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A CHARACTERIZATION OF w-ARTINIAN MODULES

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Abstract. Let R be a commutative ring with identity and let M be a w-module over R. Denote by $\mathscr{F}_M$ the set of all w-submodules of M such that $(M/N)_w$ is w-cofinitely generated. Then it is shown that M is w-Artinian if and only if $\mathscr{F}_M$ is closed under arbitrary intersections, if and only if $\mathscr{F}_M$ satisfies the descending chain condition.

1. Introduction

Throughout this paper, we assume that R is a commutative ring with identity and any R-module is unitary.

Vamos [6] was the first to define and study "finitely embedded modules" as the dual of "finitely generated modules" to characterize Artinian modules. An R-module M is said to be *finitely embedded* (later called by Jans as *cofinitely generated* [4] and by Anderson and Fuller as *finitely cogenerated* [1]) if $E(M) = E(S_1) \oplus \cdots \oplus E(S_n)$, where each S_i is a simple R-submodule of M and $E(N)$ denotes the injective envelope of a module N. There are many characterizations of cofinitely generated modules, for example, a module M is cofinitely generated if and only if for each chain $\mathscr{C} = \{N_i \mid i \in I\}$ of nonzero submodules of M such that $\bigcap \mathscr{S} = 0$, there is an index $j \in I$ such that $N_j = 0$ [5, Theorema 1.1].

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Let M be an R-module. Denote by

$$\Lambda_M = \{N \leq M \mid M/N \text{ is cofinitely generated}\}.$$

In [5, Theorema 2.2], Pẽna characterized Artinian modules as follows: M is Artinian if and only if Λ_M is closed under arbitrary intersections, if and only if Λ_M satisfies the descending chain condition.

This paper is intended to give a characterization of w-Artinian module, which is the w-analog of [5, Theorema 2.2]. To do so, we first introduce the w-theory briefly.

Let J be a finitely generated ideal of R. If the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism, then J is called a *GV-ideal*, denoted by $J \in$GV(R). Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

Thus $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. And M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). Clearly R is a GV-torsion-free R-module ([8, Corollary 1.5]). A GV-torsion-free module M is called a *w-module* if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$. The w-envelope of a GV-torsion-free module M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}.$$

It is easy to see that M is a w-module if and only if $M_w = M$. A w-module M is of *w-finite type* if $M = N_w$ for some finitely generated submodule N of M.

The following theorem [7, Theorem 6.1.17] is used throughout the paper without mentioning it.

Theorem 1.1. *The following statements are equivalent for a GV-torsion-free module M.*

(1) *M is a w-module.*
(2) *If $0 \to M \to N \to C \to 0$ is an exact sequence in which N is a w-module, then C is GV-torsion-free.*
(3) *There exists an exact sequence $0 \to M \to N \to C \to 0$ such that N is a w-module and C is GV-torsion-free.*

Following [10, Definition 2.1], a w-module M is said to be *w-cofinitely generated* if for any set $\{M_i \mid i \in \Omega\}$ of w-submodules of M satisfying $\bigcap_{i\in\Omega} M_i = 0$, there exists a finite subset Λ of Ω such that $\bigcap_{j\in\Lambda} M_j = 0$.

Recall that a nonzero w-module M is called *w-simple* if M has no nontrivial w-submodules [7, Definition 6.5.1]. A w-module M is w-simple if and only if $M = (Rx)_w$ for some nonzero element $x \in M$ [7, Proposition 6.5.2]. The *w-socle* of a w-module M, denoted by $w\text{-soc}(M)$, is the (direct) sum of its all w-simple submodules [9, Definition 1.1]. A w-module M is called a *w-Artinian module* if M satisfies the DCC on w-submodules of M [7, Definition 6.9.1], equivalently, M has the minimal condition on w-submodules of M [7, Theorem 6.9.2].

Any undefined terminology is standard, as in [7].

2. Results

In this section, first we give a characterization of w-cofinitely generated modules, which is the w-analog of [5, Theorem 1.1]. Then we characterize w-Artinian modules.

THEOREM 2.1. *The following conditions are equivalent for a w-module M.*

(1) *M is w-cofinitely generated.*
(2) *For each chain $\mathscr{C} = \{N_i \mid i \in I\}$ of w-submodules of M such that $\bigcap \mathscr{C} = 0$, there exists an index $j \in I$ such that $N_j = 0$.*

Proof. (1) $\Rightarrow$ (2) This follows directly from the definition of w-cofinitely generated modules.

(2) $\Rightarrow$ (1) Suppose that $M \neq 0$ and let $S := w\text{-soc}(M) = \bigoplus_{k \in K} S_k$, where each S_i is a w-simple submodule of M and K is an indexed set. Denote by $\mathscr{L}_w$ the set of nonzero w-submodules of M. By (2) and applying Zorn's Lemma (descending version) to $\mathscr{L}_w$, M has a minimal element, and hence $S \neq 0$. Now let N be a nonzero w-submodule of M and let $\mathscr{L}_w(N)$ denote the set of nonzero w-submodule of N. Again by (2) and applying Zorn's Lemma (descending version) to $\mathscr{L}_w(N)$, N has a minimal element, and so every nonzero w-submodule of M contains a w-simple submodule. Then it is easy to see that M is an essential extension of S.

Now we will prove that K is finite, and hence S is of w-finite type. Assume on the contrary that K is infinite. Then there exists an infinitely countable subset, say $\{k_1, k_2, \dots\}$, of K. So we have the following nonzero w-submodules of M: $M_n := \bigoplus_{j \geq n} S_{k_j}$ for each $n \geq 1$. Then

$\{M_n \mid n \geq 1\}$ is a descending chain of nonzero w-submodules of M, and so by (2) we have that $\bigcap_{n\geq 1} M_n \neq 0$, which quickly generates a contradiction.

