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ON THE UNIQUENESS OF CERTAIN TYPE OF SHIFT POLYNOMIALS SHARING A SMALL FUNCTION

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Abstract. In this article, we consider the uniqueness problem of the shift polynomials $f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ and $f^n(z)(f(z)-1)^m\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$, where $f(z)$ is a transcendental entire function of finite order, $c_j (j = 1, 2, ..., s)$ are distinct finite complex numbers and $n(\geq 1)$, $m(\geq 1)$, s and $\mu_j (j = 1, 2, ..., s)$ are integers. With the concept of weakly weighted sharing and relaxed weighted sharing we obtain some results which extend and generalize some results due to P. Sahoo [Commun. Math. Stat. 3 (2015), 227-238].

1. Introduction, Definitions and Results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. We assume that the reader is familiar with the standard notations in Nevanlinna's value distribution theory of meromorphic functions as explained in [6], [7] and [16]. For a nonconstant meromorphic function h, we denote by $T(r, h)$ the Nevanlinna characteristic function of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$ as $r \to \infty$, possibly outside of a set of finite linear

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measure. We say that the meromorphic function $\alpha(z)$ is a small function of f, if $T(r,\alpha(z)) = S(r,f)$.

Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM (counting multiplicities). On the other hand, if $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM (ignoring multiplicities). We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of distinct a-points of f with multiplicities not greater than k. We denote by $N_{k)}(r,a;f)$ the counting function of zeros of $f - a$ with multiplicity less or equal to k, and by $\overline{N}_{k)}(r,a;f)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,a;f)$ be the counting function of zeros of $f - a$ with multiplicity at least k and $\overline{N}_{(k}(r,a;f)$ the corresponding one for which multiplicity is not counted. Set

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}_{(2}(r,a;f) + ... + \overline{N}_{(k}(r,a;f).$$

Let $N_E(r,a;f,g)$ ($\overline{N}_E(r,a;f,g)$) be the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ with the same multiplicities and $N_0(r,a;f,g)$ ($\overline{N}_0(r,a;f,g)$) the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ ignoring multiplicities. If

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_E(r,a;f,g) = S(r,f) + S(r,g),$$

then we say that f and g share a "CM". On the other hand, if

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_0(r,a;f,g) = S(r,f) + S(r,g),$$

then we say that f and g share a "IM".

We now explain in the following definition the notion of weakly weighted sharing which was introduced by Lin and Lin [9].

Definition 1.1. [9] Let f and g share a "IM" and k be a positive integer or ∞. $\overline{N}^{E}_{k)}(r,a;f,g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, and both of their multiplicities are not greater than k. $\overline{N}^{0}_{(k}(r,a;f,g)$ denotes the reduced counting function of those a-points of

f which are a-points of g, both of their multiplicities are not less than k.

DEFINITION 1.2. [9] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or ∞. If

$$\overline{N}_{k)}(r, a; f) - \overline{N}^{E}_{k)}(r, a; f, g) = S(r, f),$$
$$\overline{N}_{k)}(r, a; g) - \overline{N}^{E}_{k)}(r, a; f, g) = S(r, g),$$
$$\overline{N}_{(k+1}(r, a; f) - \overline{N}^{0}_{(k+1}(r, a; f, g) = S(r, f),$$
$$\overline{N}_{(k+1}(r, a; g) - \overline{N}^{0}_{(k+1}(r, a; f, g) = S(r, g),$$

or if $k = 0$ and

$$\overline{N}(r, a; f) - \overline{N}_0(r, a; f, g) = S(r, f),$$
$$\overline{N}(r, a; g) - \overline{N}_0(r, a; f, g) = S(r, g),$$

then we say f and g weakly share a with weight k and we write f and g share "(a, k)" to mean that f, g weakly share a with weight k.

Now it is clear from definition 1.2 that weakly weighted sharing is a scaling between IM and CM.

In 2007, A. Banerjee and S. Mukherjee [1] introduced a new type of sharing which is weaker than weakly weighted sharing and is defined as follows.

