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η-RICCI SOLITONS ON KENMOTSU MANIFOLDS ADMITTING GENERAL CONNECTION

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Abstract. The object of the present paper is to study η-Ricci soliton on Kenmotsu manifold with respect to general connection.

1. Introduction

Throughout our paper, we denote Schouten-Van Kampen connection, general connection, Zamkovoy connection, generalized Tanaka-Webster connection, quarter-symmetric connection, Levi-civita connection by the symbols ∇^s, ∇^G, ∇^z, ∇^T, ∇^q, ∇ respectively.

Recently, Biswas and Baishya([3], [4]) introduced and studied a new connection, named general connection in the context of Sasakian geometry. The general connection ∇^G is defined as

$$\nabla^G_X Y = \nabla_X Y + k_1 \left[\left(\nabla_X \eta\right)(Y)\xi - \eta(Y)\nabla_X \xi \right] + k_2 \eta(X)\phi Y, \tag{1}$$

for all $U, V \in \chi(M)$ and the pair (λ, μ) being real constants. The beauty of such connection ∇^G lies in the fact that it has the flavour of

(i) quarter symmetric metric connection([11], [5]) for $(k_1, k_2) \equiv (0, -1)$;

(ii) Schouten-Van Kampen connection [26] for $(k_1, k_2) \equiv (1, 0)$;

(iii) Tanaka Webster connection [29] for $(k_1, k_2) \equiv (1, -1)$ and

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(iv) Zamkovoy connection [33] for $(k_1, k_2) \equiv (1, 1)$.

In 1982, Hamilton [24] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([19], [20]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}. \tag{2}$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that ([25], [13], [23] [16])

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{3}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively. As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [13]. They have studied Ricci soliton of real hypersurfaces in a non-flat complex space form and defined η-Ricci soliton, which satisfies the equation

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{4}$$

where λ and μ are real numbers. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ). Recenty, η-Ricci solitons have been studied by various authors for details we refer ([9], [7], [18], [17], [28] and the reference therein).

This paper is structered as follows: After introduction, a short description of Kenmotsu manifold is given in section 2. In section 3, we have studied some properties of Kenmotsu manifold admitting general connection. Section 4 deals with η-Ricci solitons and Ricci solitons on Kenmotsu manifolds with respect to the general connection admitting some curvature restrictions. Finally in section 6, we have given an non trivial example of η-Ricci solitions and found out the relation between the scalars λ and μ on Kenmotsu manifolds with respect to the general connection.

2. Preliminaries

Let M be an $n(=2m+1)$-dimensional differentiable manifold, it said to be an almost contact Riemannian manifold if either its structural group can be reduced to $U(n)\times\{I\}$ or there is an almost contact metric structure (ϕ,ξ,η,g) consisting of a vector field ξ, $(1,1)$ tensor field ϕ, 1-form η and Riemannian metric g satisfying

$$\phi^2X = -X+\eta(X)\xi, \tag{5}$$

$$\eta(\xi) = 1, \eta(\phi X)=0,\ \phi\xi=0. \tag{6}$$

In Kenmotsu manifolds (M^n,g) the following relations hold ([17], [28], [14], [30], [1]).

$$g(X,\phi Y) = -g(\phi X,Y), g(X,\xi)=\eta(X), \forall X,Y\in TM \tag{7}$$

$$g(\phi X,\phi Y) = g(X,Y)-\eta(X)\eta(Y), g(QX,Y)=S(X,Y) \tag{8}$$

$$S(\phi X,\phi Y)=S(X,Y)+(n-1)\eta(X)\eta(Y) \tag{9}$$

$$(\nabla_X\phi)Y=-g(X,\phi Y)\xi\ -\ \eta(Y)\phi(X), \tag{10}$$

$$\nabla_X\xi = X-\eta(X)\xi, \tag{11}$$

$$(\nabla_X\eta)Y = g(X,Y)-\eta(X)\ \eta(Y). \tag{12}$$

Further, for Kenmotsu manifold with structure $(\phi,\ \xi,\ \eta,\ g)$, following relations holds

