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A NOTE ON ALMOST RICCI SOLITON AND GRADIENT ALMOST RICCI SOLITON ON PARA-SASAKIAN MANIFOLDS

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Abstract. The object of the offering exposition is to study almost Ricci soliton and gradient almost Ricci soliton in 3-dimensional para-Sasakian manifolds. At first, it is shown that if (g, V, λ) be an almost Ricci soliton on a 3-dimensional para-Sasakian manifold M, then it reduces to a Ricci soliton and the soliton is expanding for λ=-2. Besides these, in this section, we prove that if V is pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold is of constant sectional curvature -1. Moreover, it is proved that if a 3-dimensional para-Sasakian manifold admits gradient almost Ricci soliton under certain conditions then either the manifold is of constant sectional curvature -1 or it reduces to a gradient Ricci soliton. Finally, we consider an example to justify some results of our paper.

1. Introduction

A Riemannian or pseudo-Riemannian manifold (M, g) obeys a Ricci soliton equation, (see Hamilton [10]) if there exists a complete vector field V, called potential vector field satisfying

$$\frac{1}{2}\pounds_V g + S = \lambda g, \tag{1.1}$$

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where $\pounds_V$ is the Lie derivative operator, λ is a real scalar and S is the Ricci tensor. It will be named *shrinking*, *steady* or *expanding* according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Otherwise, it will be called *indefinite*. The Ricci soliton has been studied by several authors such as ([3], [6], [7], [10], [13], [18], [20]) and many others. As a generalization of Ricci soliton, the notion of almost Ricci soliton was introduced by Pigola et. al. [15], where basically they modified the definition of Ricci soliton by affixing the condition on the parameter λ to be a variable function in (1.1). In the present paper, we study the para-Sasakian manifold admitting an almost Ricci soliton which plays an important role in coeval mathematics.

When the vector field V is the gradient of a smooth function $f : M^n \to \mathbb{R}$, then the manifold will be called gradient almost Ricci soliton. In this case the antecedent equation takes the form

$$\nabla^2 f + S = \lambda g, \tag{1.2}$$

where $\nabla^2 f$ stands for the Hessian of f.

Over and above, the almost Ricci soliton will be called trivial if the vector field X is trivial, or the potential f is constant, otherwise, it will be a non-trivial almost Ricci soliton. In this context, we mention that when $n \geq 3$ and X is Killing vector field an almost Ricci soliton will be a Ricci soliton, since in this case, we have an Einstein manifold, from which we can take up Schur's lemma to presume that λ is constant. Since the soliton function, λ is not necessarily constant, surely comparison with soliton theory will be modified. In particular, the rigidity result contained in Theorem 1.3 of [15] indicates that almost Ricci soliton should reveal a fair board generalization of the productive concept of classical soliton. In truth, we refer the reader to [15] to see some of these changes. In the way to fathom the geometry of almost Ricci soliton, Barros and Ribeiro Jr. proved in [2] that a compact gradient almost Ricci soliton with non-trivial conformal vector field is isometric to a Euclidean sphere. In the same paper, they proved an integral formula for the compact case, which was applied to prove various rigidity results, for more trifles see [2].

The existence of Ricci almost soliton has been verified by Pigola et. al. [15] on some certain class of warped product manifolds. Some characterization of Ricci almost soliton on a compact Riemannian manifold can be found in([1], [2]). It is fascinating to note that if the potential vector field V of the Ricci almost soliton (M, g, V, λ) is Killing then the

soliton becomes trivial, provided the dimension of $M > 2$. Moreover, if V is conformal then M^n is isometric to Euclidean sphere S^n. Thus the Ricci almost soliton can be considered as a generalization of Einstein metric as well as Ricci soliton. The almost Ricci solitons have been studied by several authors such as ([8], [9], [15], [17])and many others. Motivated from the above studies, we make the contribution to investigate an almost Ricci soliton and gradient almost Ricci soliton in a 3-dimensional para-Sasakian manifold.
The present paper is constructed as follows: In section 2, we recall some basic facts and formulas of para-Sasakian manifolds which we will need throughout the paper. In section 3, we prove that if (g, V, λ) be an almost Ricci soliton on a 3-dimensional para-Sasakian manifold M, then it reduces to a Ricci soliton and the soliton is expanding for λ=-2. Besides these, in this section, we prove that if V is pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold is of constant sectional curvature -1. Finally, in section 4, it is proved that if a 3-dimensional para-Sasakian manifold admits gradient almost Ricci soliton under certain conditions then either the manifold is of constant sectional curvature -1 or it reduces to a gradient Ricci soliton. Then we consider an example to verify the results of our paper. This paper terminates with a small bibliography which by no means is exhaustive but contains only those references which have been consulted during the preparation of the present paper.