DEFINITION 1.3. [1] We denote by $\overline{N}(r, a; f \mid= p; \mid= q)$ the reduced counting function of common a-points of f and g with multiplicities p and q, respectively.

DEFINITION 1.4. [1] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or ∞. Suppose that f and g share a "IM". If for $p \neq q$,

$$\sum_{p,q \leq k} \overline{N}(r, a; f \mid= p; g \mid= q) = S(r),$$

then we say that f and g share a with weight k in a relaxed manner and in that case we write f and g share $(a, k)^*$.

Recently, the topic of shift equation and shift product in the complex plane $\mathbb{C}$ has attracted many mathematicians, a large number of papers have focused on value distribution of shifts and shift operator analogues of Nevanlinna theory (including [3], [4], [5], [8] and [13]) and many people paid their attention to the uniqueness of shifts and shift polynomials of meromorphic function and obtained many interesting results. In this direction J.L. Zhang [17] considered the zeros of certain type of shift polynomials and proved the following result for small functions.

THEOREM A. *Let $f(z)$ and $g(z)$ be two transcendental entire function of finite order, $\alpha(z)$ $(\not\equiv 0)$ be a small function with respect to $f(z)$ and c be a nonzero complex constant. If $n \geq 2$ is an integer then $f^n(z)(f(z)-1)f(z+c)-\alpha(z)$ has infinitely many zeros.*

In the same paper the author also proved the following uniqueness result.

THEOREM B. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z)=g(z)$.*

In 2014, using the idea of weakly weighted sharing and relaxed weighted sharing C. Meng [12] obtained the following uniqueness theorems which improve and supplement Theorem B in different directions.

THEOREM C. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share "$(\alpha(z),2)$", then $f(z)\equiv g(z)$.*

THEOREM D. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n \geq 10$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $(\alpha(z),2)^*$, then $f(z)\equiv g(z)$.*

THEOREM E. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n \geq 16$ is an integer. If*

$$\overline{E}_{2)}\Big(\alpha(z), f^n(z)(f(z)-1)f(z+c)\Big)=\overline{E}_{2)}\Big(\alpha(z), g^n(z)(g(z)-1)g(z+c)\Big),$$

then $f(z)\equiv g(z)$.

In 2015, P. Sahoo [14] studied the uniqueness problem of shift polynomials of the form $f^n(z)(f^m(z)-1)f(z+c)$ and $f^n(z)(f(z)-1)^m f(z+c)$ and proved the following results which generalize Theorems C-E.

Theorem F. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant, n and $m(\geq 1)$ are integers such that $n\geq m+6$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share "$(\alpha(z),2)$", then $f(z)\equiv tg(z)$ where $t^m=1$.*

Theorem G. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant, n and $m(\geq 1)$ are integers such that $n\geq 2m+8$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $(\alpha(z),2)^*$, then $f(z)\equiv tg(z)$ where $t^m=1$.*

Theorem H. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant, n and $m(\geq 1)$ are integers such that $n\geq 4m+12$. If*

$$\overline{E}_{2)}\Big(\alpha(z),f^n(z)(f^m(z)-1)f(z+c)\Big)=\overline{E}_{2)}\Big(\alpha(z),g^n(z)(g^m(z)-1)g(z+c)\Big),$$

then $f(z)\equiv tg(z)$ where $t^m=1$.

Theorem I. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant, n and $m(\geq 1)$ are integers satisfying $n+m\geq 10$. If $f^n(z)(f(z)-1)^m f(z+c)$ and $g^n(z)(g(z)-1)^m g(z+c)$ share "$(\alpha(z),2)$", then either $f(z)\equiv g(z)$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)$ is given by*

$$R(w_1,w_2)=w_1^n(w_1-1)^m w_1(z+c)-w_2^n(w_2-1)^m w_2(z+c).$$

Theorem J. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant, n and $m(\geq 1)$ are integers satisfying $n+m\geq 13$. If $f^n(z)(f(z)-1)^m f(z+c)$ and $g^n(z)(g(z)-1)^m g(z+c)$ share $(\alpha(z),2)^*$, then the conclusions of Theorem I hold.*

Theorem K. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant, n and $m(\geq 1)$ are integers satisfying $n+m \geq 19$. If*

$$\overline{E}_{2)}\Big(\alpha(z), f^n(z)(f(z)-1)^m f(z+c)\Big) = \overline{E}_{2)}\Big(\alpha(z), g^n(z)(g(z)-1)^m g(z+c)\Big),$$

then the conclusions of Theorem I hold.

