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MINIMAL SURFACES IN $\mathbb{R}^4$ FOLIATED BY CONIC SECTIONS AND PARABOLIC ROTATIONS OF HOLOMORPHIC NULL CURVES IN $\mathbb{C}^4$

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Abstract. Using the complex parabolic rotations of holomorphic null curves in $\mathbb{C}^4$, we transform minimal surfaces in Euclidean space $\mathbb{R}^3$ to a family of degenerate minimal surfaces in Euclidean space $\mathbb{R}^4$. Applying our deformation to holomorphic null curves in $\mathbb{C}^3$ induced by helicoids in $\mathbb{R}^3$, we discover new minimal surfaces in $\mathbb{R}^4$ foliated by hyperbolas or straight lines. Applying our deformation to holomorphic null curves in $\mathbb{C}^3$ induced by catenoids in $\mathbb{R}^3$, we rediscover the Hoffman-Osserman catenoids in $\mathbb{R}^4$ foliated by ellipses or circles.

1. Introduction

It is a fascinating fact that any simply connected minimal surface in $\mathbb{R}^3$ admits a 1-parameter family of minimal isometric deformations, called the associate family. Rotating the holomorphic null curve in $\mathbb{C}^3$ induced by the given minimal surface in $\mathbb{R}^3$ realizes such isometric deformation of minimal surfaces. For instance, catenoids (foliated by circles) belong to the associate families of helicoids (foliated by lines) in $\mathbb{R}^3$.

R. Schoen [30] characterized catenoids as the only complete, embedded minimal surfaces in $\mathbb{R}^3$ with finite topology and two ends. F. López and A. Ros [15] used the so called López-Ros deformation (See Example 2.4) of holomorphic null curves in $\mathbb{C}^3$ to prove that planes and catenoids are the only embedded complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature and genus zero. The López-Ros deformation has various interesting geometric applications [1, 15, 16, 25, 26, 29]. Meeks and Rosenberg [18] proved that helicoids are the unique complete, simply connected, properly embedded, nonplanar minimal surfaces in $\mathbb{R}^3$ with one end.

I. Castro and F. Urbano [3] established the uniqueness of n-dimensional Lagrangian catenoid [8, Section III. 3. B.] in $\mathbb{R}^{2n}$. In particular, in $\mathbb{R}^4$, the

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Lagrangian catenoid can be identified as the holomorphic curve in $\mathbb{C}^2$:

$$\left\{ \left(\zeta, \frac{\lambda}{\zeta} \right) \in \mathbb{C}^2 \mid \zeta \in \mathbb{C} - \{0\} \right\}$$

for a constant $\lambda \in \mathbb{C}-\{0\}$. Lagrangian catenoids are the only non-planar special Lagrangian surfaces foliated by circles. S.-H. Park [24] showed that a non-planar circle-foliated minimal surface in $\mathbb{R}^{n\geq 3}$ should be a part of Lagrangian catenoid in $\mathbb{R}^4$ or classical minimal surfaces in $\mathbb{R}^3$: catenoids and Riemann's minimal surface [17, 27] foliated by circles or lines.

R. Osserman [23, Theorem 9.4] established that catenoids and Enneper's surfaces are the only complete minimal surfaces in $\mathbb{R}^3$ with total curvature -4π. The Lagrangian catenoids in $\mathbb{R}^4$ have total curvature -4π. More generally, D. Hoffman and R. Osserman [9, Chapter 6] classified all complete minimal surfaces in $\mathbb{R}^{n\geq 3}$ with total curvature -4π. The Hoffman-Osserman list includes a family of minimal surfaces in $\mathbb{R}^4$ foliated by ellipses or circles.

We first review the classical deformations of minimal surfaces in $\mathbb{R}^{n\geq 3}$ induced by deformations of holomorphic null curves in $\mathbb{C}^n$. We use the complex parabolic rotations (Lemma 4.1) of holomorphic null curves in $\mathbb{C}^4$ to show that the deformation of classical catenoids in $\mathbb{R}^3$ to the Hoffman-Osserman catenoids in $\mathbb{R}^4$ can be generalized to deformations of arbitrary minimal surfaces in $\mathbb{R}^3$ to a family of degenerate minimal surfaces in $\mathbb{R}^4$ (Theorem 5.1). As an application of our deformation, generalizing helicoids in $\mathbb{R}^3$ foliated by lines, we construct new minimal surfaces in $\mathbb{R}^4$ foliated by hyperbolas or lines (Example 6.1). We establish the existence of minimal surfaces in $\mathbb{R}^4$ foliated by ellipses, which converges to circles at infinity (Corollary 6.3). Finally, we also provide examples of minimal surfaces in $\mathbb{R}^4$ foliated by parabolas (Example 6.5).

2. Deformations of minimal surfaces in $\mathbb{R}^n$

Since two dimensional minimal surfaces in real Euclidean space is parametrized by conformal harmonic mappings, they can be constructed from holomorphic null curves in complex Euclidean space. Let's review classical deformations of simply connected minimal surfaces in $\mathbb{R}^n$ induced by deformations of the holomorphic null curves in $\mathbb{C}^n$.

Example 2.1 (Associate family of minimal surfaces in $\mathbb{R}^3$)**.** Let $\mathbf{X} : \Sigma \to \mathbb{R}^3$ denote a conformal harmonic immersion of a simply connected minimal surface Σ induced by the holomorphic null curve (ϕ_1, ϕ_2, ϕ_3):

$$\mathbf{X}(\zeta) = \mathbf{X}(\zeta_0) + \left(\mathrm{Re} \int_{\zeta_0}^{\zeta} \phi_1(\zeta) d\zeta, \mathrm{Re} \int_{\zeta_0}^{\zeta} \phi_2(\zeta) d\zeta, \mathrm{Re} \int_{\zeta_0}^{\zeta} \phi_3(\zeta) d\zeta \right),$$

or concisely,

$$\mathbf{X}(\zeta) = \left(\mathrm{Re} \int \phi_1(\zeta) d\zeta, \mathrm{Re} \int \phi_2(\zeta) d\zeta, \mathrm{Re} \int \phi_3(\zeta) d\zeta \right).$$

Given a real angle constant θ, we rotate the initial holomorphic null curve (ϕ_1, ϕ_2, ϕ_3) to associate by the new holomorphic null curve $\left(\widetilde{\phi_1}, \widetilde{\phi_2}, \widetilde{\phi_3}\right)$:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{\phi_1} \\ \widetilde{\phi_2} \\ \widetilde{\phi_3} \end{bmatrix} = \begin{bmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}.$$

The holomorphic null curve $\left(\widetilde{\phi_1}, \widetilde{\phi_2}, \widetilde{\phi_3}\right)$ induces the minimal surface Σ^{θ} is given by the conformal harmonic immersion

$$\mathbf{X}^{\theta}(\zeta) = \left(\mathrm{Re} \int \widetilde{\phi_1}(\zeta) d\zeta, \mathrm{Re} \int \widetilde{\phi_2}(\zeta) d\zeta, \mathrm{Re} \int \widetilde{\phi_3}(\zeta) d\zeta\right).$$

The minimal surface $\Sigma^{\frac{\pi}{2}}$ is called the conjugate surface of $\Sigma^0 = \Sigma$. Under the conjugate transformation $\Sigma \mapsto \Sigma^{\frac{\pi}{2}}$, the lines of curvature on Σ map to the asymptotic lines on $\Sigma^{\frac{\pi}{2}}$ and the asymptotic lines on Σ map to the lines of curvature on $\Sigma^{\frac{\pi}{2}}$.