$$R(X,Y)\xi = \eta(X)\ Y-\eta(Y)X, \tag{13}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{14}$$

$$R(X,\xi)Y = g(X,Y)\xi-\eta(Y)X, \tag{15}$$

$$R(\xi,X)Y = \eta(Y)X-\ g(X,Y)\xi \tag{16}$$

$$Q\xi=-(n-1)\xi, \tag{17}$$

where S and Q are Ricci tensor and Ricci operator.

3. Kenmotsu manifold admitting general connection

By the help of (5), (11) and (12) the relation (1) reduces to

$$\nabla^G_X Y = \nabla_X Y + k_1 \left[g\left(X, Y\right)\xi - \eta\left(Y\right)X\right] + k_2\eta\left(X\right)\phi Y. \tag{18}$$

Substituting Y by ξ in (18) and using (5), (11)

$$\nabla^G_X \xi = \left(1 - k_1\right)\left(X - \eta\left(X\right)\xi\right). \tag{19}$$

Now on an acount of (5), (6), (7), (10), (11), (12) and (18), we get the following

$$\nabla^G_X \eta\left(Y\right) = \eta\left(\nabla_X Y\right) + g\left(X, Y\right) - \eta\left(X\right)\eta\left(Y\right), \tag{20}$$

$$\nabla^G_X \left(\phi Y\right) = \nabla_X \left(\phi Y\right) + k_1 g\left(X, \phi Y\right)\xi - k_2\eta\left(X\right)Y + k_2\eta\left(X\right)\eta\left(Y\right)\xi, \tag{21}$$

$$\begin{aligned} \nabla^G_X g\left(Y, Z\right) &= g\left(\nabla_X Y, Z\right) + k_1\eta\left(Z\right)g\left(X, Y\right) - k_1\eta\left(Y\right)g\left(X, Z\right) \\ &\quad + k_2\eta\left(X\right)g\left(\phi Y, Z\right) + g\left(Y, \nabla_X Z\right) + \lambda\eta\left(Y\right)g\left(X, Z\right) \\ &\quad - k_1\eta\left(Z\right)g\left(Y, X\right) + k_2\eta\left(X\right)g\left(Y, \phi Z\right). \end{aligned} \tag{22}$$

Now we know that

$$R^G\left(X, Y\right)Z = \nabla^G_X \nabla^G_Y Z - \nabla^G_Y \nabla^G_X Z - \nabla^G_{[X,Y]} Z. \tag{23}$$

By using (18), (19), (20), (21) and (22) we obtain the following

$$\begin{aligned} \nabla^G_{[X,Y]} Z &= \nabla_{[X,Y]} Z + k_2\eta\left(\nabla_X Y\right)\phi Z - k_2\eta\left(\nabla_Y X\right)\phi Z \\ &\quad + k_1\left[g\left(\nabla_X Y, Z\right)\xi - g\left(\nabla_Y X, Z\right)\xi - \eta\left(Z\right)\nabla_X Y + \eta\left(Z\right)\nabla_Y X\right], \end{aligned} \tag{24}$$

