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ON SOME NEW FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX AND CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this study, some new inequalities of Hermite-Hadamard type for convex and co-ordinated convex functions via Riemann-Liouville fractional integrals are derived. It is also shown that the results obtained in this paper are the extension of some earlier ones.

1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications have drawn attention much interest in elementary mathematics. Several mathematicians have devoted their efforts to generalize, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significance in the literature (see, e.g., [16, p.137], [9]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex

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function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. For the further study of this area, one can consult [1]- [7], [13], [14].

In [10], the authors gave an inequality (1.1) for twice differentiable functions and they raised the succeeding problem:

do there exist real numbers $q,\ Q$ such that

$$f\left(\frac{a+b}{2}\right) \leq q \leq \frac{1}{b-a}\int_a^b f(x)dx \leq Q \leq \frac{f(a)+f(b)}{2}?$$

where f is convex function.

After that, in [11], Farissi gave a favorable answer to the above-given problem and found the following values of q and Q:

$$f\left(\frac{a+b}{2}\right) \leq q(\omega) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq Q(\omega) \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

where

$$\begin{aligned} q(\omega) &= \omega f\left(\frac{\omega b+(2-\omega)a}{2}\right)+(1-\omega)f\left(\frac{(1+\omega)b+(1-\omega)a}{2}\right), \\ Q(\omega) &= \frac{1}{2}\left(f(\omega b+(1-\omega)a)+\omega f(a)+(1-\omega)f(b)\right). \end{aligned}$$

Inspired by this work of Farissi, Chen gave these values of Hermite-Hadamard inequalities for co-ordinated convex functions as follows:

Theorem 1. [8] *Let $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be a co-ordinated convex function, then we have following inequality for all $\omega, \mu \in [0,1]$*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq q(\omega,\mu) \leq \int_a^b\int_c^d f(x,y)\,dydx \leq Q(\omega,\mu) \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \end{aligned} \tag{1.3}$$

where

$$
\begin{aligned}
& q(\omega,\mu) \\
= \; & \omega\mu f\left(\frac{\omega b+(2-\omega)a}{2},\frac{\mu d+(2-\mu)c}{2}\right) \\
& +\omega(1-\mu)f\left(\frac{\omega b+(2-\omega)a}{2},\frac{(1+\mu)d+(1-\mu)c}{2}\right) \\
& +(1-\omega)\mu f\left(\frac{(1+\omega)b+(1-\omega)a}{2},\frac{\mu d+(2-\mu)c}{2}\right) \\
& +(1-\omega)(1-\mu)f\left(\frac{(1+\omega)b+(1-\omega)a}{2},\frac{(1+\mu)d+(1-\mu)c}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& Q(\omega,\mu) \\
= \; & \frac{\omega\mu}{4}f(a,c)+\frac{\omega(1-\mu)}{4}f(a,d)+\frac{(1-\omega)\mu}{4}f(b,c)+\frac{(1-\omega)(1-\mu)}{4}f(b,d) \\
& +\frac{f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)}{4}+\frac{\omega}{4}f(a,\mu d+(1-\mu)c) \\
& +\frac{1-\omega}{4}f(b,\mu d+(1-\mu)c)+\frac{\mu}{4}f(\omega b+(1-\omega)a,c) \\
& +\frac{1-\mu}{4}f(\omega b+(1-\omega)a,d).
\end{aligned}
$$

The main objective of this paper is to give the fractional variant of inequalities (1.2) and (1.3).

2. Preliminaries

In this section, we review the definitions of Rieman Liouville fractional integrals for single and two variables functions.

DEFINITION 1. [12] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and J^{α}_{b-} of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad a<x$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here $\Gamma(\alpha)$ is Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

In [18], Sarikaya et al. gave the following Hermite-Hadamard inequalities concerned with the last fractional integrals.

THEOREM 2. [18] *Let $f : [a,b] \to R$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \leq \frac{f(a)+f(b)}{2}, \tag{2.1}$$

where $\alpha > 0$.