Regarding the results of P. Sahoo stated above it is natural to ask the following question which is the motive of the present paper.

Question 1.1. What can be said about the relationship between two entire functions $f(z)$ and $g(z)$ if one replace the difference polynomial $f^n(z)(f^m(z)-1)f(z+c)$ by $f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ in Theorems F-H and $f^n(z)(f(z)-1)^m f(z+c)$ by $f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ in Theorems I-K, where $f(z)$ is a transcendental entire function of finite order, $c_j (j=1,2,...,s)$, $n(\geq 1)$, $m(\geq 1)$, s and $\mu_j (j=1,2,...,s)$ are integers?

For the sake of simplicity we also use the notation $\sigma = \sum_{j=1}^{s} \mu_j$.

In the paper, our main concern is to find the possible answer of the above question. We prove following theorems which extend and generalize Theorems F-K. The following theorems are the main results of the paper.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c_j $(j=1,2,...,s)$ be distinct finite complex numbers and $n(\geq 1)$, $m(\geq 1)$, s and $\mu_j (j=1,2,...,s)$ are integers satisfying $n \geq \max\{m+\sigma+5, m+5\sigma\}$. If $f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ and $g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$ share "$(\alpha(z), 2)$", then $f(z) = tg(z)$ where $t^{m+\sigma} = 1$.*

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c_j $(j=1,2,...,s)$ be distinct finite complex numbers and $n(\geq 1)$, $m(\geq 1)$, s and $\mu_j(j=1,2,...,s)$ are integers satisfying $n\geq \max\{2m+2\sigma+6, m+5\sigma\}$. If $f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ and $g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$ share $(\alpha(z),2)^*$, then $f(z)=tg(z)$ where $t^{m+\sigma}=1$.*

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c_j $(j=1,2,...,s)$ be distinct finite complex numbers and $n(\geq 1)$, $m(\geq 1)$, s and $\mu_j(j=1,2,...,s)$ are integers satisfying $n\geq \max\{4m+4\sigma+8, m+5\sigma\}$. If $\overline{E}_{2)}\left(\alpha(z), f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)=\overline{E}_{2)}\left(\alpha(z), g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)$, then $f(z)=tg(z)$ where $t^{m+\sigma}=1$.*

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c_j $(j=1,2,...,s)$ be distinct finite complex numbers and $n(\geq 1)$, $m(\geq 1)$, s and $\mu_j(j=1,2,...,s)$ are integers satisfying $n+m\geq \sigma+9$. If $f^n(z)(f(z)-1)^m\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ and $g^n(z)(g(z)-1)^m\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$ share "$(\alpha(z),2)$", then either $f(z)\equiv g(z)$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)$ is given by*

$$R(w_1,w_2)=w_1^n(w_1-1)^m\prod_{j=1}^{s} w_1(z+c_j)^{\mu_j}-w_2^n(w_2-1)^m\prod_{j=1}^{s} w_2(z+c_j)^{\mu_j}.$$

Theorem 1.5. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c_j $(j=1,2,...,s)$ be distinct finite* *complex numbers and* $n(\geq 1)$, $m(\geq 1)$, s *and* $\mu_j (j = 1, 2, ..., s)$ *are integers satisfying* $n + m \geq 2\sigma + 11$. *If* $f^n(z)(f(z) - 1)^m \prod_{j=1}^{s} f(z + c_j)^{\mu_j}$ *and* $g^n(z)(g(z) - 1)^m \prod_{j=1}^{s} g(z + c_j)^{\mu_j}$ *share* $(\alpha(z), 2)^*$, *then the conclusions of 1.4 hold.*