Therefore S is of w-finite type and essential in M. By [10, Theorem 2.4], M is w-cofinitely generated. □

A w-module M is said to be *w-subdirectly irreducible* provided that the intersection V of all nonzero w-submodules of M is nonzero, that is, M has a unique minimal w-submodule V contained in every nonzero w-submodule. Clearly if M is such a module, then $E(M) = E(S)$ for some w-simple submodule S of M, and so M is w-cofinitely generated by [10, Theorem 2.4]. If N is a proper w-submodule of M so that $(M/N)_w$ is a w-subdirectly irreducible module, then we say that N is a w-subdirectly irreducible submodule of M. By the straightforward application of Zorn's Lemma, one proves the following result, which is the w-analog of [2, 2.17C].

Lemma 2.2. *(w-version of Birkhoff's theorem) Let M be a w-module and N be a proper w-submodule of M. Then for any $x \in M \setminus N$ there exists a w-submodule $N_x \supseteq N$ maximal with respect to excluding x. Furthermore N_x is a w-subdirectly irreducible submodule and N is the intersection of all such N_x.*

Proof. One easily checks that the set $\mathscr{S}$ of all w-submodules that contain N and exclude x is inductive. Hence $\mathscr{S}$ contains a maximal element N_x by Zorn's Lemma. Furthermore, every w-submodule of M that properly contains N_x also contains x, and hence $(M/N_x)_w$ is w-subdirectly irreducible. Obviously N is the intersection of the sets $\{N_x\}_{x\in M\setminus N}$. □

Denote by τ_w the hereditary torsion theory induced by a (Gabriel) topology

$$\{I \leq R \mid I_w = R\}.$$

Let M be an R-module. Then a submodule N of M is said to be *τ_w-pure in M* if M/N is GV-torsion-free. An R-module M is called *τ_w-cofinitely generated* if for any set $\{M_i \mid i \in \Omega\}$ of τ_w-pure submodules of M satisfying $\bigcap_{i\in\Omega} M_i = \mathrm{tor}_{\mathrm{GV}}(M)$, there exists a finite subset Ω_0 of Ω such that $\bigcap_{i\in\Omega_0} M_i = \mathrm{tor}_{\mathrm{GV}}(M)$.

Let M be a w-module. Denote by $\mathscr{F}_M$ the set of w-submodules of M such that $(M/N)_w$ is w-cofinitely generated. For each w-submodule N

of M, we will use the following notation:

$$\mathscr{F}_M(N) := \{L \in \mathscr{F}_M \mid N \subseteq L\}.$$

Proposition 2.3. *Let M be a w-module and let N be a w-submodule of M. If $\mathscr{F}_M(N)$ satisfies the descending chain condition, then $N \in \mathscr{F}_M$. In particular, if $\mathscr{F}_M$ satisfies this condition, then M is w-cofinitely generated.*

Proof. Suppose on the contrary that $N \notin \mathscr{F}_M$. Then $N \neq M$, and so there exists $x \in M \setminus N$. By Lemma 2.2, there exists $L_1 \in \mathscr{F}_M$ such that $N \subseteq L_1$ and $x \notin L_1$. Hence one gets $N \subsetneq L_1 \subsetneq M$. Now, as before, if $x_1 \in L_1$ such that $x_1 \notin N$, then there exists $L_2 \in \mathscr{F}_{L_1}$ such that $N \subseteq L_2$ and $x_1 \notin L_2$. Now consider the following exact sequence

$$0 \to L_1/L_2 \to M/L_2 \to M/L_1 \to 0.$$

By [10, Proposition 2.10], L_1/L_2 and M/L_1 are τ_w-cofinitely generated. By [3, Proposition 1.6], M/L_2 is τ_w-cofinitely generated. Again by [10, Proposition 2.10], $(M/L_2)_w$ is w-cofinitely generated, and so $L_2 \in \mathscr{F}_M$. Also note that $N \subsetneq L_2 \subsetneq L_1 \subsetneq M$.

Continuing in this way, one could construct a strictly descending chain in $\mathscr{F}_M(N)$, which is a contradiction. Therefore $N \in \mathscr{F}_M$. □

Now we give a characterization of w-Artinian modules in terms of $\mathscr{F}_M$.

Theorem 2.4. *The following statements are equivalent for a w-module M.*

(1) *M is a w-Artinian module.*
(2) *$\mathscr{F}_M$ is closed under arbitrary intersections.*
(3) *$\mathscr{F}_M$ satisfies the descending chain condition.*

Proof. (1) ⇒ (2) This follows from Proposition 2.3.

(2) ⇒ (3) Assume (2) holds and let

$$M = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$$

be a descending chain in $\mathscr{F}_M$. Then by (2), $L := \bigcap_{i \geq 0} L_i \in \mathscr{F}_M$. Hence one has the following descending chain of GV-torsion-free submodules of M/L:

$$M/L = L_0/L \supseteq L_1/L \supseteq L_2/L \supseteq \cdots$$

Note that M/L is τ_w-cofinitely generated. Moreover, as $\bigcap_{i\geq 0}(L_i/L) = 0$, there exists a nonnegative integer n such that $L_n = L$. Therefore $L_i = L$ for each $i \geq n$.

(3) $\Rightarrow$ (1) By (3) and Proposition 2.3, $\mathscr{F}_M$ is the set of all w-submodules of M. Now the assertion follows immediately from (3). □

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