2. Para-Sasakian manifolds

In this section, we gather the formulas and results of the para-Sasakian manifold which will be required in later sections. To know more fact about paracontact metric geometry, we may refer to ([4], [14]) and references therein. Several years ago, the notion of Paracontact metric structures was introduced in [14]. Since the publication of [14], paracontact metric manifolds have been studied by many authors in recent years. The importance of para-Sasakian geometry has been pointed out especially in the last years by several papers highlighting the exchanges with the theory of para-Kähler manifolds and its role in semi-Riemannian geometry and mathematical physics ([5, 11, 12]).

Let M be an $2n+1$-dimensional differentiable manifold of class C^∞ in which there are given a $(1,1)$-type tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 X = X - \eta(X)\xi,\ \phi\xi = 0,\ \eta(\xi) = 1,\ \eta(\phi X) = 0. \tag{2.1}$$

Then (ϕ, ξ, η) is called an almost paracontact structure and M an almost paracontact manifold. Moreover, if M admits a semi-Riemannian metric g such that

$$g(\xi, X) = \eta(X),\ g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

then (ϕ, ξ, η, g) is called an almost paracontact metric structure and M an almost paracontact metric manifold [16].
We can now define the fundamental 2-form of the almost paracontact metric manifold by $\Phi(X, Y) = g(X, \phi Y)$. If $d\eta(X, Y) = g(X, \phi Y)$, then (M, ϕ, ξ, η, g) is said to be paracontact metric manifold.
A normal paracontact metric manifold is called a para-Sasakian manifold. In a para-Sasakian manifold the following relations hold :

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.3}$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.4}$$

$$\nabla_X \xi = -\phi X, \tag{2.5}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.6}$$

$$S(X, \xi) = -(n-1)\eta(X),\ Q\xi = -(n-1)\xi, \tag{2.7}$$

$$S(\phi X, \phi Y) = -S(X, Y) - (n-1)\eta(X)\eta(Y), \tag{2.8}$$

for any vector fields X, Y, Z where Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ of the manifold.
An almost paracontact metric manifold M is said to be η-Einstein if there exist smooth functions a and b, such that

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.9}$$

for all $X, Y \in TM$. If $b = 0$, then M becomes an Einstein manifold.

3. Almost Ricci soliton

The infamous Riemannain curvature tensor of a three dimensional semi-Riemannian manifold is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \tag{3.1}$$

for any vector fields X, Y, Z where r is the scalar curvature of the manifold. Replacing $Y{=}Z{=}\xi$ in the above equation and using (2.3) and (2.7) we obtain(see [12]

$$QX = \frac{1}{2}[(r+2)X - (r+6)\eta(X)\xi]. \tag{3.2}$$

In view of (3.2) the Ricci tensor is written as

$$S(X,Y) = \frac{1}{2}[(r+2)g(X,Y) - (r+6)\eta(X)\eta(Y)]. \tag{3.3}$$

Now before introducing the detailed proof of our main theorem, we first state the following result [12]:

Lemma 3.1. *For a 3-dimensional para-Sasakian manifold* $(M^3, \phi, \xi, \eta, g)$, *we have*

$$\xi r = 0 \tag{3.4}$$

where r denotes the scalar curvature of M.

We consider a 3-dimensional para-Sasakian manifold M admitting an almost Ricci soliton defined by(1.1). Using (3.3) in (1.1) we write

$$(\pounds_V g)(Y,Z) = (2\lambda - r - 2)g(Y,Z) + (r+6)\eta(Y)\eta(Z). \tag{3.5}$$

Differentiating the above equation with respect to X and making use (2.7) we obtain

$$(\nabla_X \pounds_V g)(Y,Z) = [2(X\lambda) - (Xr)]g(Y,Z) + (Xr)\eta(Y)\eta(Z) - (r+6)\{g(X,\phi Y)\eta(Z) + \eta(Y)g(X,\phi Z)\}. \tag{3.6}$$

Now we recall the following well-known formula(Yano [19]):

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y),$$

for any vector fields X, Y, Z on M. From this we can easily deduce:

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y). \tag{3.7}$$

Since $\pounds_V\nabla$ is symmetric tensor of type (1,2), it follows from (3.7) that

$$\begin{aligned} & g((\pounds_V\nabla)(X,Y),Z) \\ (3.8)\quad & = \frac{1}{2}(\nabla_X\pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y\pounds_V g)(X,Z) - \frac{1}{2}(\nabla_Z\pounds_V g)(X,Y). \end{aligned}$$