THEOREM 1.6. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order and* $\alpha(z)(\not\equiv 0, \infty)$ *be a small function with respect to both* $f(z)$ *and* $g(z)$. *Suppose that* c_j $(j = 1, 2, ..., s)$ *be distinct finite complex numbers and* $n(\geq 1)$, $m(\geq 1)$, s *and* $\mu_j (j = 1, 2, ..., s)$ *are integers satisfying* $n+m \geq 4\sigma+15$. *If* $\overline{E}_{2)}\left(\alpha(z), f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)$ $= \overline{E}_{2)}\left(\alpha(z), g^n(z)(g(z) - 1)^m \prod_{j=1}^{s} g(z + c_j)^{\mu_j}\right)$, *then the conclusions of 1.4 hold.*

2. Lemmas

Let F and G be two nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right).$$

LEMMA 2.1. [3] *Let* $f(z)$ *be a transcendental meromorphic function of finite order, then*

$$T(r, f(z + c)) = T(r, f) + S(r, f).$$

LEMMA 2.2. [11] *Let* f *be a meromorphic function of finite order* ρ *and let* $c(\neq 0)$ *be a fixed nonzero complex constant. Then*

$$\begin{aligned} N(r, 0; f(z + c)) &\leq N(r, 0; f) + S(r, f), \\ N(r, \infty; f(z + c)) &\leq N(r, \infty; f) + S(r, f), \\ \overline{N}(r, 0; f(z + c)) &\leq \overline{N}(r, 0; f) + S(r, f), \\ \overline{N}(r, \infty; f(z + c)) &\leq \overline{N}(r, \infty; f) + S(r, f), \end{aligned}$$

outside of possible exceptional set with finite logarithmic measure.

Lemma 2.3. [2] *Let f be an entire function of finite order and $F = f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$. Then*

$$T(r,F) = (n+m+\sigma)T(r,f) + S(r,f).$$

Lemma 2.4. *Let f be an entire function of finite order and $F = f^n(z)(f(z)-1)^m\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$. Then*

$$T(r,F) = (n+m+\sigma)T(r,f) + S(r,f).$$

Proof. Applying the same method of Lemma 2.3, we can easily prove it. □

Lemma 2.5. [1] *Let F and G be two nonconstant meromorphic functions that share "$(1,2)$" and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) &\leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ &\quad -\sum_{p=3}^{\infty} \overline{N}\big(r,0;\frac{G'}{G} \mid\geq p\big) + S(r,F) + S(r,G), \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2.6. [1] *Let F and G be two nonconstant meromorphic functions that share $(1,2)^*$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) &\leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ &\quad +\overline{N}(r,0;F) + \overline{N}(r,\infty;F) - m(r,1;G) \\ &\quad +S(r,F) + S(r,G), \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2.7. [10] *Let F and G be two nonconstant entire functions, and $p \geq 2$ an integer. If $\overline{E}_{p)}(1,F) = \overline{E}_{p)}(1,G)$ and $H \not\equiv 0$, then*

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G),$$

and the same inequality is true for $T(r,G)$.

Lemma 2.8. [15] *Let F and G be two nonconstant meromorphic functions and $H \equiv 0$. If*

$$\limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G)}{T(r)} < 1$$

where $T(r) = \max\{T(r,F), T(r,G)\}$, $r \in I$ *and* I *is a set with infinite linear measure, then either* $F \equiv G$ *or* $FG \equiv 1$.

LEMMA 2.9. [2] *Let* f *and* g *be transcendental entire functions of finite order. Suppose that* c_j $(j = 1, 2, ..., s)$ *be distinct finite complex numbers and* n, m, s *and* $\mu_j (j = 1, 2, ..., s)$ *are integers. If* $n \geq m + 5\sigma$ *and* $.f^n(z)(f^m(z) - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j} = g^n(z)(g^m(z) - 1)\prod_{j=1}^{s} g(z + c_j)^{\mu_j}$, *then* $f(z) = tg(z)$, *where* $t^m = t^{n+\sigma} = 1$.