Example 2.2 (Lawson's isometric deformation [12] of minimal surfaces in $\mathbb{R}^3$ to minimal surfaces in $\mathbb{R}^6$)**.** Given a simply connected minimal surface Σ in $\mathbb{R}^3$, we construct Lawson's two-parameter family of minimal surfaces in $\mathbb{R}^6$, which are isometric to Σ. Suppose that the minimal surface Σ is obtained by integrating the holomorphic null curve (ϕ_1, ϕ_2, ϕ_3) in $\mathbb{C}^3$. First, given a real angle constant β, we obtain the minimal surface Σ_β in $\mathbb{R}^6$ induced by the holomorphic null curve in $\mathbb{C}^6$:

$$(\varphi_1, \ldots, \varphi_6) = \cos\beta\,(\phi_1, 0, \phi_2, 0, \phi_3, 0) + \sin\beta\,(0, -i\phi_1, 0, -i\phi_2, 0, -i\phi_3).$$

We notice that the minimal surface $\Sigma_{\frac{\pi}{4}}$ in $\mathbb{R}^6$ can be identified with the holomorphic curve $\frac{1}{\sqrt{2}}(\phi_1, \phi_2, \phi_3)$ in $\mathbb{C}^3$. Second, given another real angle constant α, we rotate the holomorphic null curve $(\varphi_1, \ldots, \varphi_6)$ to get the holomorphic null curve

$$(\widetilde{\varphi_1}, \ldots, \widetilde{\varphi_6}) = e^{-i\alpha}(\varphi_1, \ldots, \varphi_6).$$

Integrating this, we obtain the minimal surface $\Sigma_{(\alpha,\beta)}$ in $\mathbb{R}^6$, which is isometric to Σ in $\mathbb{R}^3$. Lawson [12, Theorem 1] proved that this is the only way to deform a minimal surface Σ in $\mathbb{R}^3$ isometrically to minimal surfaces in Euclidean space $\mathbb{R}^{n\geq 3}$.

Example 2.3 (Goursat's transformation of minimal surfaces in $\mathbb{R}^3$ [5–7, 13])**.** The Goursat deformation transforms minimal surfaces in $\mathbb{R}^3$ to minimal surfaces in $\mathbb{R}^3$. Let $\mathbf{X}(\zeta) : \Sigma \to \mathbb{R}^3$ be the immersion of a simply connected minimal surface Σ induced by the holomorphic null curve (ϕ_1, ϕ_2, ϕ_3).

For each $t \in \mathbb{R}$, we associate the holomorphic null curve $\left(\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3\right)$ by the linear map:

$$\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix} \mapsto \begin{bmatrix}\widetilde{\phi}_1\\ \widetilde{\phi}_2\\ \widetilde{\phi}_3\end{bmatrix} = \begin{bmatrix}\cosh t & -i\sinh t & 0\\ i\sinh t & \cosh t & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix}.$$

This induces the conformal harmonic map of the minimal surface

$$\mathbf{X}^t(\zeta) = \left(\mathrm{Re}\int \widetilde{\phi}_1(\zeta)d\zeta, \mathrm{Re}\int \widetilde{\phi}_2(\zeta)d\zeta, \mathrm{Re}\int \widetilde{\phi}_3(\zeta)d\zeta\right).$$

The nullity of the holomorphic curve $\left(\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3\right)$ comes from the identity, for $t \in \mathbb{R}$,

$$\begin{aligned} &{z_1}^2 + {z_2}^2 = {\widetilde{z_1}}^{\,2} + {\widetilde{z_2}}^{\,2},\\ &\begin{bmatrix}\widetilde{z_1}\\ \widetilde{z_2}\end{bmatrix} = \begin{bmatrix}\cosh t & -i\sinh t\\ i\sinh t & \cosh t\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \begin{bmatrix}\cos(it) & -\sin(it)\\ \sin(it) & \cos(it)\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}.\end{aligned}$$

See also [28, Section 2] by P. Romon.

Example 2.4 (López-Ros deformation [15] of minimal surfaces in $\mathbb{R}^3$)**.** In terms of Weierstrass datum of holomorphic null curves, we rewrite Goursat's deformation more geometrically. (For instance, see [26, Section 2.1.1].) Let Σ be a simply connected minimal surface in $\mathbb{R}^3$, up to translations, parametrized by the conformal harmonic map

$$(1)\qquad (\mathbf{x}_1(\zeta), \mathbf{x}_2(\zeta), \mathbf{x}_3(\zeta)) = \left(\mathrm{Re}\int \phi_1(\zeta)d\zeta, \mathrm{Re}\int \phi_2(\zeta)d\zeta, \mathrm{Re}\int \phi_3(\zeta)d\zeta\right),$$

where the curve $(\phi_1,\, \phi_2,\, \phi_3)$ is determined by the Weierstrass data $(G(\zeta)$, $\Psi(\zeta)d\zeta)$:

$$(\phi_1,\, \phi_2,\, \phi_3) = \left(\frac{1}{2}\left(1 - G^2\right)\Psi,\, \frac{i}{2}\left(1 + G^2\right)\Psi,\, G\Psi\right).$$

Geometrically, the meromorphic function G is the complexified Gauss map under the stereographic projection of the induced unit normal on the minimal surface. Given a constant $\lambda > 0$, taking $t = -\ln\lambda$ in Goursat transfromation (in Example 2.3), we have the linear deformation of holomorphic null curves:

$$\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix} \mapsto \begin{bmatrix}\widetilde{\phi}_1\\ \widetilde{\phi}_2\\ \widetilde{\phi}_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\left(\frac{1}{\lambda}+\lambda\right) & -\frac{i}{2}\left(\lambda-\frac{1}{\lambda}\right) & 0\\ \frac{i}{2}\left(\lambda-\frac{1}{\lambda}\right) & \frac{1}{2}\left(\frac{1}{\lambda}+\lambda\right) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix}.$$

This induces the conformal harmonic map of the minimal surface Σ^λ :

$$\left(\mathbf{x}_1^\lambda(\zeta), \mathbf{x}_2^\lambda(\zeta), \mathbf{x}_3^\lambda(\zeta)\right) = \left(\mathrm{Re}\int \phi_1^\lambda(\zeta)d\zeta, \mathrm{Re}\int \phi_2^\lambda(\zeta)d\zeta, \mathrm{Re}\int \phi_3^\lambda(\zeta)d\zeta\right).$$

The Goursat deformation of holomorphic null curves yields the deformation of the Weierstrass datum of the so called López-Ros deformation:

$$(G(\zeta), \Psi(\zeta)d\zeta) \mapsto \left(G^\lambda(\zeta),\, \Psi^\lambda(\zeta)d\zeta\right) = \left(\lambda G(\zeta), \frac{1}{\lambda}\Psi(\zeta)d\zeta\right).$$

Under the López-Ros deformation $\Sigma \mapsto \Sigma^\lambda$, the lines of curvature maps into the lines of curvature and the asymptotic lines maps into the asymptotic lines. Since this deformation preserves the height differential

$$\phi_3\, d\zeta = G(\zeta)\, \Psi(\zeta)d\zeta = (\lambda G(\zeta)) \left(\frac{1}{\lambda}\Psi(\zeta)d\zeta\right) = \phi_3^\lambda\, d\zeta,$$

we immediately find that the third component of the conformal harmonic map is also preserved (up to vertical translations):

$$\mathbf{x}_3(\zeta) = \mathbf{x}_3^\lambda(\zeta).$$

Another important and useful property of the López-Ros deformation is that if a component of a horizontal level set $\Sigma \cap \{\mathbf{x}_3 = \text{constant}\}$ is convex, then the same property holds for the related component at the corresponding height on Σ^λ. The López-Ros deformation admits number of interesting applications [1, 15, 16, 25, 26, 29].

3. Gauss map of degenerate minimal surfaces in $\mathbb{R}^4$

Given a conformal harmonic immersion $\mathbf{X} = (x_0, x_1, x_2, x_3) : \Sigma \to \mathbb{R}^4$ with the local coordinate ζ, we associate the complex curve $\phi(\zeta) : \Sigma \to \mathbb{C}^4$ defined by

$$\phi = (\phi_0, \phi_1, \phi_2, \phi_3) := 2\frac{d\mathbf{X}}{d\zeta}. \tag{2}$$

We find that the harmonicity of $\mathbf{X}$ guarantees that the curve ϕ is holomorphic and that the conformality of the immersion $\mathbf{X}$ implies that ϕ lies on the complex null cone

$$\mathcal{Q}_2 = \left\{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \mid {z_0}^2 + {z_1}^2 + {z_2}^2 + {z_3}^2 = 0\right\}. \tag{3}$$

Such immersion $\mathbf{X}(\zeta)$ can be recovered, up to translations, by the integration

$$\mathbf{X} = \left(\mathrm{Re}\int \omega_0,\, \mathrm{Re}\int \omega_1,\, \mathrm{Re}\int \omega_2,\, \mathrm{Re}\int \omega_3\right), \tag{4}$$

where we introduce holomorphic one forms $(\omega_0, \omega_1, \omega_2, \omega_3) = (\phi_0\, d\zeta, \phi_1\, d\zeta, \phi_2\, d\zeta, \phi_3 d\zeta)$. The induced metric on the surface Σ reads

$$g_\Sigma = \frac{1}{2}\left(|\phi_0|^2 + |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right) |d\zeta|^2.$$