Using (3.6) in (3.8) we get

$$\begin{aligned} 2g((\pounds_V\nabla)(X,Y),Z) &= [2(X\lambda) - (Xr)]g(Y,Z) + (Xr)\eta(Y)\eta(Z) \\ &\quad +[2(Y\lambda) - (Yr)]g(X,Z) + (Yr)\eta(X)\eta(Z) \\ &\quad -[2(Z\lambda) - (Zr)]g(X,Y) - (Zr)\eta(X)\eta(Y) \\ &\quad -2(r+6)g(X,\phi Y)\eta(Z). \end{aligned} \tag{3.9}$$

After substituting $X = Y = e_i$ in the above equation and removing Z from both sides, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking $\sum i$, $1 \le i \le 3$, we have

$$(\pounds_V\nabla)(e_i,e_i) = -D\lambda, \tag{3.10}$$

where $X\alpha = g(D\alpha, X)$, D denotes the gradient operator with respect to g.

Now differentiating(1.1) and using it in (3.7) we can easily determine

$$g((\pounds_V\nabla)(X,Y),Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{3.11}$$

Taking $X = Y = e_i$ (where $\{e_i\}$ is an orthonormal frame) in (3.11) and summing over i we obtain

$$(\pounds_V\nabla)(e_i,e_i) = 0, \tag{3.12}$$

for all vector fields Z. Associating (3.10) and (3.12) yields

$$D\lambda = 0. \tag{3.13}$$

This implies that λ is constant. This leads to the following theorem:

THEOREM 3.1. *An almost Ricci soliton on a 3-dimensional para-Sasakian manifold reduces to a Ricci soliton.*

Following the above theorem and removing Z from both sides of (3.9) yields

$$\begin{aligned} 2(\pounds_V\nabla)(X,Y) &= -(Xr)Y + (Xr)\eta(Y)\xi - (Yr)X + (Yr)\eta(X)\xi \\ &\quad +g(X,Y)Dr - \eta(X)\eta(Y)Dr - 2(r+6)g(X,\phi Y)\xi. \end{aligned} \tag{3.14}$$

Setting $Y = \xi$ in the above equation and using (3.4) we obtain

$$2(\pounds_V\nabla)(X,\xi) = 0. \tag{3.15}$$

Taking covariant derivative of (3.15) along an arbitrary vector field Y we get

$$2(\nabla_Y \pounds_V \nabla)(X,\xi) \ + \ 2(\pounds_V \nabla)(X,\phi Y) = 0. \tag{3.16}$$

If, we apply the following formula:

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z)$$

in the above equation we have

$$(\pounds_V R)(X,\xi)\xi \ = 0. \tag{3.17}$$

Taking Lie derivative of (2.3) along V we obtain

$$\begin{aligned}(\pounds_V R)(X,\xi)\xi + R(X,\pounds_V\xi)\xi \ &+ \ R(X,\xi)\pounds_V\xi \\ = (\pounds_V \eta(X))\xi + \eta(X)\pounds_V\xi.\end{aligned} \tag{3.18}$$

Using (2.3), (2.6) and (3.17) in the above equation we infer

$$g(X,\pounds_V\xi)\xi \ -2\eta(\pounds_V\xi)\eta(X)\xi = (\pounds_V\eta(X))\xi. \tag{3.19}$$

Now setting $Z=\xi$ in (3.5) it follows that $(\pounds_V g)(Y,\xi) = (2\lambda+4)\eta(Y)$. Lie-differentiating the equation (2.2) along V and by virtue of the last equation we have

$$(\pounds_V\eta)(X) - g(\pounds_V\xi, X) - (2\lambda+4)\eta(X) = 0. \tag{3.20}$$

Putting $X=\xi$ in the foregoing equation gives

$$\eta(\pounds_V\xi) = -(2\lambda+4). \tag{3.21}$$

By the help of (3.20) and (3.21), equation (3.19) provides $\lambda=-2$. Thus we can state the following:

Theorem 3.2. *Let (M^3,ϕ,ξ,η,g) ba a para-Sasakian manifold. If g represents an almost Ricci solitons, then the soliton is expanding for $\lambda=-2$.*

Now let the potential vector field V be pointwise collinear with ξ i.e., $V=b\xi$, where b is a function on M. Then from (1.1) we have

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) = 2\lambda g(X,Y). \tag{3.22}$$