EXAMPLE 1. A function $f(x) = x^2$ is a convex function. The above inequality (2.1) holds for the given $f(x)$.

SOLUTION 1. For $\alpha = \frac{1}{2}$, $a = 1$, and $b = 2$, we have

$$\begin{aligned} \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] &= 2.36, \\ f\left(\frac{a+b}{2}\right) &= 2.25, \end{aligned}$$

and

$$\frac{f(a)+f(b)}{2} = 2.5.$$

Thus, the inequality (2.1) is true.

In [17], Sarikaya offered the following Riemann-Liouville fractional integrals and associated inequalities of Hermite-Hadamard type:

DEFINITION 2. [17] Let $f \in L_1([a,b] \times [c,d])$. Then Riemann-Liouville integrals $J_{a+,c+}^{\alpha,\beta}$, $J_{a+,d-}^{\alpha,\beta}$, $J_{b-,c+}^{\alpha,\beta}$ and $J_{b-,d-}^{\alpha,\beta}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$

are defined by

$$J_{a+,c+}^{\alpha,\beta}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}f(t,s)\,dtds,$$

$x>a,\ y>c,$

$$J_{a+,d-}^{\alpha,\beta}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}f(t,s)\,dtds,$$

$x>a,\ y<d,$

$$J_{b-,c+}^{\alpha,\beta}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}f(t,s)\,dtds,$$

$x<b,\ y>c$

and

$$J_{b-,d-}^{\alpha,\beta}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}f(t,s)\,dtds,$$

$x<b,\ y<d,$

respectively. Here Γ is a gamma function,

$$J_{a+,c+}^{0,0}f(x,y) = J_{a+,d-}^{0,0}f(x,y) = J_{b-,c+}^{0,0}f(x,y) = J_{b-,d-}^{0,0}f(x,y) = f(x,y).$$

Theorem 3. [17] *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a co-oedinated convex function on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $0\le a<b,\ 0\le c<d$ and $f\in L_1(\Delta)$. Then we have following inequalities for double fractional integrals:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\Big[J_{a+,c+}^{\alpha,\beta}f(b,d)+J_{a+,d-}^{\alpha,\beta}f(b,c) \\ &\quad +J_{b-,c+}^{\alpha,\beta}f(a,d)+J_{b-,d-}^{\alpha,\beta}f(a,c)\Big] \qquad (2.2)\\ &\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

3. Key Results

For brevity, we use the following notations in upcoming new results:

$$\Delta_1 = [a, \omega b+(1-\omega)a],\ \Delta_2 = [\omega b+(1-\omega)a, b].$$

and

$$\Delta = [a,b]\times[c,d] = \Delta_3\cup\Delta_4\cup\Delta_5\cup\Delta_6$$

where

$$\begin{aligned}\Delta_3 &= [a, \omega b + (1-\omega) a] \times [c, \mu d + (1-\mu) c], \\ \Delta_4 &= [a, \omega b + (1-\omega) a] \times [\mu d + (1-\mu) c, d], \\ \Delta_5 &= [\omega b + (1-\omega) a, b] \times [c, \mu d + (1-\mu) c], \\ \Delta_6 &= [\omega b + (1-\omega) a, b] \times [\mu d + (1-\mu) c, d].\end{aligned}$$

THEOREM 4. *Suppose that $f : I \to \mathbb{R}$ is a convex function, then following inequalities hold for all $\omega \in [0,1]$,*

$$f\left(\frac{a+b}{2}\right) \le q(\omega) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} I^\alpha(f) \le Q(\omega) \le \frac{f(a)+f(b)}{2},$$

where

$$\begin{aligned}I^\alpha(f) &= \frac{1}{\omega^{\alpha-1}}\left[J_{a+}^\alpha f(\omega b + (1-\omega) a) + J_{(\omega b+(1-\omega)a)-}^\alpha f(a)\right] \\ &\quad + \frac{1}{(1-\omega)^{\alpha-1}}\left[J_{b-}^\alpha f(\omega b + (1-\omega) a) + J_{(\omega b+(1-\omega)a)+}^\alpha f(b)\right], \\ q(\omega) &= \omega\left(f\left(\frac{\omega b + (2-\omega) a}{2}\right) + (1-\omega) f\left(\frac{(1+\omega) b + (1-\omega) a}{2}\right)\right), \\ Q(\omega) &= \frac{1}{2}\left(f(\omega b + (1-\omega) a) + \omega f(a) + (1-\omega) f(b)\right)\end{aligned}$$

and $\alpha > 0$.