We introduce the Gauss map of minimal surfaces in $\mathbb{R}^4$. Inside the complex projective space $\mathbb{CP}^3$, we take the complex null cone

$$\mathcal{Q}_2 := \{z = [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid {z_0}^2 + {z_1}^2 + {z_2}^2 + {z_3}^2 = 0\}.$$

Definition (Gauss map of minimal surfaces in $\mathbb{R}^4$ [9, Chapter 3, p. 35])**.** Let Σ be a minimal surface in $\mathbb{R}^4$. Consider a conformal harmonic immersion $\mathbf{X} : \Sigma \to \mathbb{R}^4$ with the local coordinate ζ. The Gauss map of Σ is the map $\mathcal{G} : \Sigma \to \mathcal{Q}_2 \subset \mathbb{CP}^3$ defined by

$$\mathcal{G}(\zeta) := \left[\frac{\partial \mathbf{X}}{\partial \zeta} \right] = [\, \phi_0 : \phi_1 : \phi_2 : \phi_3 \,] \in \mathbb{CP}^3.$$

Following [9, Chapter 4], we shall review the notion of degeneracy of Gauss map of minimal surfaces in $\mathbb{R}^4$. Let Σ be a minimal surface, which *lies fully* in $\mathbb{R}^4$, in the sense that it does not lie in any proper affine subspace of $\mathbb{R}^4$. The surface Σ is *degenerate* when the image of its Gauss map lies in a hyperplane of $\mathbb{CP}^3$. Otherwise, it is *non-degenerate*. A degenerate minimal surface is *k-degenerate*, if $k \in \{1, 2, 3\}$ is the largest integer such that the Gauss map image lies in a projective subspace of codimension k in $\mathbb{CP}^3$. We recall the fundamental theorem [9, Proposition 4.6] that the surface Σ is 2-degenerate if and only if there exists an orthogonal complex structure on $\mathbb{R}^4$ such that it becomes a complex analytic curve lying fully in $\mathbb{C}^2$.

Bernstein's beautiful theorem says that the only entire minimal graphs in $\mathbb{R}^3$ are planes. More generally, Osserman solved the codimension two generalization of Bernstein type problem, and showed that examples of 1-degenerate and 2-generate minimal surfaces in $\mathbb{R}^4$ naturally appear in the classification of entire minimal graphs.

Example 3.1 (Osserman's entire, non-planar, minimal graph in $\mathbb{R}^4$ [23, Chapter 5])**.** We prepare a complex constant $\mu = a - ib$ with $a \in \mathbb{R}$ and $b > 0$. For any entire holomorphic function $\mathbf{F} : \mathbb{C} \to \mathbb{C}$, we define the minimal surface Σ with the patch

$$\mathbf{X}\,(\zeta) = \left(\operatorname{Re} \int \widehat{\phi}_0\,(\zeta)\, d\zeta,\ \operatorname{Re} \int \widehat{\phi}_1\,(\zeta)\, d\zeta,\ \operatorname{Re} \int \widehat{\phi}_2\,(\zeta)\, d\zeta,\ \operatorname{Re} \int \widehat{\phi}_3\,(\zeta)\, d\zeta \right),$$

where the holomorphic curve $\left(\widehat{\phi}_0,\ \widehat{\phi}_1 = \mu \widehat{\phi}_0,\ \widehat{\phi}_2,\ \widehat{\phi}_3 \right)$ reads

$$\left(1,\ \mu,\ \tfrac{1}{2} \left(e^{\mathbf{F}(\zeta)} - \left(1 + \mu^2\right) e^{-\mathbf{F}(\zeta)} \right),\ \tfrac{i}{2} \left(e^{\mathbf{F}(\zeta)} + \left(1 + \mu^2\right) e^{-\mathbf{F}(\zeta)} \right)\right).$$

One can check that Σ becomes the entire graph $(x_1, x_2, \mathbf{A}(x_1, x_2), \mathbf{B}(x_1, x_2))$ defined on the whole $x_1 x_2$-plane. Osserman proved that any entire, non-planar, minimal graphs in $\mathbb{R}^4$ should admit the above representation with the entire holomorphic function $\mathbf{F}$. When $\mu \in \{i, -i\}$, the minimal surface Σ becomes the complex analytic curve.

Remark 3.2. As known in [9, Theorem 4.7], degenerate minimal surfaces in $\mathbb{R}^4$ admit a general representation formula, which is analogous to the classical Enneper-Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$.

4. Complex parabolic rotations of holomorphic null curves in $\mathbb{C}^4$

Given a holomorphic null curve ϕ in $\mathbb{C}^4$, for any linear mapping $\mathcal{M} \in \mathbf{O}\,(4, \mathbb{C})$, we can associate new holomorphic null curve $\widetilde{\phi} = M\phi$. The purpose of this section is to construct the so-called parabolic rotations of holomorphic null curves in $\mathbb{C}^4$ to construct explicit deformations of minimal surfaces in $\mathbb{R}^3$ to degenerate minimal surfaces in $\mathbb{R}^4$.

Lemma 4.1 (Complex parabolic rotations of holomorphic null curves in $\mathbb{C}^4$)**.** *Given a non-constant holomorphic curve $\phi(\zeta) : \Sigma \to \mathbb{C}^4$ and a constant $c \in \mathbb{C}$, we associate the holomorphic curve $\widehat{\phi} : \Sigma \to \mathbb{C}^4$ by the linear transformation*

$$\begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \mapsto \begin{bmatrix} \widehat{\phi_0} \\ \widehat{\phi_1} \\ \widehat{\phi_2} \\ \widehat{\phi_3} \end{bmatrix} = \begin{bmatrix} 1 & -c & -c\,i & 0 \\ c & 1-\frac{c^2}{2} & -\frac{c^2}{2}i & 0 \\ c\,i & -\frac{c^2}{2}i & 1+\frac{c^2}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}. \tag{5}$$

If the curve ϕ lies on the null cone $\mathcal{Q}_2$, then the curve $\widehat{\phi}$ also lies on $\mathcal{Q}_2$.

Proof. It is straightforward to check the algebraic identity

$$\widehat{\phi_0}^{\,2} + \widehat{\phi_1}^{\,2} + \widehat{\phi_2}^{\,2} = {\phi_0}^2 + {\phi_1}^2 + {\phi_2}^2. \tag{6}$$

Since $\widehat{\phi_3} = \phi_3$, this implies

$$\widehat{\phi_0}^{\,2} + \widehat{\phi_1}^{\,2} + \widehat{\phi_2}^{\,2} + \widehat{\phi_3}^{\,2} = {\phi_0}^2 + {\phi_1}^2 + {\phi_2}^2 + {\phi_3}^2.$$

Since the curve ϕ lies on the null cone $\mathcal{Q}_2$, the curve $\widehat{\phi}$ also lies on $\mathcal{Q}_2$. $\square$

The identity (6) in the proof of Lemma 4.1 is the key idea of our deformation (5) in Lemma 4.1. We shall illustrate ideas behind the identity (6) and Lemma 4.1.

Remark 4.2 (Real light cone $x^2+y^2-z^2=0$ in $\mathbb{L}^3$, complex cone ${z_0}^2+{z_1}^2+{z_2}^2 = 0$ in $\mathbb{C}^3$, parabolic rotations, and Wick rotation). We explain that the algebraic identity (6) in the proof of Lemma 4.1 can be obtained by the complexification of parabolic rotational isometries in $\mathbb{L}^3$ via Wick rotation. Let $\mathbb{L}^3$ denote the Lorentz-Minkowski $(2+1)$-space, which is the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $dx^2+dy^2-dz^2$. The light cone sitting in $\mathbb{L}^3$ given by the quadratic variety

$$\left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2-z^2=0\right\}$$

is invariant under parabolic rotations $\mathcal{L}_{t\in\mathbb{R}}$ with respect to the null line spanned by light-like vector $(1,0,1)$. The isometry $\mathcal{L}_t$ is given by (for instance, see [19, Section 2])

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} = \mathcal{L}_t \begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{pmatrix} = \begin{bmatrix} 1-\frac{t^2}{2} & t & \frac{t^2}{2} \\ -t & 1 & t \\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{7}$$

for each $t \in \mathbb{R}$. We have the real quadratic form identity

$$x^2+y^2-z^2 = \widehat{x}^{\,2} + \widehat{y}^{\,2} - \widehat{z}^{\,2}, \text{ where } \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} = \mathcal{L}_t \begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{pmatrix}. \tag{8}$$