3. Proof of the Theorems

Proof of Theorem 1.1. Let

$$F(z) = \frac{f^n(z)(f^m(z) - 1)\prod_{j=1}^{s} f(z + c_j)^{\mu_j}}{\alpha(z)},$$

$$G(z) = \frac{g^n(z)(g^m(z) - 1)\prod_{j=1}^{s} g(z + c_j)^{\mu_j}}{\alpha(z)}.$$

Then F and G are transcendental meromorphic functions that share "$(1, 2)$" except the zeros and poles of $\alpha(z)$. From Lemma 2.3 we get

$$T(r, F) = (n + m + \sigma)T(r, f) + S(r, f), \tag{3.1}$$

$$T(r, G) = (n + m + \sigma)T(r, g) + S(r, g). \tag{3.2}$$

If possible we may assume that $H \not\equiv 0$. Then using Lemmas 2.1, 2.2 and 2.5 we deduce that

$$\begin{aligned}
& T(r,F)+T(r,G) \\
\leq\ & 2N_2(r,0;F)+2N_2(r,0;G)+2N_2(r,\infty;F)+2N_2(r,\infty;G) \\
& +S(r,F)+S(r,G) \\
\leq\ & 4\overline{N}(r,0;f)+4\overline{N}(r,0;g)+2N(r,1;f^m)+2N(r,1;g^m) \\
& +2N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)+2N\left(r,0;\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right) \\
& +S(r,f)+S(r,g) \\
\leq\ & (2m+2\sigma+4)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \qquad (3.3)
\end{aligned}$$

Therefore from (3.1), (3.2) and (3.3) we obtain

$$(n-m-\sigma-4)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

a contradiction with the assumption that $n \geq m+\sigma+5$. Thus, we must have $H \equiv 0$.
Since

$$\begin{aligned}
& \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G) \\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,0;g)+N(r,1;f^m)+N(r,1;g^m) \\
& +N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)+N\left(r,0;\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right) \\
& +S(r,f)+S(r,g) \\
\leq\ & (m+\sigma+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g) \\
\leq\ & T(r),
\end{aligned}$$

where $T(r)=\max\{T(r,F),T(r,G)\}$, by Lemma 2.8, we deduce that either $F \equiv G$ or $FG \equiv 1$. Let $FG \equiv 1$. Then

$$f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j} g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}=\alpha^2$$

i.e.

$$f^n(z)(f(z)-1)(f^{m-1}(z)+f^{m-2}(z)+...+1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$$

$$g^n(z)(g(z)-1)(g^{m-1}(z)+g^{m-2}(z)+...+1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}=\alpha^2$$

It can be easily viewed from above that $N(r,0;f)=S(r,f)$ and $N(r,1;f)=S(r,f)$. Thus we obtain

$$\delta(0,f)+\delta(1,f)+\delta(1,f)=3,$$

which is not possible. Therefore, we must have $F\equiv G$ and then

$$f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\equiv g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}.$$

Therefore, by Lemma 2.9 it follows that $f(z)\equiv tg(z)$, where t is a constant satisfying $t^m=1$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let F and G be defined as in the proof of Theorem 1.1. Then F and G are transcendental meromorphic functions that share $(1,2)^*$ except the zeros and poles of $\alpha(z)$. We assume, if possible, that $H\not\equiv 0$. Using Lemma 2.1, 2.2 and 2.6 we deduce that

$$\begin{aligned}
&T(r,F)+T(r,G)\\
&\leq 2N_2(r,0;F)+2N_2(r,0;G)+2N_2(r,\infty;F)+2N_2(r,\infty;G)\\
&\quad+\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\\
&\quad-m(r,1;F)-m(r,1;G)+S(r,F)+S(r,G)\\
&\leq 5\overline{N}(r,0;f)+5\overline{N}(r,0;g)+3N(r,1;f^m)+3N(r,1;g^m)\\
&\quad+3N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)+3N\left(r,0;\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)\\
&\quad+S(r,f)+S(r,g)\\
&\leq (3m+3\sigma+5)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \qquad (3.4)
\end{aligned}$$