Proof. From inequalities in (2.1) over the Δ_1, we have

$$\begin{aligned}& f\left(\frac{\omega b + (2-\omega) a}{2}\right) \\ \le\ & \frac{\Gamma(\alpha+1)}{2\omega^\alpha (b-a)^\alpha}\left[J_{a+}^\alpha f(\omega b + (1-\omega) a) + J_{(\omega b+(1-\omega)a)-}^\alpha f(a)\right] \\ \le\ & \frac{f(a) + f(\omega b + (1-\omega) a)}{2}.\end{aligned} \tag{3.1}$$

Again from inequalities in (2.1) over Δ_2, we find that

$$\begin{aligned}& f\left(\frac{(1+\omega) b + (1-\omega) a}{2}\right) \\ \le\ & \frac{\Gamma(\alpha+1)}{2(1-\omega)^\alpha (b-a)^\alpha}\left[J_{(\omega b+(1-\omega)a)+}^\alpha f(b) + J_{b-}^\alpha f(\omega b + (1-\omega) a)\right] \\ \le\ & \frac{f(b) + f(\omega b + (1-\omega) a)}{2}.\end{aligned} \tag{3.2}$$

Multiplying (3.1), (3.2) by ω and $(1-\omega)$, respectively. After that, adding the resultant inequalities, we obtain that

$$q(\omega) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} I^{\alpha}(f) \leq Q(\omega). \tag{3.3}$$

Since f is convex function, so we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \\ = \; & f\left(\omega\left(\frac{\omega b+(1-\omega)a+a}{2}\right)+(1-\omega)\left(\frac{\omega b+(1-\omega)a+b}{2}\right)\right) \\ \leq \; & \omega f\left(\frac{\omega b+(1-\omega)a+a}{2}\right)+(1-\omega) f\left(\frac{\omega b+(1-\omega)a+b}{2}\right) \\ \leq \; & \frac{1}{2}\left(f(\omega b+(1-\omega)a)+\omega f(a)+(1-\omega) f(b)\right) \\ \leq \; & \frac{f(a)+f(b)}{2}. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we conclude the desired inequality. □

Remark 1. Under the hypothesis of Theorem 4 with $\alpha=1$, we have [11, Theorem 1.1].

Corollary 1. *Under the same conditions and notations stated in Theorem 4, we have the following new inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \sup_{\omega\in[0,1]} q(\omega) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} I^{\alpha}(f) \leq \inf_{\omega\in[0,1]} Q(\omega) \leq \frac{f(a)+f(b)}{2}.$$

Theorem 5. *Let $f:\Delta\to\mathbb{R}$ be a co-ordinated convex function and $f\in L(\Delta)$, then the following inequalities satisfy for all $\omega,\mu\in[0,1]$:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq \; & q(\omega,\mu) \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} I^{\alpha,\beta}(f) \leq Q(\omega,\mu) \\ \leq \; & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \end{aligned}$$

where

$$I^{\alpha,\beta}(f)=\frac{1}{\omega^{\alpha-1}\mu^{\beta-1}}\left\{J^{\alpha,\beta}_{a+,c+}f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)\right.$$

$$+J^{\alpha,\beta}_{a+,(\mu d+(1-\mu)c)-}f(\omega b+(1-\omega)a,c)+J^{\alpha,\beta}_{(\omega b+(1-\omega)a)-,c+}f(a,\mu d+(1-\mu)c)$$