So far, we viewed x, y, z, $\widehat{x}$, $\widehat{y}$, $\widehat{z}$ as real variables. However, clearly, the algebraic identity (8) also holds when we treat them as complex variables. We perform the so-called Wick rotation. Replace z by $-iz$, and $\widehat{z}$ by $-i\widehat{z}$ in (6) to have the transformation

$$\begin{bmatrix} x \\ y \\ -iz \end{bmatrix} \mapsto \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ -i\widehat{z} \end{bmatrix} = \begin{bmatrix} 1-\frac{t^2}{2} & t & \frac{t^2}{2} \\ -t & 1 & t \\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ -iz \end{bmatrix},$$

which can be rewritten as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} = \mathcal{R}_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1-\frac{t^2}{2} & t & -\frac{t^2}{2}i \\ -t & 1 & -ti \\ -\frac{t^2}{2}i & ti & 1+\frac{t^2}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Now, the real variables identity (8) induces the complex quadratic form identity

$$x^2+y^2+z^2 = \widehat{x}^{\,2}+\widehat{y}^{\,2}+\widehat{z}^{\,2}, \text{ where } \begin{bmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{bmatrix} = \mathcal{R}_t \begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{pmatrix}. \tag{9}$$

Finally, taking $(x, y, z) = (z_1, z_0, z_2)$, $(\widehat{x}, \widehat{y}, \widehat{z}) = (\widehat{z}_1, \widehat{z}_0, \widehat{z}_2)$, and $t = c$, we have (10)

$${z_0}^2+{z_1}^2+{z_2}^2 = \widehat{z}_0^{\;2}+\widehat{z}_1^{\;2}+\widehat{z}_2^{\;2}, \text{ where } \begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \end{bmatrix} = \begin{bmatrix} 1 & -c & -c\,i \\ c & 1-\frac{c^2}{2} & -\frac{c^2}{2}i \\ c\,i & -\frac{c^2}{2}i & 1+\frac{c^2}{2} \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix},$$

which gives the algebraic identity (6). We see that $(z_1, z_0, z_2) \mapsto (\widehat{z}_1, \widehat{z}_0, \widehat{z}_2)$ becomes the well-defined linear transformation from the complex null cone to itself. See also [20, Section 3. Complex rotations and minimal surfaces] and [11, Section 2. Examples].

Remark 4.3 (Parabolic rotations of null curves in $\mathbb{C}^4$ in terms of Segre coordinates). We begin with the algebraic identity

$${z_0}^2+{z_1}^2+{z_2}^2+{z_3}^2 = (z_1+iz_2)(z_1-iz_2)-(z_3+iz_0)(-z_3+iz_0). \tag{11}$$

Using the Segre transformation

$$(t_1, t_2, t_3, t_0) = (z_1+iz_2, z_1-iz_2, z_3+iz_0, -z_3+iz_0),$$

we are able to identify the null cone ${z_0}^2+{z_1}^2+{z_2}^2+{z_3}^2 = 0$ as the determinant variety

$$\mathcal{S} = \left\{ \begin{bmatrix} t_1 & t_0 \\ t_3 & t_2 \end{bmatrix} \in M_2(\mathbb{C}) \;\middle|\; \det \begin{bmatrix} t_1 & t_0 \\ t_3 & t_2 \end{bmatrix} = 0 \right\}. \tag{12}$$

Given a pair (L, R) of complex constants, we find the implication

$$\begin{bmatrix} t_1 & t_0 \\ t_3 & t_2 \end{bmatrix} \in \mathcal{S} \;\Rightarrow\; \begin{bmatrix} \widehat{t_1} & \widehat{t_0} \\ \widehat{t_3} & \widehat{t_2} \end{bmatrix} := \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} t_1 & t_0 \\ t_3 & t_2 \end{bmatrix} \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \in \mathcal{S}. \tag{13}$$

Writing $(\widehat{t}_1, \widehat{t}_2, \widehat{t}_3, \widehat{t}_0) = (\widehat{z}_1 + i\widehat{z}_2, \widehat{z}_1 - i\widehat{z}_2, \widehat{z}_3 + i\widehat{z}_0, -\widehat{z}_3 + i\widehat{z}_0)$, we see that this implication induces the linear map from the quadratic cone ${z_0}^2 + {z_1}^2 + {z_2}^2 + {z_3}^2 = 0$ to itself:

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{z}_3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{i}{2}(L+R) & \frac{1}{2}(L+R) & 0 \\ \frac{i}{2}(L+R) & 1+\frac{LR}{2} & \frac{i}{2}LR & -\frac{1}{2}(L-R) \\ -\frac{1}{2}(L+R) & \frac{i}{2}LR & 1-\frac{LR}{2} & -\frac{i}{2}(L-R) \\ 0 & \frac{1}{2}(L-R) & \frac{i}{2}(L-R) & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{14}$$

In particular, taking $(L, R) = (-ci, -ci)$, we obtain the linear transformation

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{z}_3 \end{bmatrix} = \begin{bmatrix} 1 & -c & -c\,i & 0 \\ c & 1-\frac{c^2}{2} & -\frac{c^2}{2}i & 0 \\ c\,i & -\frac{c^2}{2}i & 1+\frac{c^2}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}, \tag{15}$$

which recovers the deformation (5) in Lemma 4.1.

5. From minimal surfaces in $\mathbb{R}^3$ to minimal surfaces in $\mathbb{R}^4$

Theorem 5.1 (Deformations of non-planar minimal surfaces in $\mathbb{R}^3$ to degenerate minimal surfaces in $\mathbb{R}^4$)**.** *Let Σ be a simply connected non-planar minimal surface in $\mathbb{R}^3$, up to translations, parametrized by the conformal harmonic immersion*

$$\mathbf{X}(\zeta) = \left(\mathrm{Re} \int \phi_1(\zeta)\, d\zeta,\ \mathrm{Re} \int \phi_2(\zeta)\, d\zeta,\ \mathrm{Re} \int \phi_3(\zeta)\, d\zeta \right), \tag{16}$$

where the holomorphic null curve $\phi = (\phi_1,\ \phi_2,\ \phi_3)$ admits the Weierstrass data $(G(\zeta), \Psi(\zeta)d\zeta)$:

$$\phi = \left(\frac{1}{2}\left(1 - G^2\right)\Psi,\ \frac{i}{2}\left(1 + G^2\right)\Psi,\ G\Psi \right). \tag{17}$$

Given a constant $c \in \mathbb{C}$, there exists a minimal surface Σ^c in $\mathbb{R}^4$, up to translations, parametrized by the conformal harmonic immersion

$$\mathbf{X}^c(\zeta) = \left(\mathrm{Re} \int \widehat{\phi}_0(\zeta)\, d\zeta,\ \mathrm{Re} \int \widehat{\phi}_1(\zeta)\, d\zeta,\ \mathrm{Re} \int \widehat{\phi}_2(\zeta)\, d\zeta,\ \mathrm{Re} \int \widehat{\phi}_3(\zeta)\, d\zeta \right), \tag{18}$$

where the holomorphic curve $\widehat{\phi} = \left(\widehat{\phi}_0,\ \widehat{\phi}_1,\ \widehat{\phi}_2,\ \widehat{\phi}_3\right)$ is determined by

$$\widehat{\phi} = \left(c\,G^2\Psi,\ \frac{1}{2}\left(1 + \left(c^2 - 1\right)G^2\right)\Psi,\ \frac{i}{2}\left(1 + \left(c^2 + 1\right)G^2\right)\Psi,\ G\Psi \right). \tag{19}$$

The induced metric on Σ^c by the patch $\mathbf{X}^c$ reads

$$g_{\Sigma^c} = \frac{1}{4}\left(|\Psi|^2 \left|1 + c^2G^2\right|^2 \left(1 + \frac{|G|^2}{|1 + icG|^2}\right)\left(1 + \frac{|G|^2}{|1 - icG|^2}\right)\right) |d\zeta|^2.$$

Due to the identity $\widehat{\phi}_0 + c\widehat{\phi}_1 + ic\widehat{\phi}_2 = 0$, the minimal surface Σ^c in $\mathbb{R}^4$ is degenerate.

