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APPROXIMATE IDENTITY OF CONVOLUTION BANACH ALGEBRAS

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Abstract. A weight ω on the positive half real line $[0, \infty)$ is a positive continuous function such that $\omega(s+t) \leq \omega(s)\omega(t)$, for all $s, t \in [0, \infty)$, and $\omega(0) = 1$. The weighted convolution Banach algebra $L^1(\omega)$ is the algebra of all equivalence classes of Lebesgue measurable functions f such that $\|f\| = \int_0^\infty |f(t)|\,\omega(t)\mathrm{dt} < \infty$, under pointwise addition, scalar multiplication of functions, and the convolution product $(f * g)(t) = \int_0^t f(t-s)g(s)\mathrm{ds}$. We give a sufficient condition on a weight function $\omega(t)$ in order that $L^1(\omega)$ has a bounded approximate identity.

1. Radical of Algebras

Let A be an algebra with or without unit over the complex field $\mathbb{C}$. Given $a \in A$ and subsets E, F of A, we denote by RF, $E(1-a)$, $(1-a)E$ respectively the sets $\{xy : x, y \in E\}$, $\{x - xa : x \in E\}$, $\{x - ax : x \in E\}$.

A *left ideal* of A is a linear subspace J of A such that $AJ \subset J$. An element u of A is a *right modular unit* for a linear subspace E of A if $A(1-u) \subset E$. A *modular left ideal* is a left ideal for which there exists a right modular unit. A left ideal of J of A is proper if $J \neq A$, *maximal* if it is proper and not contained in any other proper left ideal, *maximal modular* if it is proper and modular and not contained in any other such left ideal. Similar definitions apply to *right ideals*, *left modular units*, etc., in terms of the inclusions $JA \subset J$, $(1-u)A \subset E$. A *two-sided ideal* is a linear subspace that is both a left ideal and a right ideal. It is clear that a left ideal of A is a subalgebra of A, and that if A has a unit element 1, then 1 is a right modular unit for every linear subspace of A.

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Definition 1.1. *The(Jacobson) radical of A is the intersection of the maximal modular left ideals of A. It is denoted by* rad(A)*, and, by the usual convention,* rad$(A) = A$ *if there are no maximal modular left ideals of A. A is said to be semi-simple if* rad$(A) = \{0\}$*, and to be a radical algebra if* rad$(A) = A$.

It is well known that the radical is the intersection of all maximal right modular ideals of A. And it is easy to see that, if A has a unit e then

$$\begin{aligned} \mathrm{rad}(A) &= \{a \in A : e - ba \in \mathrm{Inv}A \quad \text{for all} \quad b \in A\} \\ &= \{a \in A : e - ab \in \mathrm{Inv}A \quad \text{for all} \quad b \in A\}, \end{aligned}$$

where Inv$A$ is the set of all invertible elements of A. This is found in [2].

Let A be a Banach algebra. Then every maximal modular left or right ideal of A is closed, so the radical of A is closed, in fact, rad(A) is a closed two-sided ideal of the Banach algebra A and $A/\mathrm{rad}(A)$ is a semi-simple Banach algebra.

Definition 1.2. *Let A be a Banach algebra. An element $x \in A$ is said to be quasi-nilpotent if*

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = 0.$$

Since $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ is the spectral radius of x, an element of a Banach algebra A is quasi-nilpotent if and only if the spectral radius of x is zero. That is, the spectrum of x contains a single element 0. It is well known that if $x \in \mathrm{rad}(A)$ then x is quasi-nilpotent. But, in a noncommutative Banach algebra, a quasi-nilpotent element need not be in the radical. In fact, Kaplansky proved that every noncommutative C*-algebra contains a non-zero nilpotent element. Since every C*-algebra is semi-simple, this nilpotent element does not belong to the radical. But in a commutative Banach algebra the radical coincides with the set of quasi-nilpotent elements [2].

2. Approximate Identity of Radical Convolution Algebras

Throughout this paper, we shall be concerned with complex valued functions on the positive half real line $\mathbb{R}^+ = [0, \infty)$. In each case, we shall suppose that we are considering Lebesgue measure functions, and, as usual when defining spaces of integrable functions, we equate functions equal almost everywhere.

Let $L^1_{loc}(\mathbb{R}^+)$ denote the set of all *locally integrable* functions on $\mathbb{R}^+$. That is, $f \in L^1_{loc}(\mathbb{R}^+)$ if and only if $\int_K |f|\, dt < \infty$ for each compact subset K of $\mathbb{R}^+$. If $f, g \in L^1_{loc}(\mathbb{R}^+)$, their *convolution product* is $f * g$ where

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds \qquad (t \in \mathbb{R}^+).$$

Then clearly $f * g = g * f$.