Therefore, using (3.1) and (3.2) we obtain from (3.4)

$$(n-2m-2\sigma-5)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction since $n \geq 2m+2\sigma+6$. Thus, we must have $H \equiv 0$. Then the result follows from the proof of Theorem 1.1. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let F and G be defined as in the proof of Theorem 1.1. Then F and G are transcendental meromorphic functions such that

$$\begin{aligned}&\overline{E}_{2)}\left(\alpha(z), f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)\\ &= \overline{E}_{2)}\left(\alpha(z), g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)\end{aligned}$$

except the zeros and poles of $\alpha(z)$.
Since

$$\begin{aligned}&2N_2(r,0;F)+2N_2(r,0;G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G)\\ \leq\ & 7\overline{N}(r,0;f)+7\overline{N}(r,0;g)+5N(r,1;f^m)+5N(r,1;g^m)\\ &+5N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)+5N\left(r,0;\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)\\ &+S(r,f)+S(r,g)\\ \leq\ & (5m+5\sigma+7)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),\end{aligned}$$

from Lemmas 2.7 and 2.8 and proceeding similarly as in the proof of Theorem 1.1, the conclusion of Theorem 1.3 follows. □

Proof of Theorem 1.4. Let

$$F(z)=\frac{f^n(z)(f(z)-1)^m\prod_{j=1}^{s} f(z+c_j)^{\mu_j}}{\alpha(z)},$$

$$G(z)=\frac{g^n(z)(g(z)-1)^m\prod_{j=1}^{s} g(z+c_j)^{\mu_j}}{\alpha(z)}.$$

Then F and G are transcendental meromorphic functions that share "$(1,2)$" except the zeros and poles of $\alpha(z)$. From Lemma 2.3 we get

$$T(r,F)=(n+m+\sigma)T(r,f)+S(r,f), \tag{3.5}$$

$$T(r,G) = (n+m+\sigma)T(r,g) + S(r,g). \tag{3.6}$$

If possible we may assume that $H \not\equiv 0$. Then using Lemmas 2.1, 2.2 and 2.5 we deduce that

$$\begin{aligned}
& T(r,F) + T(r,G) \\
\leq\ & 2N_2(r,0;F) + 2N_2(r,0;G) + 2N_2(r,\infty;F) + 2N_2(r,\infty;G) \\
& +S(r,F) + S(r,G) \\
\leq\ & 4\overline{N}(r,0;f) + 4\overline{N}(r,0;g) + 4N(r,1;f) + 4N(r,1;g) \\
& +2N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right) + 2N\left(r,0;\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right) \\
& +S(r,f) + S(r,g) \\
\leq\ & (2\sigma+8)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \qquad (3.7)
\end{aligned}$$

Therefore from (3.5), (3.6) and (3.7) we obtain

$$(n+m-\sigma-8)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

a contradiction with the assumption that $n+m \geq \sigma+9$. Thus, we must have $H \equiv 0$.
Since

$$\begin{aligned}
& \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\
\leq\ & \overline{N}(r,0;f) + \overline{N}(r,0;g) + N(r,1;f) + N(r,1;g) \\
& +N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right) + N\left(r,0;\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right) \\
& +S(r,f) + S(r,g) \\
\leq\ & (\sigma+2)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g) \\
\leq\ & T(r),
\end{aligned}$$

where $T(r) = \max\{T(r,F), T(r,G)\}$, by Lemma 2.8, we deduce that either $F \equiv G$ or $FG \equiv 1$. Let $FG \equiv 1$. Then

$$f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j} g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j} = \alpha^2.$$

It can be easily seen from above that $N(r,0;f) = S(r,f)$ and $N(r,1;f) = S(r,f)$. Thus we obtain

$$\delta(0,f) + \delta(1,f) + \delta(1,f) = 3,$$

which is not possible. Therefore, we must have $F \equiv G$ and then

(3.8)
$$f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j} = g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j},$$