$$
\begin{aligned}
&\nabla^G_X \nabla^G_Y Z \\
= \;& \nabla_X (\nabla_Y Z) + k_1 g(X, \nabla_Y Z)\xi - k_1 \eta(\nabla_Y Z) X + k_2 \eta(X)\phi \nabla_Y Z \\
&+k_1 g(\nabla_X Y, Z)\xi + k_1^2 \eta(Z) g(X,Y)\xi - k_1^2 \eta(Y) g(X,Z)\xi \\
&+k_1 g(Y, \nabla_X Z)\xi + k_1^2 \eta(Y) g(X,Z)\xi - k_1^2 \eta(Z) g(Y,X)\xi \\
&+k_1(1-k_1) g(Y,Z) X - k_1(1-k_1) g(Y,Z)\eta(X)\xi \\
&+k_1 k_2 \eta(X) g(Y, \phi Z)\xi - k_1 \eta(\nabla_X Z) Y - k_1 g(X,Z) Y \\
&+k_1 \eta(X)\eta(Z) Y + k_1 k_2 \eta(X) g(\phi Y, Z)\xi - k_1 \eta(Z)\nabla_X Y \\
&-k_1^2 \eta(Z) g(X,Y)\xi + k_1 \eta(Z)\eta(Y) X - k_1 k_2 \eta(Z)\eta(X)\phi Y \\
&+k_2 \eta(\nabla_X Y)\phi Z + k_2 g(X,Y)\phi Z - k_2 \eta(X)\eta(Y)\phi Z + k_2 \eta(Y)\nabla_X \varphi Z \\
&+k_1 k_2 \eta(Y) g(X, \varphi Z)\xi - k_2^2 \eta(X)\eta(Y) Z + k_2^2 \eta(X)\eta(Y)\eta(Z)\xi.
\end{aligned}
\tag{25}
$$

Interchanging Y and X in (25)

$$
\begin{aligned}
&\nabla^G_Y \nabla^G_X Z \\
= \;& \nabla_Y (\nabla_X Z) + k_1 g(Y, \nabla_X Z)\xi - k_1 \eta(\nabla_X Z) Y + k_2 \eta(Y)\phi \nabla_X Z \\
&+k_1 g(\nabla_Y X, Z)\xi + k_1^2 \eta(Z) g(Y,X)\xi - k_1^2 \eta(X) g(Y,Z)\xi \\
&+k_1 g(X, \nabla_Y Z)\xi + k_1^2 \eta(X) g(Y,Z)\xi - k_1^2 \eta(Z) g(X,Y)\xi \\
&+k_1 k_2 \eta(Y) g(X, \phi Z)\xi + k_1(1-k_1) g(X,Z) Y \\
&-k_1(1-k_1) g(X,Z)\eta(Y)\xi - k_1 \eta(\nabla_Y Z) X \\
&-k_1 g(Y,Z) X + k_1 \eta(Y)\eta(Z) X + k_1 k_2 \eta(Y) g(\phi X, Z)\xi \\
&-k_1 \eta(Z)\nabla_Y X - k_1^2 \eta(Z) g(Y,X)\xi + k_1 \eta(Z)\eta(X) Y - k_1 k_2 \eta(Z)\eta(Y)\phi X \\
&+k_2 \eta(\nabla_Y X)\phi Z + k_2 g(Y,X)\phi Z - k_2 \eta(Y)\eta(X)\phi Z + k_2 \eta(X)\nabla_Y \varphi Z \\
&+k_1 k_2 \eta(X) g(Y, \varphi Z)\xi - k_2^2 \eta(Y)\eta(X) Z + k_2^2 \eta(Y)\eta(X)\eta(Z)\xi.
\end{aligned}
\tag{26}
$$

Now in reference of (24), (25) and (26) we get from (23)

$$
\begin{aligned}
&R^G(X,Y)Z \\
= \;& R(X,Y)Z + (k_1 k_2 - k_2)\left[\eta(Y) g(X, \phi Z)\xi - \eta(X) g(Y, \varphi Z)\xi\right] \\
&+(k_1 k_2 - k_2)\left[\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y\right] \\
&+k_1(1-k_1)\left[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\right] \\
&+k_1[2-k_1] g(Y,Z) X - k_1[2-k_1] g(X,Z) Y.
\end{aligned}
\tag{27}
$$