For all natural number n, let

$$p_n(f) = \int_0^n |f(t)|\, dt.$$

If $f, g \in L^1_{loc}(\mathbb{R}^+)$, we apply Fubini's theorem to show that $f * g \in L^1_{loc}(\mathbb{R}^+)$

$$\begin{aligned} p_n(f * g) &= \int_0^n \left| \int_0^t f(t-s)g(s)\, ds \right| dt \\ &\leq \iint_{0 \leq s \leq t \leq n} |f(t-s)g(s)|\, ds\, dt \\ &= \iint_{u \geq 0,\, v \geq 0,\, u+v \leq n} |f(u)g(v)|\, du\, dv \\ &= \int_0^n \int_0^n |f(u)g(v)|\, du\, dv \\ &= p_n(f)p_n(g). \end{aligned}$$

Then, p_n is a semi norm so that $L^1_{loc}(\mathbb{R}^+)$ is a commutative Fréchet algebra, and, in fact, it is a radical Fréchet algebra.

Definition 2.1. *Let ω be a continuous function $\mathbb{R}^+$ into $\mathbb{R}^+$ such that*

$$\omega(0) = 1, \quad \omega(t) > 0, \quad \omega(s+t) \leq \omega(s)\omega(t) \quad (s, t \in \mathbb{R}^+).$$

Such a function ω is said to be a weight function.

Let

$$L^1(\omega) = \{f \in L^1_{loc}(\mathbb{R}^+) : \|f\| = \int_0^\infty |f(t)|\omega(t)\, dt < \infty\}.$$

Then with the same convolution multiplication as above, each $L^1(\omega)$ is a commutative Banach algebra without identity with respect to the indicated norm.

PROPOSITION 2.2. *Let $\omega(t)$ be a weight function. Then, $\lim_{t\to\infty} \omega(t)^{\frac{1}{t}}$ always exists and*

$$\lim_{t\to\infty} \omega(t)^{\frac{1}{t}} = \inf_{t>0} \omega(t)^{\frac{1}{t}}.$$

Proof. Let $f(t) = \ln \omega(t)$. Then $f(t)$ is a continuous subadditive function, that is, $f(s+t) \leq f(s) + f(t)$. Let $\alpha > \inf_{t>0} \frac{1}{t} f(t)$ be given, and choose $s > 0$ such that $\frac{1}{s} f(s) < \alpha$. We set

$$m = \sup\{f(t) : s \leq t \leq 2s\},$$

then $m < \infty$ by the continuity of f. For any positive number n and real number t with

$$(n+1)s \leq t \leq (n+2)s,$$

since $s \leq t - ns \leq 2s$, we have

$$f(t) \leq f(ns) + f(t-ns) \leq nf(s) + m.$$

And so

$$\begin{aligned}\frac{1}{t} f(t) &< \frac{n}{t} f(s) + \frac{m}{t} \\ &\leq \frac{ns}{t}\alpha + \frac{m}{t}.\end{aligned}$$

Letting $t \to \infty$ and $n \to \infty$ together so that $(n+1)s \leq t \leq (n+2)s$, we have

$$\limsup_{t\to\infty} \frac{1}{t} f(t) \leq \alpha.$$

Hence we have

$$\lim_{t\to\infty} \ln(\omega(t)^{\frac{1}{t}}) = \inf_{t>0} \ln(\omega(t)^{\frac{1}{t}}).$$

Since $\ln(\cdot)$ is continuous, we have

$$\lim_{t\to\infty} \omega(t)^{\frac{1}{t}} = \inf_{t>0} \omega(t)^{\frac{1}{t}}.$$

□

For $f \in L^1_{loc}(\mathbb{R}^+) \setminus \{0\}$, let $\alpha(f) = \inf(\operatorname{supp} f)$, where supp$f$ is the support of f, that is the closure of the set where f is non-zero. Hence, $\alpha(f) \geq a$ if and only if $f = 0$ almost everywhere on $[0, a]$. For zero function 0, we define $\alpha(0) = \infty$.

THEOREM 2.3. *(Titchmarsh) Let f, $g \in L^1_{loc}(\mathbb{R}^+)$. Then,*

$$\alpha(f * g) = \alpha(f) + \alpha(g).$$

The proof of Titchmarsh convolution theorem is found in [4].

COROLLARY 2.4. *The algebra $L^1_{loc}(\mathbb{R}^+)$ and its subalgebras are integral domains. Especially $L^1(\omega)$ is an integral domain.*

If $a > 0$, we set

$$M_a = \{f \in L^1(\omega) : \alpha(f) \geq a\}.$$

Then it is easy to show that M_a is a closed ideal in Banach algebra $L^1(\omega)$. These ideals are called the *standard ideal* of $L^1(\omega)$.

Let $\omega(t)$ be a weight function on $\mathbb{R}^+$, and set $\rho = \lim_{t\to\infty} \omega(t)^{\frac{1}{t}}$.