$$\left.+J^{\alpha,\beta}_{(\omega b+(1-\omega)a)-,(\mu d+(1-\mu)c)-}f(a,c)\right\}$$

$$+\frac{1}{\omega^{\alpha-1}(1-\mu)^{\beta-1}}\left\{J^{\alpha,\beta}_{a+,(\mu d+(1-\mu)c)+}f(\omega b+(1-\omega)a,d)\right.$$

$$+J^{\alpha,\beta}_{a+,d-}f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)+J^{\alpha,\beta}_{(\omega b+(-\omega)a)-,(\mu d+(1-\mu)c)+}f(a,d)$$

$$\left.+J^{\alpha,\beta}_{(\omega b+(-\omega)a)-,d-}f(a,\mu d+(1-\mu)c)\right\}$$

$$+\frac{1}{(1-\omega)^{\alpha-1}\mu^{\beta-1}}\left\{J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,c+}f(b,\mu d+(1-\mu)c)\right.$$

$$+J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,(\mu d+(1-\mu)c)-}f(b,c)+J^{\alpha,\beta}_{b-,c+}f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)$$

$$\left.+J^{\alpha,\beta}_{b-,(\mu d+(1-\mu)c)-}f(\omega b+(1-\omega)a,c)\right\}$$

$$+\frac{1}{(1-\omega)^{\alpha-1}(1-\mu)^{\beta-1}}\left\{J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,(\mu d+(1-\mu)c)+}f(b,d)\right.$$

$$+J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,d-}f(b,\mu d+(1-\mu)c)+J^{\alpha,\beta}_{b-,(\mu d+(1-\mu)c)+}f(\omega b+(1-\omega)a,d)$$

$$\left.+J^{\alpha,\beta}_{b-,d-}f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)\right\},$$

and $\alpha,\beta>0$.

Proof. From inequalities given in(2.2) for $\Delta_3,\ \Delta_4,\ \Delta_5,\ \Delta_6$ with $\omega\neq 0,1$ and $\mu\neq 0,1$, we get that

$$f\left(\frac{\omega b+(2-\omega)a}{2},\frac{\mu d+(2-\mu)c}{2}\right) \tag{3.5}$$

$$\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\omega^{\alpha}\mu^{\beta}(b-a)^{\alpha}(d-c)^{\beta}}$$

$$\times\Big[J^{\alpha,\beta}_{a+,c+}f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)$$

$$+J^{\alpha,\beta}_{a+,(\mu d+(1-\mu)c)-}f(\omega b+(1-\omega)a,c)$$

$$+J^{\alpha,\beta}_{(\omega b+(1-\omega)a)-,c+}f(a,\mu d+(1-\mu)c)$$

$$+J^{\alpha,\beta}_{(\omega b+(1-\omega)a)-,(\mu d+(1-\mu)c)-}f(a,c)\Big]$$

$$\leq \frac{1}{4}\left[f(a,c)+f(a,\mu d+(1-\mu)c)+f(\omega b+(1-\omega)a,c)\right.$$

$$\left.+f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)\right],$$

$$f\left(\frac{\omega b+(2-\omega)a}{2},\frac{(1+\mu)d+(1-\mu)c}{2}\right) \tag{3.6}$$

$$\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\omega^{\alpha}(1-\mu)^{\beta}(b-a)^{\alpha}(d-c)^{\beta}}$$

$$\times\Big[J^{\alpha,\beta}_{a+,(\mu d+(1-\mu)c)+}f(\omega b+(1-\omega)a,d)$$

$$+J^{\alpha,\beta}_{a+,d-}f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)$$

$$+J^{\alpha,\beta}_{(\omega b+(-\omega)a)-,(\mu d+(1-\mu)c)+}f(a,d)$$

$$+J^{\alpha,\beta}_{(\omega b+(-\omega)a)-,d-}f(a,\mu d+(1-\mu)c)\Big]$$

$$\leq \frac{1}{4}\left[f(a,\mu d+(1-\mu)c)+f(a,d)\right.$$

$$\left.+f(\omega b+(-\omega)a,\mu d+(1-\mu)c)+f(\omega b+(-\omega)a,d)\right],$$