On contracting (27), we obtain the Ricci tensor S^G of a Kenmotsu manifold with respect to the general connection ∇^G as

$$\begin{aligned} S^G(Y,Z) &= S(Y,Z) + k_2(1-k_1)\, g(Y,\varphi Z) \\ &+ k_1(1-k_1)\,\eta(Y)\,\eta(Z) + \left[2nk_1 - nk_1^2 - 3k_1 + 2k_1^2\right] g(Y,Z). \end{aligned} \tag{28}$$

This gives

$$\begin{aligned} Q^G Y &= QY - k_2(1-k_1)\,\phi Y \\ &+ \left[2nk_1 - nk_1^2 - 3k_1 + 2k_1^2\right] Y + k_1(1-k_1)\,\eta(Y)\,\xi. \end{aligned} \tag{29}$$

Again contracting (28) over Y and Z we obtain

$$r^G = r + k_1(1-k_1) + n\left[2nk_1 - nk_1^2 - 3k_1 + 2k_1^2\right]. \tag{30}$$

Replacing Y by ξ in (28) we get

$$S^G(Y,\xi) = (-n+1)(1-k_1)^2\,\eta(Y). \tag{31}$$

By the help of (13), (15), (16) and (27) we obtain the the following

$$\begin{aligned} R^G(\xi,Y)Z &= (1-k_1)^2\,\eta(Z)Y - (1-k_1)\; g(Y,Z)\xi \\ &- k_2(k_1-1)\left[g(Y,\varphi Z)\,\xi + \eta(Z)\,\phi Y\right] \\ &+ k_1(1-k_1)\,\eta(Z)\,\eta(Y)\,\xi, \end{aligned} \tag{32}$$

$$\begin{aligned} R^G(Y,Z)\xi &= (1-k_1)^2\,\eta(Y)Z - (1-k_1)^2\,\eta(Z)Y \\ &+ k_2(k_1-1)\left[\eta(Z)\,\phi Y - \eta(Y)\,\phi Z\right], \end{aligned} \tag{33}$$

$$\begin{aligned} &R^G(Y,\xi)Z \\ &= (1-k_1)\,g(Y,Z)\,\xi - (1-k_1)^2\,\eta(Z)\,Y \\ &+ k_2(k_1-1)\left[g(Y,\phi Z)\,\xi + \eta(Z)\,\phi Y\right] - k_1(1-k_1)\,\eta(Z)\,\eta(Y)\,\xi. \end{aligned} \tag{34}$$

Thus we can state the following

THEOREM 3.1. *Let M be an n-dimensional Kenmotsu manifold admitting general connection ∇^G. Then (i) the curvature tensor R^G of ∇^G is given by (27), (ii) the Ricci tensor S^G of ∇^G is given by (28) and (iii) the scalar curvature r^G of ∇^G is given by (30).*

4. η-Ricci solitions on Kenmotsu manifolds admitting general connection

We consider a Kenmotsu manifold with respect to general connection admitting an η-Ricci soliton (g,ξ,λ,μ). Then from (4), it is obvious that

$$\left(\pounds_{\xi}^{G}g\right)(X,Y)+2S^{G}(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)=0. \tag{35}$$

Now, we express the Lie derivative along ξ on M with respect to general connection as follows:

$$\begin{aligned}\left(\pounds_{\xi}^{G}g\right)(X,Y) &= \pounds_{\xi}^{G}g(X,Y)-g\left(\pounds_{\xi}^{G}X,Y\right)-g\left(X,\pounds_{\xi}^{G}Y\right)\\ &= \pounds_{\xi}^{G}g(X,Y)-g([\xi,X],Y)-g(X,[\xi,Y]).\end{aligned} \tag{36}$$

By the help of (1) and (36), we obtain

$$\begin{aligned}&\left(\pounds_{\xi}^{G}g\right)(X,Y)\\ &= \nabla_{\xi}^{G}g(X,Y)-g\left(\nabla_{\xi}^{G}X-\nabla_{X}^{G}\xi-k_1(X-\eta(X)\xi-k_2\phi X),Y\right)\\ &\quad -g\left(X,\nabla_{\xi}^{G}Y-\nabla_{Y}^{G}\xi-k_1(Y-\eta(Y)\xi-k_2\phi Y)\right).\end{aligned} \tag{37}$$