Proof. To prove that Σ^c is a minimal surface in $\mathbb{R}^4$, we show that the holomorphic curve

$$\left(c\,G^2\Psi,\, \frac{1}{2}\left(1+\left(c^2-1\right)G^2\right)\Psi,\, \frac{i}{2}\left(1+\left(c^2+1\right)G^2\right)\Psi,\, G\Psi\right)$$

is null. Take $\phi_0 = 0$. Regard Σ in $\mathbb{R}^3$ induced by the holomorphic null curve (ϕ_1, ϕ_2, ϕ_3) in $\mathbb{C}^3$ as a minimal surface in $\mathbb{R}^4$ induced by the holomorphic null curve in $\mathbb{C}^4$:

$$(\phi_0, \phi_1, \phi_2, \phi_3) = \left(0, \frac{1}{2}\left(1-G^2\right)\Psi,\, \frac{i}{2}\left(1+G^2\right)\Psi,\, G\Psi\right).$$

Then, by Lemma 4.1, we find that the holomorphic curve in $\mathbb{C}^4$:

$$\begin{bmatrix} 1 & -c & -c\,i & 0 \\ c & 1-\frac{c^2}{2} & -\frac{c^2}{2}i & 0 \\ c\,i & -\frac{c^2}{2}i & 1+\frac{c^2}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} c\,G^2\Psi \\ \frac{1}{2}\left(1+\left(c^2-1\right)G^2\right)\Psi \\ \frac{i}{2}\left(1+\left(c^2+1\right)G^2\right)\Psi \\ G\Psi \end{bmatrix} = \begin{bmatrix} \widehat{\phi_0} \\ \widehat{\phi_1} \\ \widehat{\phi_2} \\ \widehat{\phi_3} \end{bmatrix}$$

should be also null. The conformal factor of the conformal metric on Σ^c is equal to

$$\frac{1}{2}\left(\sum_{k=0}^{3} |\,\widehat{\phi}_k|^2\right) = \frac{1}{4}\left(|\Psi|^2\left|1+c^2G^2\right|^2\left(1+\frac{|G|^2}{|1+icG|^2}\right)\left(1+\frac{|G|^2}{|1-icG|^2}\right)\right).$$

The definition of $\widehat{\phi}$ gives the following equality, which implies the degeneracy of Σ^c:

$$\widehat{\phi}_0 + c\widehat{\phi}_1 + ic\widehat{\phi}_2 = \phi_0 = 0.$$

□

Remark 5.2. We used the parabolic rotations of holomorphic null curves in $\mathbb{C}^4$ in Lemma 4.1 to construct deformations (Theorem 5.1) of minimal surfaces in $\mathbb{R}^3$ to a family of minimal surfaces in $\mathbb{R}^4$. These deformations of simply connected minimal surfaces in $\mathbb{R}^3$ to minimal surfaces in $\mathbb{R}^3$ or $\mathbb{R}^4$ are non-isometric deformations, in general. Indeed, H. Lawson [12, Theorem 1] used E. Calabi's Theorem [2] to determine when minimal surfaces in $\mathbb{R}^{n\geq 3}$ are isometric to minimal surfaces in $\mathbb{R}^3$. See also [22, Theorem 1.2].

Remark 5.3. We point out that the holomorphic null curves in $\mathbb{C}^4$ also naturally appears in the theory of superconformal surfaces in $\mathbb{R}^4$. For instance, see [4,21].

Taking the constant $c = \tan\theta \in \mathbb{R}$ in Theorem 5.1 and rotating coordinate system in the ambient space $\mathbb{R}^4$, we have the following deformation:

Corollary 5.4 (Degenerate minimal surfaces in $\mathbb{R}^4$)**.** *Let Σ be a simply connected minimal surface in $\mathbb{R}^3$, up to translations, parametrized by the conformal harmonic immersion*

$$\mathbf{X}(\zeta) = \left(\mathrm{Re}\int \phi_1(\zeta)\,d\zeta,\ \mathrm{Re}\int \phi_2(\zeta)\,d\zeta,\ \mathrm{Re}\int \phi_3(\zeta)\,d\zeta\right), \tag{20}$$

where the holomorphic null curve $\phi = (\phi_1, \phi_2, \phi_3)$ admits the Weierstrass data $(G(\zeta), \Psi(\zeta)d\zeta)$:

$$\phi = \left(\frac{1}{2}\left(1 - G^2\right)\Psi,\ \frac{i}{2}\left(1 + G^2\right)\Psi,\ G\Psi\right). \tag{21}$$

For each angle constant $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, there exists a degenerate minimal surface $\Sigma^{\tan\theta}$ in $\mathbb{R}^4$, up to translations, parametrized by the conformal harmonic immersion

(22)

$$\mathbf{X}^{\tan\theta}(\zeta) = \left(\mathrm{Re}\int \widetilde{\phi}_0(\zeta)\,d\zeta,\ \mathrm{Re}\int \widetilde{\phi}_1(\zeta)\,d\zeta,\ \mathrm{Re}\int \widetilde{\phi}_2(\zeta)\,d\zeta,\ \mathrm{Re}\int \widetilde{\phi}_3(\zeta)\,d\zeta\right),$$

where the holomorphic curve $\left(\widetilde{\phi}_0 = -i(\sin\theta)\,\widehat{\phi}_2,\ \widetilde{\phi}_1,\ \widetilde{\phi}_2,\ \widehat{\phi}_3\right)$ is determined by

(23)

$$\widetilde{\phi} = \left(\frac{\sin\theta}{2}\left(1 + \frac{G^2}{\cos^2\theta}\right)\Psi,\ \frac{\cos\theta}{2}\left(1 - \frac{G^2}{\cos^2\theta}\right)\Psi,\ \frac{i}{2}\left(1 + \frac{G^2}{\cos^2\theta}\right)\Psi,\ G\Psi\right).$$

Proof. We take $c = \tan\theta$ in Theorem 5.1. With respect to the standard frame $\mathbf{e}_0 = (1,0,0,0)$, $\mathbf{e}_1 = (0,1,0,0)$, $\mathbf{e}_2 = (0,0,1,0)$, $\mathbf{e}_3 = (0,0,0,1)$, the surface $\Sigma^{\tan\theta}$ admits the patch

$$\left(\mathrm{Re}\int \widehat{\phi}_0(\zeta)\,d\zeta\right)\mathbf{e}_0 + \left(\mathrm{Re}\int \widehat{\phi}_1(\zeta)\,d\zeta\right)\mathbf{e}_1 + \left(\mathrm{Re}\int \widehat{\phi}_2(\zeta)\,d\zeta\right)\mathbf{e}_2 + \left(\mathrm{Re}\int \widehat{\phi}_3(\zeta)\,d\zeta\right)\mathbf{e}_3,$$

where the holomorphic curve $\widehat{\phi} = \left(\widehat{\phi}_0,\ \widehat{\phi}_1,\ \widehat{\phi}_2,\ \widehat{\phi}_3\right)$ reads

$$\widehat{\phi} = \left(\tan\theta\, G^2\Psi,\ \frac{1}{2}\left(1 + \left(-1 + \tan^2\theta\right)G^2\right)\Psi,\ \frac{i}{2}\left(1 + \left(1 + \tan^2\theta\right)G^2\right)\Psi,\ G\Psi\right).$$

With the new frame $\mathbf{E}_0 = \cos\theta\,\mathbf{e}_0 + \sin\theta\,\mathbf{e}_1$, $\mathbf{E}_1 = -\sin\theta\,\mathbf{e}_0 + \cos\theta\,\mathbf{e}_1$, $\mathbf{E}_2 = \mathbf{e}_2$, $\mathbf{E}_3 = \mathbf{e}_3$, the minimal surface $\Sigma^{\tan\theta}$ admits the patch

$$\left(\mathrm{Re}\int \widetilde{\phi}_0(\zeta)\,d\zeta\right)\mathbf{E}_0 + \left(\mathrm{Re}\int \widetilde{\phi}_1(\zeta)\,d\zeta\right)\mathbf{E}_1 + \left(\mathrm{Re}\int \widetilde{\phi}_2(\zeta)\,d\zeta\right)\mathbf{E}_2 + \left(\mathrm{Re}\int \widetilde{\phi}_3(\zeta)\,d\zeta\right)\mathbf{E}_3.$$

□

6. Minimal surfaces in $\mathbb{R}^4$ foliated by conic sections

We present applications of our deformations of minimal surfaces in $\mathbb{R}^3$ to produce new minimal surfaces in $\mathbb{R}^4$. Applying our deformation to holomorphic null curves in $\mathbb{C}^3 \subset \mathbb{C}^4$ induced by helicoids in $\mathbb{R}^3$, we discover minimal surfaces in $\mathbb{R}^4$ foliated by hyperbolas or lines. Applying our deformation to holomorphic null curves in $\mathbb{C}^3$ induced by catenoids in $\mathbb{R}^3$, we can rediscover the Hoffman-Osserman minimal surfaces in $\mathbb{R}^4$ foliated by ellipses or circles.

