, there exists a cycle $\mathfrak{m}$ of the number field K such that

$$K^\times U_{\mathfrak{m}} \subseteq K^\times \mathsf{s}_K\left({\mathscr{N}_K^{\varphi_K}}^{\text{Langlands}}\mathscr{N}_{E/K}^{\infty}(\mathscr{W}\!\mathscr{A}_E^{\varphi_E})\right).$$

Now, the subgroup $\mathsf{s}_K^{-1}(K^\times){\mathscr{N}_K^{\varphi_K}}^{\text{Langlands}}\mathscr{N}_{E/K}^{\infty}(\mathscr{W}\!\mathscr{A}_E^{\varphi_E})$ of $\mathscr{W}\!\mathscr{A}_K^{\varphi_K}$ is open and of finite index in $\mathscr{W}\!\mathscr{A}_K^{\varphi_K}$, and corresponds to a subfield $E_!$ of E (abelian over K) under the global non-abelian existence theorem in Langlands form. Moreover:

– Let E/K be any finite Galois extension. Then $E_!$ is a subfield of $R_{\mathfrak{m}}$ for some cycle $\mathfrak{m}$ of K.

The proofs of the above two statements are similar to the proofs of Theorems 11 and 12 of [13].

5.11. The set of primes in K that split in a finite extension E/K

Let E be a finite Galois extension of the number field K. As usual, denote the set of finite primes υ in K that split completely in E by $\mathrm{Spl}(E/K)$.

We again assume that Conjecture 4.5 holds in this subsection. Then, from the facts stated in Subsection 5.7 combined with the basic properties of the arrow (5.6) stated in Subsection 5.5, the following result follows:

- A prime υ in K splits completely in E/K if and only if $q_\upsilon({}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})} \times \mathrm{SU}(2)) \subset \ker(\mathsf{NR}_{E/K}^{\varphi_K}{}^{\mathrm{Langlands}}) = \mathscr{N}_K^{\varphi_K}{}^{\mathrm{Langlands}} \mathscr{N}_{E/K}^{\infty}(\mathscr{W}\mathscr{A}_E^{\varphi_E})$.

Thus, the set $\mathrm{Spl}(E/K)$ is characterized in terms of the base field K by

$$\mathrm{Spl}(E/K) = \left\{\upsilon \in \mathbb{h}_K \mid q_\upsilon({}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})} \times \mathrm{SU}(2)) \subset \ker(\mathsf{NR}_{E/K}^{\varphi_K}{}^{\mathrm{Langlands}})\right\}.$$

This completes the discussion of the basic properties of the global non-abelian norm-residue symbol in Langlands form following closely [13].

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Kâzim İlhan İkeda
Department of Mathematics
Yeditepe University
26 Ağustos Yerleşimi
İnönü Mah., Kayişdaği Cad.
34755, Ataşehir, Istanbul, Turkey

Current address:
Department of Mathematics
Boğaziçi University
34342, Bebek, Istanbul, Turkey
and
Feza Gürsey Center for Physics and Mathematics
Boğaziçi University-Kandilli Campus
Rasathane Cad., Kandilli Mah.
34684, Istanbul, Turkey
Email address: kazimilhan.ikeda@boun.edu.tr