Using (18) and (19), the relation (37) reduces to

$$\left(\pounds_{\xi}^{G}g\right)(X,Y)=2g(X,Y)-2\eta(X)\eta(Y). \tag{38}$$

By virtue of (38), the equation (35) takes the following form

$$S^{G}(X,Y)=[1-\mu]\eta(X)\eta(Y)-[1+\lambda]g(X,Y). \tag{39}$$

Setting $X=Y=\xi$ in (39), we get

$$\mu+\lambda=(n-1)(1-k_1)^2. \tag{40}$$

Thus we can conclude that

THEOREM 4.1. *If (g,ξ,λ,μ) is an η-Ricci soliton on Kenmotsu manifold with respect to quarter symmetric metric connection, then the η-Ricci soliton on M is expanding, steady or shrinking according as $(n-1)\gtreqless\mu$.*

THEOREM 4.2. *If (g,ξ,λ,μ) is an η-Ricci soliton on Kenmotsu manifold with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection, then the η-Ricci soliton on M is expanding, steady or shrinking according as $\mu\lesseqgtr 0$.*

Definition 4.3. A Kenmotsu manifold is said to be quasi-conformal like flat with respect to general connection if

$$\omega^{G}(X,Y)Z=0, \tag{41}$$

where ω^G is the quasi-conformal like curvature tensor with respect to general connection and is given([2]) by

$$\begin{aligned}\omega^{G}(X,Y)Z &= R^{G}(X,Y)Z+a\left[S^{G}(Y,Z)X-S^{G}(X,Z)Y\right]\\ &\quad -\frac{cr^{G}}{n}\left(\frac{1}{n-1}+a+b\right)\left[g(Y,Z)X-g(X,Z)Y\right]\\ &\quad +b\left[g(Y,Z)Q^{G}X-g(X,Z)Q^{G}Y\right],\end{aligned} \tag{42}$$

for all X, Y & $Z\in\chi(M)$, the set of all vector field of the manifold M, where scalar triple $(a,\ b,\ c)$ are real constants. The beauty of such *curvature tensor* lies in the fact that it has the flavour of Riemann curvature tensor R^G if the scalar triple $(a,\ b,\ c)\equiv(0,0,0)$, conformal curvature tensor $C^{\ G}$([10]) if $(a,b,c)\equiv\left(-\frac{1}{n-2},-\frac{1}{n-2},1\right)$, conharmonic curvature tensor L^G ([12]) if $(a,b,c)\equiv(-\frac{1}{n-2},-\frac{1}{n-2},0)$, concircular curvature tensor E^G ([8], p. 84) if $(a,\ b,\ c)\equiv(0,0,1)$, projective curvature tensor P^G([8], p. 84) if $(a,b,c)\equiv(-\frac{1}{n-1},0,0)$ and m-projective curvature tensor H^G [21], if $(a,\ b,\ c)\equiv(-\frac{1}{2n-2},-\frac{1}{2n-2},0)$, the W_1^G-curvature tensor [22] if, $(a,b,c)=(\frac{1}{(n-1)},0,0)$,the W_2^G-curvature tensor [21], if $(a,b,c)=(0,-\frac{1}{(n-1)},0)$, the W_4^G-curvature tensor [22], if $(a,b,c)=(0,0,\frac{n}{r})$.