Using (2.5) in (3.22), we get

$$(Xb)\eta(Y) + (Yb)\eta(X) + 2S(X,Y) = 2\lambda g(X,Y). \tag{3.23}$$

Putting $Y=\xi$ in (3.23) and using (2.7) yields

$$(Xb) + (\xi b)\eta(X) - 4\eta(X) = 2\lambda\eta(X). \tag{3.24}$$

Putting $X=\xi$ in (3.24) we obtain

$$(\xi b) = 2+\lambda. \tag{3.25}$$

Putting the value of ξb in (3.24) yields

$$db = (2+\lambda)\eta. \tag{3.26}$$

Operating (3.26) by d and using Poincare lemma $d^2 \equiv 0$, we obtain

$$0 = d^2 b = (2+\lambda)d\eta + d\lambda\eta. \tag{3.27}$$

Taking wedge product of (3.27) with η, we have

$$(2+\lambda)\eta \wedge d\eta = 0. \tag{3.28}$$

Since $\eta \wedge d\eta \neq 0$ in a 3-dimensional para-Sasakian manifold, therefore

$$\lambda = -2. \tag{3.29}$$

Using (3.29) in (3.26) gives $db = 0$ i.e., b =constant. Therefore from (3.23) we infer

$$S(X,Y) = -2g(X,Y), \tag{3.30}$$

that is the manifold is an Einstein manifold and hence from (3.1) it follows that the manifold is of constant sectional curvature -1.

Thus we can state the following:

THEOREM 3.3. *Let $(M^3, \phi, \xi, \eta, g)$ ba a para-Sasakian manifold. If g represents an almost Ricci solitons and V is pointwise collinear with ξ, then V is constant multiple of ξ and the manifold is of constant sectional curvature -1.*

4. Gradient Almost Ricci soliton

This section is devoted to studying 3-dimensional para-Sasakian manifolds admitting gradient almost Ricci soliton. For a gradient almost Ricci soliton, we have

$$\nabla_Y Df = -QY + \lambda Y. \tag{4.1}$$

where D denotes the gradient operator of g.
Differentiating (4.1) covariantly in the direction of X yields

$$\nabla_X \nabla_Y Df = -\nabla_X QY + (X\lambda)Y + \lambda \nabla_X Y. \tag{4.2}$$

Similarly we get

$$\nabla_Y \nabla_X Df = -\nabla_Y QX + (Y\lambda)X + \lambda \nabla_Y X, \tag{4.3}$$

and

$$\nabla_{[X,Y]} Df = -Q[X,Y] + \lambda[X,Y]. \tag{4.4}$$

In view of (4.2),(4.3) and (4.4) we have

$$\begin{aligned} R(X,Y)Df &= \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df \\ &= -(\nabla_X Q)Y + (\nabla_Y Q)X + (X\lambda)Y - (Y\lambda)X. \end{aligned} \tag{4.5}$$

In view of (3.2) we obtain

$$\begin{aligned} R(X,Y)Df &= \frac{(Yr)}{2}X - \frac{(Xr)}{2}Y - \frac{(Yr)}{2}\eta(X)\xi + \frac{(Xr)}{2}\eta(Y)\xi \\ &+ (\frac{r}{2} + 3)[\eta(X)\phi Y - \eta(Y)\phi X] + (X\lambda)Y - (Y\lambda)X. \end{aligned} \tag{4.6}$$

This reduces to

$$g(R(X,Y)\xi, Df) = (Y\lambda)\eta(X) - (X\lambda)\eta(Y). \tag{4.7}$$

Using (2.3) in the above equation we obtain

$$\eta(X)(Yf) - \eta(Y)(Xf) = (Y\lambda)\eta(X) - (X\lambda)\eta(Y). \tag{4.8}$$

Putting $Y = \xi$ in (4.8) we have

$$d(\lambda - f) = \xi(\lambda - f)\eta. \tag{4.9}$$

Operating (4.9) by d and using Poincare lemma $d^2 \equiv 0$, we obtain

$$d[\xi(\lambda - f)]\eta \wedge d\eta = 0. \tag{4.10}$$

Since in a 3-dimensional para-Sasakian manifold $\eta \wedge d\eta \neq 0$, we have

$$\xi(\lambda - f) = constant. \tag{4.11}$$

Now contracting Y in (4.6) and using $\xi r = 0$ we obtain

$$S(X, Df) = \frac{1}{2}(Xr) - 2(X\lambda). \tag{4.12}$$

Comparing (3.3) and (4.12) we have

$$\frac{1}{2}(Xr) - 2(X\lambda) = \frac{(r+2)}{2}(Xf) - \frac{(r+6)}{2}\eta(X)(\xi f). \tag{4.13}$$

Substituting $X = \xi$ and using $\xi r = 0$ in (4.13) we obtain

$$\xi(\lambda - f) = 0. \tag{4.14}$$

In view of (4.9) and (4.14) we get

$$(\lambda - f) = constant. \tag{4.15}$$

Suppose the soliton function λ is invariant under the characteristic vector field ξ and the scalar curvature is constant. Then from (4.13) we have

$$(r + 6)(X\lambda) = 0, \tag{4.16}$$

which implies that either $r = -6$ or $\lambda = constant$.
If r=-6, then from (3.3) we get $S = -2g$, that is the manifold is an Einstein manifold and hence from (3.1) it follows that the manifold is of constant sectional curvature -1.