$$f\left(\frac{(1+\omega)b+(1-\omega)a}{2},\frac{\mu d+(2-\mu)c}{2}\right) \tag{3.7}$$

$$\begin{aligned}
&\le \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(1-\omega)^{\alpha}\mu^{\beta}(b-a)^{\alpha}(d-c)^{\beta}} \\
&\quad\times\Big[J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,c+}f(b,\mu d+(1-\mu)c) \\
&\quad+J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,(\mu d+(1-\mu)c)-}f(b,c) \\
&\quad+J^{\alpha,\beta}_{b-,c+}f(\omega b+(1-\omega)a,\mu d+(1-\mu)c) \\
&\quad+J^{\alpha,\beta}_{b-,(\mu d+(1-\mu)c)-}f(\omega b+(1-\omega)a,c)\Big] \\
&\le \frac{1}{4}\left[f(\omega b+(1-\omega)a,c)+f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)\right. \\
&\quad\left.+f(b,c)+f(b,\mu d+(1-\mu)c)\right],
\end{aligned}$$

$$f\left(\frac{(1+\omega)b+(1-\omega)a}{2},\frac{(1+\mu)d+(1-\mu)c}{2}\right) \tag{3.8}$$

$$\begin{aligned}
&\le \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(1-\omega)^{\alpha}(1-\mu)^{\beta}(b-a)^{\alpha}(d-c)^{\beta}} \\
&\quad\times\Big[J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,(\mu d+(1-\mu)c)+}f(b,d) \\
&\quad+J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,d-}f(b,\mu d+(1-\mu)c) \\
&\quad+J^{\alpha,\beta}_{b-,(\mu d+(1-\mu)c)+}f(\omega b+(1-\omega)a,d) \\
&\quad+J^{\alpha,\beta}_{b-,d-}f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)\Big]
\end{aligned}$$

$$\leq \frac{1}{4}\left[f\left(\omega b+(1-\omega)a, \mu d+(1-\mu)c\right)+f\left(\omega b+(1-\omega)a, d\right)\right.$$
$$\left.+f\left(b, \mu d+(1-\mu)c\right)+f(b,d)\right].$$

Multiplying (3.5), (3.6), (3.7) and (3.8) by $\omega\mu$, $\omega(1-\mu)$, $(1-\omega)\mu$ and $(1-\omega)(1-\mu)$, respectively. After that, adding the resultant inequalities, we found that

$$\begin{aligned}
(3.9)\quad & \omega\mu f\left(\frac{\omega b+(2-\omega)a}{2}, \frac{\mu d+(2-\mu)c}{2}\right) \\
& +\omega(1-\mu) f\left(\frac{\omega b+(2-\omega)a}{2}, \frac{(1+\mu)d+(1-\mu)c}{2}\right) \\
& +(1-\omega)\mu f\left(\frac{(1+\omega)b+(1-\omega)a}{2}, \frac{\mu d+(2-\mu)c}{2}\right) \\
& +(1-\omega)(1-\mu) f\left(\frac{(1+\omega)b+(1-\omega)a}{2}, \frac{(1+\mu)d+(1-\mu)c}{2}\right) \\
= \; & q(\omega,\mu) \\
\leq \; & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\
& \times\left[\frac{1}{\omega^{\alpha-1}\mu^{\beta-1}}\left\{J^{\alpha,\beta}_{a+,c+} f\left(\omega b+(1-\omega)a, \mu d+(1-\mu)c\right)\right.\right. \\
& +J^{\alpha,\beta}_{a+,(\mu d+(1-\mu)c)-} f\left(\omega b+(1-\omega)a, c\right) \\
& +J^{\alpha,\beta}_{(\omega b+(1-\omega)a)-,c+} f\left(a, \mu d+(1-\mu)c\right) \\
& \left.+J^{\alpha,\beta}_{(\omega b+(1-\omega)a)-,(\mu d+(1-\mu)c)-} f(a,c)\right\} \\
& +\frac{1}{\omega^{\alpha-1}(1-\mu)^{\beta-1}}\left\{J^{\alpha,\beta}_{a+,(\mu d+(1-\mu)c)+} f\left(\omega b+(1-\omega)a, d\right)\right. \\
& +J^{\alpha,\beta}_{a+,d-} f\left(\omega b+(1-\omega)a, \mu d+(1-\mu)c\right) \\
& +J^{\alpha,\beta}_{(\omega b+(-\omega)a)-,(\mu d+(1-\mu)c)+} f(a,d)
\end{aligned}$$