Contracting Y over Z in the above relation, we have

$$\begin{aligned}S^{G}(X,W) &= -\frac{ar^{G}}{[1-a+bn-b]}g(X,W)\\ &\quad +\frac{cr^{G}}{n}\left(\frac{1}{n-1}+a+b\right)\frac{(n-1)}{[1-a+bn-b]}g(X,W).\end{aligned} \tag{43}$$

Using (39) in (43), we find

$$
\begin{aligned}
&[1-\mu]\,\eta(X)\,\eta(W)-[1+\lambda]\,g(X,W) \\
&= -\frac{ar^{G}}{[1-a+bn-b]}g(X,W) \\
&\quad +\frac{cr^{G}}{n}\left(\frac{1}{n-1}+a+b\right)\frac{(n-1)}{[1-a+bn-b]}g(X,W)\,.
\end{aligned} \tag{44}
$$

Putting $X=W=\xi$ in (44), we get

$$
\begin{aligned}
[\mu+\lambda] &= \frac{a\left[r+k_1(1-k_1)+n\left(2nk_1-nk_1^2-3k_1+2k_1^2\right)\right]}{[1-a+bn-b]} \\
&\quad -\frac{c\left[r+k_1(1-k_1)+n\left(2nk_1-nk_1^2-3k_1+2k_1^2\right)\right]}{n} \\
&\quad \left(\frac{1}{n-1}+a+b\right)\frac{(n-1)}{[1-a+bn-b]}.
\end{aligned} \tag{45}
$$

Again, putting $Y=Z=\xi$, in (42) and then using (39), we get

$$
[\mu+\lambda]=-\frac{1}{b}(1-k_1)^2\left[an-a+1-nab+ab-n\right]. \tag{46}
$$

This leads to the following:

THEOREM 4.4. *If (g,ξ,λ,μ) is an η-Ricci soliton on the quasi-conformal like flat Kenmotsu manifold admitting general connection ∇^G, then the scalars λ and μ are related by (45).*

THEOREM 4.5. *Let (g,ξ,λ,μ) be a η-Ricci soliton on Kenmotsu manifold with respect to quarter symmetric metric connection. Then the following relation hold*

(i) the η-Ricci soliton on M for each of $C^G(X,\xi)\xi=0$ and $L^G(X,\xi)\xi=0$ is expanding, steady or shrinking according as $\left(\frac{n^3-4n^2+6n-3}{n-2}\right)\lesseqqgtr 0$.

(ii) the η-Ricci soliton on M having $H^G(X,\xi)\xi=0$ is expanding, steady or shrinking according as $\left(\frac{4n^3-10n^2+8n-1}{2n-2}\right)\lesseqqgtr 0$.

THEOREM 4.6. *If (g,ξ,λ,μ) is an η-Ricci soliton admitting $\omega^G(X,\xi)\xi=0$ on Kenmotsu manifold, then with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection, the η-Ricci soliton on is expanding, steady or shrinking according as $\mu\lesseqqgtr 0$.*

Now, let (g, ξ, λ, μ) be an η-Ricci soliton on Kenmotsu manifold admitting general connection such that V is pointwise collinear with ξ, that is, $V = \beta\xi$, where β is a function. Then obviously (35) holds and we have

$$\begin{aligned} 0 &= (X\beta)\,\eta(Y) + (Y\beta)\,\eta(X) + 2\beta g(X,Y) - 2\beta\eta(X)\,\eta(Y) \\ &\quad +2S^{G}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\,\eta(Y). \end{aligned} \tag{47}$$

Putting $Y = \xi$ in (47) and using (5), (6) and (31) it follows that

$$(X\beta) + (\xi\beta)\,\eta(X) - 2(n-1)(1-k_1)^2\,\eta(X) + 2\lambda\eta(X) + 2\mu\eta(X) = 0. \tag{48}$$