If $\lambda = constant$, then gradient almost Ricci soliton reduces to a gradient Ricci soliton. Hence we can state the following:

THEOREM 4.1. *If a 3-dimensional para-Sasakian manifold admits a gradient almost Ricci soliton (f, ξ, λ), then either the manifold is of constant sectional curvature -1 or it reduces to a gradient Ricci soliton, provided the soliton function λ is invariant under the characteristic vector field ξ and the scalar curvature is constant.*

5. Example

Here we consider an example of the paper [12]. In this paper the author considers the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$ and the vector fields

$$\phi e_2 = e_1 = 2y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \quad \phi e_1 = e_2 = \frac{\partial}{\partial y}, \quad \xi = e_3 = \frac{\partial}{\partial x}$$

and shows that the manifold is a para-Sasakian manifold. Also the author has obtained the expressions of the curvature tensor and the Ricci tensor respectively as follows:

$$R(e_1, e_2)\xi = 0, \quad R(e_2, \xi)\xi = -e_2, \quad R(e_1, \xi)\xi = -e_1,$$

$$R(e_1, e_2)e_2 = -3e_1, \quad R(e_2, \xi)e_2 = -\xi, \quad R(e_1, \xi)e_2 = 0,$$

$$R(e_1, e_2)e_1 = -3e_2, \quad R(e_2, \xi)e_1 = 0, \quad R(e_1, \xi)e_1 = \xi$$

and

$$\begin{aligned}
S(e_1, e_1) &= -g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) \\
&= 2 \\
&= 2g(e_1, e_1).
\end{aligned}$$

Similarly, we have

$$S(e_2, e_2) = 2g(e_2, e_2) \quad \textit{and} \quad S(e_3, e_3) = 2g(e_3, e_3).$$

Therefore,

$$r = S(e_1, e_1) - S(e_2, e_2) + S(\xi, \xi) = 2.$$

After writing $V = ae_1 + be_2 + ce_3$; a, b, c are real number and using the equation

$$(\pounds_V g)(X, Y) = \pounds_V g(X, Y) - g(\pounds_V X, Y) - g(X, \pounds_V Y)$$

we have

$$\begin{aligned}
(\pounds_{ae_1+be_2+ce_3} g)(X, Y) &= a[g(\nabla_X e_1, Y) + g(X, \nabla_Y e_1)] \\
&+ b[g(\nabla_X e_2, Y) + g(X, \nabla_Y e_2)] \\
&+ c[g(\nabla_X e_3, Y) + g(X, \nabla_Y e_3)].
\end{aligned}$$

Using the Lie derivatives, we obtain

$$(\pounds_V g)(e_1, e_1) = 0, \quad (\pounds_V g)(e_2, e_2) = 0, \quad (\pounds_V g)(e_3, e_3) = 0,$$
$$(\pounds_V g)(e_1, e_2) = (\pounds_V g)(e_2, e_1) = 0,$$
$$(\pounds_V g)(e_1, e_3) = (\pounds_V g)(e_3, e_1) = -2b,$$
$$(\pounds_V g)(e_3, e_2) = (\pounds_V g)(e_2, e_3) = 2a.$$

Hence, from the above equations for being $\pounds_V g = 0$, we get $a = b = 0$.
Again

$$(\pounds_{c\xi} g)(e_1, e_1)) + 2S(e_1, e_1) + 2\lambda g(e_1, e_1) = 0,$$
$$(\pounds_{c\xi} g)(e_2, e_2)) + 2S(e_2, e_2) + 2\lambda g(e_2, e_2) = 0,$$
$$(\pounds_{c\xi} g)(e_3, e_3)) + 2S(e_3, e_3) + 2\lambda g(e_3, e_3) = 0,$$

for $\lambda = -2$.
Thus we have

$$(\pounds_{c\xi} g)(e_i, e_j)) + 2S(e_i, e_j) + 2\lambda g(e_i, e_j) = 0,$$

for $i, j = 1, 2, 3$ and $\lambda = -2$. So, the constructed metric reduces to a Ricci soliton. Thus the **Theorem 3.1.** and **Theorem 3.2.** are verified.

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