Example 6.1 (From helicoids in $\mathbb{R}^3$ to minimal surfaces in $\mathbb{R}^4$ foliated by hyperbolas or lines)**.** Let's begin with the helicoid

$$\Sigma = \{(-\sinh u \sin v, \sinh u \cos v, v) \in \mathbb{R}^3 \mid u, v \in \mathbb{R}\}$$

foliated by horizontal lines. The global conformal coordinate $\zeta = u + iv \in \mathbb{C}$ on Σ induces the Weierstrass data

$$(G(\zeta), \Psi(\zeta)d\zeta) = \left(e^{\zeta}, -ie^{-\zeta}d\zeta\right).$$

The helicoid Σ is, up to translations, given by the conformal harmonic immersion

$$\mathbf{X}(\zeta) = \operatorname{Re}\left(\int \frac{1}{2}\left(1 - G^2\right)\Psi d\zeta, \int \frac{i}{2}\left(1 + G^2\right)\Psi d\zeta, \int G\Psi d\zeta\right).$$

Let $c = \alpha + i\beta \in \mathbb{R} + i\mathbb{R}$ be the deformation constant. Applying Theorem 5.1 to the Weierstrass data $(G(\zeta), \Psi(\zeta)) = \left(e^{\zeta}, -ie^{-\zeta}\right)$, we obtain the minimal surface Σ^c in $\mathbb{R}^4$, up to translations, parametrized by the conformal harmonic map $\mathbf{X}^c(\zeta)$:

$$\mathbf{X}^c = \operatorname{Re}\left(\int cG^2\Psi d\zeta, \int \frac{1}{2}\left(1 + \left(c^2 - 1\right)G^2\right)\Psi d\zeta, \int \frac{i}{2}\left(1 + \left(c^2 + 1\right)G^2\right)\Psi d\zeta, \int G\Psi d\zeta\right).$$

With the frame $\mathbf{e}_0 = (1,0,0,0)$, $\mathbf{e}_1 = (0,1,0,0)$, $\mathbf{e}_2 = (0,0,1,0)$, $\mathbf{e}_3 = (0,0,0,1)$, we write

$$\mathbf{X}^c = \mathbf{X}_0\mathbf{e}_0 + \mathbf{X}_1\mathbf{e}_1 + \mathbf{X}_2\mathbf{e}_2 + \mathbf{X}_3\mathbf{e}_3$$

and obtain

$$\begin{aligned}
\mathbf{X}_0(u,v) &= e^u\left(\alpha \sin v + \beta \cos v\right),\\
\mathbf{X}_1(u,v) &= e^u\left(\frac{\alpha^2 - \beta^2 - 1}{2}\sin v + \alpha\beta\cos v\right) + e^{-u}\left(\frac{\sin v}{2}\right),\\
\mathbf{X}_2(u,v) &= e^u\left(\frac{-\alpha\beta\sin v + \alpha^2 - \beta^2 + 1}{2}\cos v\right) + e^{-u}\left(\frac{-\cos v}{2}\right),\\
\mathbf{X}_3(u,v) &= v.
\end{aligned}$$

We show that the minimal surface Σ^c is foliated by hyperbolas or lines. We introduce new orthonormal frame

$$\begin{aligned}
\mathbf{E}_0 &= \frac{1}{\sqrt{\alpha^2 + \beta^2 + 1}}\left(e_0 + \alpha e_1 - \beta e_2\right),\\
\mathbf{E}_1 &= \frac{1}{\sqrt{\alpha^2 + 1}}\left(-\alpha e_0 + e_1\right),\\
\mathbf{E}_2 &= \frac{1}{\sqrt{\beta^2 + 1}}\left(\beta e_0 + e_2\right),\\
\mathbf{E}_3 &= e_3,
\end{aligned}$$

and prepare two auxiliary functions

$$\begin{aligned}\mathbf{Ch}(u) &= \frac{1}{2}\left(\sqrt{\alpha^2+\beta^2+1}\,e^u + \frac{1}{\sqrt{\alpha^2+\beta^2+1}\,e^u}\right)\\ &= \cosh\left(u + \ln\left(\sqrt{\alpha^2+\beta^2+1}\right)\right),\\ \mathbf{Sh}(u) &= \frac{1}{2}\left(\sqrt{\alpha^2+\beta^2+1}\,e^u - \frac{1}{\sqrt{\alpha^2+\beta^2+1}\,e^u}\right)\\ &= \sinh\left(u + \ln\left(\sqrt{\alpha^2+\beta^2+1}\right)\right).\end{aligned}$$

Each components of the patch $\mathbf{X}^c(u,v) = \Xi_0\mathbf{E}_0 + \Xi_1\mathbf{E}_1 + \Xi_2\mathbf{E}_2 + \Xi_3\mathbf{E}_3$ are given by

$$\begin{aligned}\Xi_0(u,v) &= \quad (\alpha\sin v + \beta\cos v)\,\mathbf{Ch}(u),\\ \Xi_1(u,v) &= -\frac{\sqrt{\alpha^2+\beta^2+1}}{\sqrt{\alpha^2+1}}\sin v\;\mathbf{Sh}(u),\\ \Xi_2(u,v) &= -\frac{\sqrt{\alpha^2+\beta^2+1}}{\sqrt{\beta^2+1}}\cos v\;\mathbf{Sh}(u),\\ \Xi_3(u,v) &= v.\end{aligned}$$

When the deformation constant $c = \alpha + i\beta \in \mathbb{R} + i\mathbb{R}$ is equal to zero, the surface $\Sigma_{c=0}$ given by the patch $\mathbf{X}^{c=0}$ recovers the helicoid. Now now on, consider the case $(\alpha,\beta) \neq (0,0)$. Fixing the last coordinate $v = v_0$ on Σ, we examine the level set $\mathcal{C}_{v_0} := \Sigma^c \cap \{\Xi_3 = v_0\}$ given by

$$\Xi_0(u,v_0)\,\mathbf{E}_0 + \Xi_1(u,v_0)\,\mathbf{E}_1 + \Xi_2(u,v_0)\,\mathbf{E}_2 + v_0\mathbf{E}_3.$$

Translating the level set $\mathcal{C}_{v_0}$ in the $\mathbf{E}_0$ direction yields the curve given by

$$\mathbf{c}(u) = \Xi_0(u,v_0)\,\mathbf{E}_0 + \Xi_1(u,v_0)\,\mathbf{E}_1 + \Xi_2(u,v_0)\,\mathbf{E}_2.$$

We introduce the new orthonormal frame

$$\begin{aligned}\epsilon_0 &= \mathbf{E}_0,\\ \epsilon_1 = \epsilon_1(v_0) &= \frac{1}{\sqrt{\frac{\sin^2 v_0}{\alpha^2+1} + \frac{\cos^2 v_0}{\beta^2+1}}}\left(\frac{\sin v_0}{\sqrt{\alpha^2+1}}\mathbf{E}_1 + \frac{\cos v_0}{\sqrt{\beta^2+1}}\mathbf{E}_2\right),\\ \epsilon_2 = \epsilon_2(v_0) &= \frac{1}{\sqrt{\frac{\sin^2 v_0}{\alpha^2+1} + \frac{\cos^2 v_0}{\beta^2+1}}}\left(\frac{\cos v_0}{\sqrt{\beta^2+1}}\mathbf{E}_1 - \frac{\sin v_0}{\sqrt{\alpha^2+1}}\mathbf{E}_2\right)\end{aligned}$$

to rewrite $\mathbf{c}(u) = \mathbf{x}(u)\epsilon_0 + \mathbf{y}(u)\epsilon_1 + \mathbf{z}(u)\epsilon_2$ with components

$$\mathbf{x} = \mathbf{c}(u)\cdot\epsilon_0 = (\alpha\sin v_0 + \beta\cos v_0)\;\mathbf{Ch}(u),$$

$$\mathbf{y} = \mathbf{c}(u) \cdot \epsilon_1 = -\sqrt{(\alpha^2+\beta^2+1)\left(\frac{\sin^2 v_0}{\alpha^2+1} + \frac{\cos^2 v_0}{\beta^2+1}\right)}\,\mathbf{Sh}(u),$$
$$\mathbf{z} = \mathbf{c}(u) \cdot \epsilon_2 = 0.$$

We distinguish two cases:

(1) When $\alpha \sin v_0 + \beta \cos v_0 \neq 0$, the level set $\mathcal{C}_{v_0}$ is congruent to the hyperbola

$$\left(\frac{\mathbf{x}}{\alpha \sin v_0 + \beta \cos v_0}\right)^2 - \left(\frac{\mathbf{y}}{\sqrt{(\alpha^2+\beta^2+1)\left(\frac{\sin^2 v_0}{\alpha^2+1} + \frac{\cos^2 v_0}{\beta^2+1}\right)}}\right)^2 = 1,$$

which has two orthogonal asymptotic lines $\mathbf{x} = \pm \frac{\alpha \sin v_0 + \beta \cos v_0}{\sqrt{\frac{\sin^2 v_0}{\alpha^2+1} + \frac{\cos^2 v_0}{\beta^2+1}}}\,\mathbf{y}$.