Putting $X = \xi$ in (48) and using (5) and (6) we have

$$(\xi\beta) - (n-1)(1-k_1)^2 + \lambda + \mu = 0. \tag{49}$$

Using (49) in (48), we get

$$(X\beta) - \left[(n-1)(1-k_1)^2 - \lambda - \mu\right]\eta(X) = 0. \tag{50}$$

Differentiating (50) covariently with respect to Y, we find

$$-\left[(n-1)(1-k_1)^2 - \lambda - \mu\right](\nabla_Y\eta)(X) = 0. \tag{51}$$

From (50) and (51), we find

$$\left[(n-1)(1-k_1)^2 - \lambda - \mu\right] d\eta = 0. \tag{52}$$

Since $d\eta \neq 0$, therefore

$$\lambda + \mu = (n-1)(1-k_1)^2. \tag{53}$$

Substituting (53) in (50), we conclude that β is a constant. Hence it is verified from (47) that

$$S^{G}(X,Y) = -(\beta+\lambda)\,g(X,Y) + (\beta-\mu)\,\eta(X)\,\eta(Y) \tag{54}$$

In view of (28), the relation (54) takes the form

$$\begin{aligned} &S(X,Y) \\ = \;& -k_2(1-k_1)\,g(X,\varphi Y) - \left[2nk_1 - nk_1^2 - 3k_1 + 2k_1^2 + \beta + \lambda\right] g(X,Y) \\ & + \left[\beta - \mu - k_1(1-k_1)\right]\eta(X)\,\eta(Y). \end{aligned} \tag{55}$$

Thus we can state

THEOREM 4.7. *If (g, ξ, λ, μ) is an η-Ricci soliton on Kenmotsu manifold with respect to general connection, such that V is pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold is a generalized η-Einstein manifold with respect to the Levi-Civita connection.*

COROLLARY 4.8. *Kenmotsu manifold with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection admitting an η-Ricci soliton (g, ξ, λ) whose potential vector field is pointwise collinear with vector field ξ, is an η-Einstein manifold with respect to the Levi-Civita connection.*

In particular, for $\mu = 0$, (53) yields

$$\lambda = (n-1)(1-k_1)^2 . \tag{56}$$

Thus we can state

THEOREM 4.9. *If Kenmotsu manifold with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection possess a Ricci soliton (g, ξ, λ) whose potential vector field is pointwise collinear with vector field ξ, then such soliton is always steady .*

THEOREM 4.10. *If Kenmotsu manifold with respect to quarter symmetric metric connection possess a Ricci soliton (g, ξ, λ) whose potential vector field is pointwise collinear with vector field ξ, then such soliton is always expanding.*

5. Example

By the help of [3] we introduce an example of 3-dimensional Kenmotsu manifold with respect to Generalised Tanaka-Webster connection. Choosing the linearly independent vector field as

$$e_1 = e^{-z}\frac{\partial}{\partial x},\ e_2 = e^{-z}\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z} \tag{57}$$

at each point of 3-dimensional manifold M, where $M = \{(x, y, z) \in R^3 : x \neq 0\}$.Let g be the Reiemannian metric defined by

$$g(e_i, e_j) = \{^{1,\ i=j}_{0, i\neq j}\ for\ i, j = 1, 2, 3 \tag{58}$$

The 1-form η is defined by $g(Y, e_3) = \eta(Y)$,and the $(1,1)$ tensor field ϕ is defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, and $\phi(e_3) = 0$.Let ∇ be the

Levi-Civita connection with respect to the Riemannian metric g. Then we have

$$[e_1, e_2] = 0, \ \ [e_1, e_3] = e_1, \ \ [e_2, e_3] = e_2. \tag{59}$$

Considering $e_3 = \xi$ and using Koszul's formula we get

$$\begin{aligned} \nabla_{e_1} e_3 &= e_1, \ \nabla_{e_1} e_2 = 0, \ \nabla_{e_1} e_1 = -e_{3,} \\ \nabla_{e_2} e_3 &= e_2, \ \nabla_{e_2} e_1 = 0, \ \nabla_{e_2} e_2 = -e_3, \\ \nabla_{e_3} e_1 &= 0, \ \nabla_{e_3} e_2 = 0, \ \nabla_{e_3} e_3 = 0. \end{aligned} \tag{60}$$