Set $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (3.8), we deduce that

$$\prod_{j=1}^{s} g(z+c_j)^{\mu_j}[g^m(h^{n+m+\sigma}-1) -^m C_1 g^{m-1}(h^{n+m+\sigma-1}-1) \\ +... + (-1)^m(h^{n+\sigma}-1)] = 0.$$

Since g is a transcendental entire function, we have $\prod_{j=1}^{s} g(z+c_j)^{\mu_j} \neq 0$. So from above we obtain

$$g^m(h^{n+m+\sigma}-1) -^m C_1 g^{m-1}(h^{n+m+\sigma-1}-1) + ... + (-1)^m(h^{n+\sigma}-1) = 0,$$

which implies $h = 1$ and hence $f = g$. If h is not a constant, then it follows from (3.8) that f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by

$$R(w_1, w_2) = w_1^n(w_1-1)^m \prod_{j=1}^{s} w_1(z+c_j)^{\mu_j} - w_2^n(w_2-1)^m \prod_{j=1}^{s} w_2(z+c_j)^{\mu_j}.$$

□

Proof of Theorem 1.5. Let F and G be defined as in the proof of Theorem 1.4. Then F and G are transcendental meromorphic functions that share $(1, 2)^*$ except the zeros and poles of $\alpha(z)$. We assume, if possible, that $H \not\equiv 0$. Then using Lemma 2.1, 2.2 and 2.6 we deduce that

$$\begin{aligned}
& T(r,F)+T(r,G) \\
\leq\ & 2N_2(r,0;F)+2N_2(r,0;G)+2N_2(r,\infty;F)+2N_2(r,\infty;G) \\
& +\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G) \\
& -m(r,1;F)-m(r,1;G)+S(r,F)+S(r,G) \\
\leq\ & 5\overline{N}(r,0;f)+5\overline{N}(r,0;g)+5N(r,1;f)+5N(r,1;g) \\
& +3N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)+3N\left(r,0;\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right) \\
& +S(r,f)+S(r,g) \\
\leq\ & (3\sigma+10)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \qquad (3.9)
\end{aligned}$$

Therefore, using (3.1) and (3.2) we obtain from (3.9)

$$(n+m-2\sigma-10)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

a contradiction since $n+m \geq 2\sigma+11$. Thus, we must have $H \equiv 0$. Then the result follows from the proof of Theorem 1.4. This completes the proof of Theorem 1.5. □

Proof of Theorem 1.6. Let F and G be defined as in the proof of Theorem 1.4. Then F and G are transcendental meromorphic functions such that

$$\begin{aligned}
& \overline{E}_{2)}\left(\alpha(z), f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right) \\
= & \overline{E}_{2)}\left(\alpha(z), g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)
\end{aligned}$$

except the zeros and poles of $\alpha(z)$.
Since

$$\begin{aligned}
& 2N_2(r,0;F)+2N_2(r,0;G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G) \\
\leq\ & 7\overline{N}(r,0;f)+7\overline{N}(r,0;g)+7N(r,1;f)+7N(r,1;g) \\
& +5N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)+5N\left(r,0;\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right) \\
& +S(r,f)+S(r,g) \\
\leq\ & (5\sigma+14)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),
\end{aligned}$$

from Lemmas 2.7 and 2.8 and proceeding similarly as in the proof of Theorem 1.4, the conclusion of Theorem 1.6 follows. □

Open Problems. In the paper, there are two open questions for further research.

QUESTION 3.1. What can we get if we consider transcendental meromorphic functions in Theorems 1.1 - 1.6?

QUESTION 3.2. Can we relax the lower bound of n in Theorems 1.1 - 1.6?

Conclusion. In this paper some results of uniqueness problems of shift polynomials of transcendental entire functions of finite order have been extended and generalized with the help of weakly weighted sharing and relaxed weighted sharing.

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