(2) When $\alpha \sin v_0 + \beta \cos v_0 = 0$, we have $\mathbf{x}_0 \equiv 0$ and $\mathbf{z}_0 \equiv 0$. This means that the level set curve $\mathcal{C}_{v_0}$ becomes a line.

Example 6.2 (From catenoids in $\mathbb{R}^3$ to the Hoffman-Osserman minimal surfaces in $\mathbb{R}^4$ foliated by ellipses or circles)**.** The catenoid of the neck radius 1 is given by

$$\mathbf{X}(\zeta) = \operatorname{Re}\left(\int \frac{1}{2}\left(1 - G^2\right)\Psi d\zeta,\ \int \frac{i}{2}\left(1 + G^2\right)\Psi d\zeta,\ \int G\Psi d\zeta\right),$$

with the Weierstrass data $(G(\zeta), \Psi(\zeta)d\zeta) = \left(\zeta, \frac{1}{\zeta^2}d\zeta\right)$, where $\zeta = u + iv \in \mathbb{C} - \{0\}$. Let $c \in \mathbb{C}$ be the deformation constant. Applying Theorem 5.1 to the Weierstrass data $(G(\zeta), \Psi(\zeta)d\zeta)$, we obtain the minimal surface Σ^c in $\mathbb{R}^4$ with the patch

$$\mathbf{X}^c(\zeta) = \operatorname{Re}\left(\int \widehat{\phi}_0\, d\zeta,\ \int \widehat{\phi}_1\, d\zeta,\ \int \widehat{\phi}_2\, d\zeta,\ \int \widehat{\phi}_3\, d\zeta\right).$$

Here, the holomorphic null curve is given by

$$\begin{bmatrix} \widehat{\phi}_0 \\ \widehat{\phi}_1 \\ \widehat{\phi}_2 \\ \widehat{\phi}_3 \end{bmatrix} = \begin{bmatrix} c\,G^2\Psi \\ \frac{1}{2}\left(1 + \left(c^2-1\right)G^2\right)\Psi \\ \frac{i}{2}\left(1 + \left(c^2+1\right)G^2\right)\Psi \\ G\Psi \end{bmatrix} = \begin{bmatrix} c \\ \frac{1}{2}\left(\frac{1}{z^2} + c^2 - 1\right) \\ \frac{i}{2}\left(\frac{1}{z^2} + c^2 + 1\right) \\ \frac{1}{z} \end{bmatrix}, \tag{24}$$

which recovers the holomorphic data [9, Proposition 6.6] of the Hoffman-Osserman minimal surfaces foliated by ellipses or circles [9, Remark 1]. The conformal harmonic immersion of the Hoffman-Osserman minimal surfaces in $\mathbb{R}^5$ should read

$$\mathbf{X} = \operatorname{Re}\left(d_1\zeta - \frac{C}{\zeta},\ d_2\zeta - i\frac{C}{\zeta},\ \alpha \log \zeta,\ d_4 z,\ d_5 z\right).$$

Here, d_1, d_2, C, d_4, d_5 are complex constants and α is a positive real constant satisfying

$$(d_1, d_2) = \left(\frac{C}{\alpha^2}\left({d_4}^2 + {d_5}^2\right) - \frac{\alpha^2}{4C}, i\left(\frac{C}{\alpha^2}\left({d_4}^2 + {d_5}^2\right) + \frac{\alpha^2}{4C}\right)\right),$$

which guarantees that the induced holomorphic null curve in $\mathbb{C}^5$:

$$\left(d_1 + \frac{C}{\zeta^2},\, d_2 + i\frac{C}{\zeta^2},\, \frac{\alpha}{\zeta},\, d_4,\, d_5 \right). \tag{25}$$

Taking the normalization $(d_1, d_2, C, d_4, d_5, \alpha) = \left(\frac{1}{2}(c^2-1), \frac{i}{2}(c^2+1), \frac{1}{2}, c, 0, 1\right)$ in the Hoffman-Osserman curve (25), we recover the holomorphic curve, which is equivalent to (24).

Corollary 6.3 (Minimal surfaces in $\mathbb{R}^4$ spanned by circles at infinity)**.** *Given a constant $\theta \in \left(0, \frac{\pi}{2}\right)$, we define the minimal surface $\Sigma^{\tan\theta}$ defined by the conformal harmonic mapping*

$$\mathbf{X}^{\tan\theta}(U, V) = \left(\frac{\sin\theta}{\cos\theta} \sinh U \cos V, \cosh U \cos V, \frac{1}{\cos\theta} \cosh U \sin V, U \right)$$

for $(U, V) \in \mathbb{R}^2$. Then, we have the following properties.

1. *For each constant height $\mathbf{x}_4 = U_0$, the level set $\mathcal{C}_{U_0} = \Sigma^{\tan\theta} \cap \{\mathbf{x}_4 = U_0\}$ is an ellipse. In particular, the neck $\mathcal{C}_0 = \Sigma^{\tan\theta} \cap \{\mathbf{x}_4 = 0\}$ is congruent to $\mathbf{x}^2 + \left(\cos^2\theta\right) \mathbf{y}^2 = 1$.*
2. *When U approaches to ∞ (or $-\infty$), the ellipse $\mathcal{C}_U$ converges to a circle.*

Proof. We recall the classical Weierstrass data of the catenoid in $\mathbb{R}^3$ with neck size 1:

$$\left(z, \frac{1}{z^2} dz\right) = \left(e^{\zeta}, e^{-\zeta} d\zeta\right) =: \left(G(\zeta), \Psi(\zeta) d\zeta\right).$$

Applying Corollary 5.4, we have the minimal surface in $\mathbb{R}^4$, up to translations, given by

$$(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \left(\mathrm{Re} \int \widetilde{\phi}_0(\zeta)\, d\zeta,\, \mathrm{Re} \int \widetilde{\phi}_1(\zeta)\, d\zeta,\, \mathrm{Re} \int \widetilde{\phi}_2(\zeta)\, d\zeta,\, \mathrm{Re} \int \widetilde{\phi}_3(\zeta)\, d\zeta \right),$$

where the holomorphic curve $\widetilde{\phi} = \left(\widetilde{\phi}_0, \widetilde{\phi}_1, \widetilde{\phi}_2, \widehat{\phi}_3\right)$ is determined by

$$\widetilde{\phi} = \left(\frac{\sin\theta}{2} \left(e^{-\zeta} + \frac{e^{\zeta}}{\cos^2\theta} \right), \frac{\cos\theta}{2} \left(e^{-\zeta} - \frac{e^{\zeta}}{\cos^2\theta} \right) \Psi,\, \frac{i}{2} \left(e^{-\zeta} + \frac{e^{\zeta}}{\cos^2\theta} \right) \Psi,\, 1 \right).$$

We write $\zeta = u + iv$ and introduce the coordinates $(U, V) := (u - \ln(\cos\theta), v)$. Applying reflection $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \to (\mathbf{x}_0, -\mathbf{x}_1, -\mathbf{x}_2, \mathbf{x}_3)$ and translation $\mathbf{x}_3 \to \mathbf{x}_3 - \ln(\cos\theta)$, the above patch recovers the desired conformal harmonic mapping $\mathbf{X}^{\tan\theta}(U, V)$. One can easily check that the level set

$$\mathcal{C}_{U_0} = \Sigma^{\tan\theta} \cap \{\mathbf{x}_4 = U_0\}$$

is congruent to the ellipse

$$\left(\frac{\mathbf{x}}{r_1}\right)^2 + \left(\frac{\mathbf{y}}{r_2}\right)^2 = 1,\ (r_1, r_2) = \left(\sqrt{\left(\frac{\sin\theta}{\cos\theta}\sinh U\right)^2 + \cosh^2 U}\,,\ \frac{1}{\cosh\theta}\cosh U\right).$$