By the help of (59) and (60), we obtain the following

$$\begin{aligned} R\left(e_1, e_2\right) e_3 &= 0; R\left(e_1, e_2\right) e_2 = -e_1; R\left(e_1, e_3\right) e_3 = -e_1; \\ R\left(e_2, e_2\right) e_3 &= 0; R\left(e_2, e_3\right) e_3 = -e_2; R\left(e_2, e_1\right) e_1 = -e_2; \\ R\left(e_3, e_2\right) e_2 &= -e_3; R\left(e_3, e_1\right) e_2 = 0; R\left(e_3, e_1\right) e_1 = -e_3; \\ R\left(e_3, e_2\right) e_1 &= R\left(e_2, e_1\right) e_3 = R\left(e_1, e_3\right) e_2 = 0. \end{aligned} \tag{61}$$

Using (18), (23), (59) and (60), we can easily calculate the following

$$\begin{aligned} \nabla^G_{e_1} e_2 &= 0; \nabla^G_{e_1} e_1 = -e_3 + k_1 e_3; \nabla^G_{e_1} e_3 = e_1 - k_1 e_1. \\ \nabla^G_{e_2} e_3 &= e_2 - k_1 e_2; \nabla^G_{e_2} e_1 = 0; \nabla^G_{e_2} e_2 = -e_3 + k_1 e_3 \\ \nabla^G_{e_3} e_2 &= +k_2 e_1; \nabla^G_{e_3} e_3 = 0; \nabla^G_{e_3} e_1 = -k_2 e_2, \end{aligned} \tag{62}$$

$$\begin{aligned} R^G\left(e_1, e_2\right) e_2 &= -e_1 + 2k_1 e_1 + k_1^2 e_1; \ R^G\left(e_1, e_2\right) e_3 = 0; \\ R^G\left(e_2, e_3\right) e_3 &= -k_2 e_1 + k_1 k_2 e_1 - e_2 + k_1 e_2; \\ R^G\left(e_3, e_1\right) e_1 &= -e_3 + k_1 e_3; \ R^G\left(e_3, e_2\right) e_2 = -e_3 + k_1 e_3; \\ R^G\left(e_2, e_1\right) e_1 &= -e_2 + 2k_1 e_2 - k_1^2 e_2; \\ R^G\left(e_1, e_3\right) e_3 &= k_2 e_2 - k_1 k_2 e_2 - e_1 + k_1 e_1; \\ R^G\left(e_1, e_3\right) e_2 &= -k_2 e_3 + k_1 k_2 e_3; R^G\left(e_2, e_1\right) e_3 = 0; \\ R^G\left(e_1, e_3\right) e_3 &= k_2 e_2 - k_1 k_2 e_2 - e_1 + k_1 e_1, \end{aligned} \tag{63}$$

$$\begin{aligned} S^G\left(e_1, e_1\right) &= k_1^2 + 3k_1 - 2; \\ S^G\left(e_2, e_2\right) &= k_1^2 + 3k_1 - 2; \\ S^G\left(e_3, e_3\right) &= -2 + 2k_1 \end{aligned} \tag{64}$$

and

$$r^G = -6 + 2k_1^2 + 8k_1. \tag{65}$$

Thus it can be seen that equation (33) is satisfied. Now from (39) and (64) we get

$$\mu + \lambda = 2(1 - k_1). \tag{66}$$

Hence the manifold under consideration satisfies Theorem 2 and Theorem 3.

THEOREM 5.1. *There exists a Kenmotsu manifold (M^3, g) with respect to quarter symmetric metric connection possessing an η-Ricci soliton (ξ, λ, μ) which is expanding, steady or shrinking according as $2 \gtreqless \mu$.*

THEOREM 5.2. *There exists a Kenmotsu manifold (M^3, g) with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection admitting an η-Ricci soliton (ξ, λ, μ) which is always steady.*

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