$$
\begin{aligned}
& +J^{\alpha,\beta}_{(\omega b+(-\omega)a)-,d-}f\left(a,\mu d+\left(1-\mu\right)c\right)\Big\} \\
& +\frac{1}{\left(1-\omega\right)^{\alpha-1}\mu^{\beta-1}}\Big\{J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,c+}f\left(b,\mu d+\left(1-\mu\right)c\right) \\
& +J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,(\mu d+(1-\mu)c)-}f\left(b,c\right) \\
& +J^{\alpha,\beta}_{b-,c+}f\left(\omega b+\left(1-\omega\right)a,\mu d+\left(1-\mu\right)c\right) \\
& +J^{\alpha,\beta}_{b-,(\mu d+(1-\mu)c)-}f\left(\omega b+\left(1-\omega\right)a,c\right)\Big\} \\
& +\frac{1}{\left(1-\omega\right)^{\alpha-1}\left(1-\mu\right)^{\beta-1}}\Big\{J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,(\mu d+(1-\mu)c)+}f\left(b,d\right) \\
& +J^{\alpha,\beta}_{(\omega b+(1-\omega)a)+,d-}f\left(b,\mu d+\left(1-\mu\right)c\right) \\
& +J^{\alpha,\beta}_{b-,(\mu d+(1-\mu)c)+}f\left(\omega b+\left(1-\omega\right)a,d\right) \\
& +J^{\alpha,\beta}_{b-,d-}f\left(\omega b+\left(1-\omega\right)a,\mu d+\left(1-\mu\right)c\right)\Big\}\Big] \\
\leq\ & \frac{\omega\mu}{4}\left[f\left(a,c\right)+f\left(a,\mu d+\left(1-\mu\right)c\right)+f\left(\omega b+\left(1-\omega\right)a,c\right)\right. \\
& \left.+f\left(\omega b+\left(1-\omega\right)a,\mu d+\left(1-\mu\right)c\right)\right] \\
& +\frac{\omega\left(1-\mu\right)}{4}\left[f\left(a,\mu d+\left(1-\mu\right)c\right)+f\left(a,d\right)\right. \\
& \left.+f\left(\omega b+\left(-\omega\right)a,\mu d+\left(1-\mu\right)c\right)+f\left(\omega b+\left(-\omega\right)a,d\right)\right] \\
& +\frac{\left(1-\omega\right)\mu}{4}\left[f\left(\omega b+\left(1-\omega\right)a,c\right)\right. \\
& f\left(\omega b+\left(1-\omega\right)a,\mu d+\left(1-\mu\right)c\right)
\end{aligned}
$$

$$
\begin{aligned}
& +f(b,c)+f(b,\mu d+(1-\mu)c)] \\
& +\frac{(1-\omega)(1-\mu)}{4}\left[f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)\right. \\
& +f(\omega b+(1-\omega)a,d) \\
& \left.+f(b,\mu d+(1-\mu)c)+f(b,d)\right] \\
= \; & \frac{\omega\mu}{4}f(a,c)+\frac{\omega(1-\mu)}{4}f(a,d) \\
& +\frac{(1-\omega)\mu}{4}f(b,c)+\frac{(1-\omega)(1-\mu)}{4}f(b,d) \\
& +\frac{f(\omega b+(1-\omega)a,\mu d+(1-\mu)c)}{4} \\
& +\frac{\omega}{4}f(a,\mu d+(1-\mu)c) \\
& +\frac{1-\omega}{4}f(b,\mu d+(1-\mu)c)+\frac{\mu}{4}f(\omega b+(1-\omega)a,c) \\
& +\frac{1-\mu}{4}f(\omega b+(1-\omega)a,d) \\
= \; & Q(\omega,\mu).
\end{aligned}
$$