When $|U| \to \infty$ (or $|\tanh U| \to 1$), the ellipse $\mathcal{C}_U$ should converge to a circle:

$$\lim_{|U|\to\infty} \frac{{r_2}^2}{{r_1}^2} = \lim_{|U|\to\infty} \frac{\frac{1}{\cos^2\theta}}{\left(\frac{\sin\theta}{\cos\theta}\tanh U\right)^2 + 1} = \frac{\frac{1}{\cos^2\theta}}{\tan^2\theta + 1} = 1. \qquad \square$$

Example 6.4 (Lagrangian catenoid in $\mathbb{R}^4$)**.** Rotating and dilating the Lagrangian catenoid

$$\left\{\left(\zeta, \frac{1}{\zeta}\right) \in \mathbb{C}^2 \mid \zeta \in \mathbb{C} - \{0\}\right\} = \left\{\left(e^w, e^{-w}\right) \in \mathbb{C}^2 \mid w \in \mathbb{C}\right\}$$

we have the holomorphic curve in $\mathbb{C}^2$:

$$\left\{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}e^w - \frac{1}{\sqrt{2}}e^{-w}, \frac{1}{\sqrt{2}}e^w + \frac{1}{\sqrt{2}}e^{-w}\right) = (\sinh w, \cosh w) \in \mathbb{C}^2 \mid w \in \mathbb{C}\right\},$$

which can be identified as a minimal surface in $\mathbb{R}^4$:

$$\Sigma = \left\{\mathbf{X}(u,v) = (\sinh u\cos v, \cosh u\sin v, \cosh u\cos v, \sinh u\sin v) \in \mathbb{R}^4 \mid (u,v) \in \mathbb{R}^2\right\},$$

where $(u,v) = (\operatorname{Re} w, \operatorname{Im} w)$. We obtain two families of planar curves on Σ:

(1) Fixing $v = v_0$, the curve $u \mapsto \mathbf{X}(u, v_0)$ can be viewed as

$$u \mapsto \mathbf{X}(u, v_0) = \cosh u\,(0, \sin v_0, \cos v_0, 0) + \sinh u\,(\cos v_0, 0, 0, \sin v_0)\,,$$

which is congruent to the hyperbola $\mathbf{x}^2 - \mathbf{y}^2 = 1$, as unit vectors $(0, \sin v_0, \cos v_0, 0)$ and $(\cos v_0, 0, 0, \sin v_0)$ are orthogonal to each other.

(2) Fixing $u = u_0$, the curve $v \mapsto \mathbf{X}(u_0, v)$ can be viewed as

$$v \mapsto \mathbf{X}(u, v_0) = \cos v\,(\sinh u_0, 0, \cosh u_0, 0) + \sinh u\,(0, \cosh u_0, 0, \sinh u_0)\,,$$

which is congruent to the circle $\mathbf{x}^2 + \mathbf{y}^2 = \cosh^2 u_0 + \sinh^2 u_0$, as two orthogonal vectors $(\sinh u_0, 0, \cosh u_0, 0)$ and $(0, \cosh u_0, 0, \sinh u_0)$ have the same length $\sqrt{\cosh^2 u_0 + \sinh^2 u_0}$.

More generally, one can easily check that, given constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, the holomorphic curve $\left\{(\lambda_1 e^w + \lambda_2 e^{-w}, \lambda_3 e^w + \lambda_4 e^{-w}) \in \mathbb{C}^2 \mid w \in \mathbb{C}\right\}$ admit similar properties.

Example 6.5 (Minimal surfaces in $\mathbb{R}^4$ foliated by parabolas)**.** Let $\mu \in \mathbb{C} - \{0\}$ be a constant. We shall see that the holomorphic curve $z = \mu w^2$ in $\mathbb{C}^2$ becomes a minimal surface Σ^μ in $\mathbb{R}^4$ foliated by a family of parabolas $\mathbf{y} = \lambda\mathbf{x}^2$ with $\lambda \in (0, |\mu|)$. With the frame $\mathbf{E}_1 = (1,0,0,0)$, $\mathbf{E}_2 = (0,1,0,0)$, $\mathbf{E}_3 = (0,0,1,0)$, $\mathbf{E}_4 = (0,0,0,1)$, we prepare the immersion $\mathbf{X}^\mu$ of the complex parabola Σ^μ, with the conformal coordinate $\zeta = u + iv \in \mathbb{C}$:

$$\mathbf{X}^\mu(u,v) = x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + x_3\mathbf{E}_3 x_3 + x_4\mathbf{E}_4$$

$$= \operatorname{Re}\zeta\, \mathbf{E}_1 + \operatorname{Im}\zeta\, \mathbf{E}_2 + \operatorname{Re}\left(\mu\zeta^2\right)\mathbf{E}_3 + \operatorname{Im}\left(\mu\zeta^2\right)\mathbf{E}_4.$$

Fix $u = u_0$. Let's look at the slice $\mathcal{C}_{u_0} = \Sigma^\mu \cap \{x_1 = u_0\}$ given by $\mathbf{c}(v) = \mathbf{X}^\mu(u_0, v)$. Writing $(a,b) = (\operatorname{Re}\mu, \operatorname{Im}\mu) \neq (0,0)$ and introducing the new orthonormal frame

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{E}_1,\\
\mathbf{e}_2 &= \mathbf{e}_2(u_0) = \frac{1}{\sqrt{1+4\left(a^2+b^2\right){u_0}^2}}\left(\mathbf{E}_2 - 2b\mathbf{E}_3 + 2a\mathbf{E}_4\right),\\
\mathbf{e}_3 &= \mathbf{e}_3(u_0) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\mathbf{E}_3 - b\mathbf{E}_4\right),\\
\mathbf{e}_4 &= \mathbf{e}_4(u_0) = \frac{1}{\sqrt{\left(a^2+b^2\right)\left(1+4{u_0}^2\right)}}\left(\mathbf{E}_2 - 2b\mathbf{E}_3 + 2a\mathbf{E}_4\right),
\end{aligned}$$

we rewrite the patch of the level set curve $\mathcal{C}_{u_0}$:

$$\mathbf{c}(v) = u_0\mathbf{e}_1 + v\sqrt{1+4\left(a^2+b^2\right){u_0}^2}\,\mathbf{e}_2 + \left(v^2 - {u_0}^2\right)\sqrt{a^2+b^2}\,\mathbf{e}_3.$$

This indicates that the slice $\mathcal{C}_{u_0}$ is congruent to the planar curve

$$(\mathbf{x}(v), \mathbf{y}(v)) = \left(v\sqrt{1+4\left(a^2+b^2\right){u_0}^2}\,,\left(v^2 - {u_0}^2\right)\sqrt{a^2+b^2}\,\right).$$

This curve represents the parabola $\mathbf{y} = \frac{\sqrt{a^2+b^2}}{1+4(a^2+b^2){u_0}^2}\mathbf{x}^2 - {u_0}^2\sqrt{a^2+b^2}$.

Remark 6.6. D. Joyce [10] constructed special Lagrangian submanifolds in $\mathbb{R}^{2n}$ evolving quadrics. In the case when $n = 2$, those examples become holomorphic curves in $\mathbb{C}^2$.

Remark 6.7. Motivated by M. Shiffman's theorems for minimal surfaces in $\mathbb{R}^3$ bounded by two convex curves [31], F. López, R. López, and R. Souam [14] generalized Riemann's minimal surfaces [17,27] in $\mathbb{R}^3$ foliated by circles and lines to maximal surfaces in Lorentz-Minkowski space $\mathbb{L}^3 = \left(\mathbb{R}^3,\, d{x_1}^2 + d{x_2}^2 - d{x_3}^2\right)$ foliated by conic sections. Unlike Euclidean space, since Lorentz-Minkowski space $\mathbb{L}^3$ admits three different rotational isometries (elliptic, hyperbolic, parabolic rotations), there exist fruitful examples of maximal surfaces foliated by conic sections.

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