Since f is co-ordinated convex function, so we obtain that

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
= \; & f\left(\begin{array}{c}\omega\frac{\omega b+(2-\omega)a}{2}+(1-\omega)\frac{(1+\omega)b+(1-\omega)a}{2}, \\ \mu\frac{\mu d+(2-\mu)c}{2}+(1-\mu)\frac{(1+\mu)d+(1-\mu)c}{2}\end{array}\right) \\
\leq \; & \omega\mu f\left(\frac{\omega b+(2-\omega)a}{2},\frac{\mu d+(2-\mu)c}{2}\right) \\
& +\omega(1-\mu)f\left(\frac{\omega b+(2-\omega)a}{2},\frac{(1+\mu)d+(1-\mu)c}{2}\right) \\
& +(1-\omega)\mu f\left(\frac{(1+\omega)b+(1-\omega)a}{2},\frac{\mu d+(2-\mu)c}{2}\right)
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
& +(1-\omega)(1-\mu) f\left(\frac{(1+\omega) b+(1-\omega) a}{2}, \frac{(1+\mu) d+(1-\mu) c}{2}\right) \\
= \;& q(\omega, \mu) .
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
& Q(\omega, \mu) \\
\leq\; & \frac{\omega \mu}{4} f(a, c)+\frac{\omega(1-\mu)}{4} f(a, d) \\
& +\frac{(1-\omega) \mu}{4} f(b, c)+\frac{(1-\omega)(1-\mu)}{4} f(b, d) \\
& +\frac{(1-\omega)(1-\mu)}{4} f(a, c)+\frac{(1-\omega) \mu}{4} f(a, d) \\
& +\frac{\omega(1-\mu)}{4} f(b, c)+\frac{\omega \mu}{4} f(b, d) \\
& +\frac{\omega \mu}{4} f(a, d)+\frac{\omega(1-\mu)}{4} f(a, c) \\
& +\frac{(1-\omega) \mu}{4} f(b, d)+\frac{(1-\omega)(1-\mu)}{4} f(b, c) \\
& +\frac{\mu \omega}{4} f(b, c)+\frac{(1-\omega) \mu}{4} f(a, c) \\
& +\frac{\omega(1-\mu)}{4} f(b, d)+\frac{(1-\omega)(1-\mu)}{4} f(a, d) \\
=\; & \frac{f(a, c)+f(a, d)+f(b, c)+f(b, d)}{4} .
\end{aligned} \tag{3.11}
$$

By (3.9), (3.10) and (3.11), we conclude the required inequality. □

Remark 2. Under the same assumptions given stated in Theorem 5 with $\alpha=\beta=1$, then we have [8, Theorem 2.1].

Remark 3. Under the same assumptions stated in Theorem 5 with $\omega=\mu=\frac{1}{2}$ and $\alpha=\beta=1$, then we have result of [15, Theorem 2.6].

Corollary 2. *Under the same conditions and notations stated in Theorem 5, we have following inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \sup_{\omega,\mu\in[0,1]} q\left(\omega,\mu\right) \leq I^{\alpha,\beta}\left(f\right) \\ &\leq \inf_{\omega,\mu\in[0,1]} Q\left(\omega,\mu\right) \\ &\leq \frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}. \end{aligned}$$

4. Conclusion

In this investigation, a new fractional version of Hermite-Hadamard type inequalities for convex and co-ordinated convex functions is derived. Some existing and new inequalities are also obtained in the special cases of the main results. The authors hope that this work may stimulate further research in different areas of pure and applied sciences.

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