THEOREM 2.5. *Let $\omega(t)$ be a weight function. If $\rho = 0$, then $L^1(\omega)$ is a radical Banach algebra.*

Proof. Let χ be the characteristic function $[a, b]$ where $0 < a < b$. Let $\chi^{*(n)}$ denote the n-times convolution product of χ . Then by induction we have,

$$\operatorname{supp} \chi^{*(n)} = [na, nb]$$

and

$$\left|\chi^{*(n)}(t)\right| \leq (b-a)^{n-1} \quad \text{for} \quad t \in [na, nb].$$

Hence we have,

$$\|\chi^{*(n)}\| \leq (b-a)^{n-1} \int_{na}^{nb} \omega(t)\, dt.$$

Let $0 < \varepsilon < \dfrac{1}{e}$ be given. Choose t_0 so that $\omega(t)^{\frac{1}{t}} < \varepsilon$ for all $t \geq t_0$. If $na > t_0$

$$\begin{aligned}\|\chi^{*(n)}\| &\leq (b-a)^{n-1} \int_{na}^{nb} \varepsilon^t\, dt \\ &\leq (b-a)^{n-1} \frac{\varepsilon^{na}}{|\ln \varepsilon|},\end{aligned}$$

so that

$$\lim_{n\to\infty} \|\chi^{*(n)}\|^{\frac{1}{n}} \leq (b-a)\varepsilon^a.$$

Since $0 < \varepsilon < \dfrac{1}{e}$ is arbitrary,

$$\lim_{n\to\infty} \|\chi^{*(n)}\|^{\frac{1}{n}} = 0.$$

Hence χ is a quasi-nilpotent. Since linear combinations of such χ are dense in $L^1(\omega)$, every element of $L^1(\omega)$ is quasi-nilpotent. Therefore, $L^1(\omega)$ is a radical Banach algebra. □

For example, if $\omega(t) = \mathrm{e}^{-t^2}$ or if $\omega(t) = \mathrm{e}^{-t\ln t}$, then $L^1(\omega)$ is a radical Banach algebra.

Let A be a Banach algebra over $\mathbb{C}$. A *left approximate identity* is a net $\{e(\lambda)\}_{\lambda\in\Lambda}$ in A such that

$$e(\lambda)x \to x \qquad (x \in A).$$

A net $\{e(\lambda)\}_{\lambda\in\Lambda}$ in A is bounded if there exists a positive constant M such that

$$\|e(\lambda)\| \leq M \qquad (\lambda \in \Lambda).$$

A *bounded left approximate identity* is a left approximate identity which is also a bounded net. *Right approximate identities* are similarly defined by replacing $e(\lambda)x$ by $xe(\lambda)$. A *(two sided) approximate identity* is a net which is both a left and a right approximate identity. That is, a net $\{e(\lambda)\}_{\lambda\in\Lambda}$ in A is a (two-sided) bounded approximate identity if $\sup_{\lambda\in\Lambda}\|e_\lambda\| < \infty$ and

$$\|x - e_\lambda\| + \|x - xe_\lambda\| \to 0 \qquad (x \in A).$$

It is well known that every C*-algebra has a two sided approximate identity consisting of self adjoint elements of norm 1.

PROPOSITION 2.6. *The Banach algebra $L^1(\omega)$ has a bounded approximate identity.*

Proof. Let χ_n be the characteristic function on $[0, \frac{1}{n}]$. Since $\omega(t)$ is continuous on $[0,1]$, there is an $M > 0$ such that $\omega(t) \leq M$ for all $t \in [0,1]$. Then

$$\|n\chi_n\| = \int_0^\infty n\chi_n\omega(t)\,dt = \int_0^{\frac{1}{n}} n\omega(t)\,dt \leq M.$$

Let $f \in L^1(\omega)$. Then,

$$\begin{aligned}
&\|n\chi_n * f - f\| \\
&\quad = \int_0^\infty |(n\chi_n * f)(t) - f(t)|\, \omega(t)\, dt \\
&\quad = \int_0^\infty \left| \int_0^t f(t-s) n\chi_n(s)\, ds - f(t) \right| \omega(t)\, dt \\
&\quad = \int_0^{\frac{1}{n}} \left| n \int_0^t f(s)\, ds - f(t) \right| \omega(t)\, dt + \int_{\frac{1}{n}}^\infty \left| n \int_{t-\frac{1}{n}}^t f(s) da - f(t) \right| \omega(t)\, dt.
\end{aligned}$$

If we apply the dominated convergence theorem, then the second part of the above integrals goes to zero as $n \to \infty$. But then,

$$\begin{aligned}
&\int_0^{\frac{1}{n}} \left| n \int_0^t f(s)\, ds - f(t) \right| \omega(t)\, dt \\
&\qquad \le n \int_0^{\frac{1}{n}} \left| \int_0^t f(s) ds \right| \omega(t)\, dt + \int_0^{\frac{1}{n}} |f(t)|\, \omega(t)\, dt.
\end{aligned}$$

Since $\int_0^t f(s)\, ds$ is a continuous function and $|f(t)|\omega(t)$ is an integrable function, the above two integrals go to zero as $n \to \infty$. Therefore, we have

$$n\chi_n * f \to f \quad \text{as} \quad n \to \infty.$$

Hence $\{n\chi_n\}$ is a bounded approximate identity of the commutative Banach algebra $L^1(\omega)$. □

THEOREM 2.7. *(Cohen's factorization Theorem) Let A be a Banach algebra and have a bounded approximate identity. Let $z \in A$ and $\delta > 0$. Then there exist $x, y \in A$ such that $z = xy$ and $\|z - x\| < \delta$.*

The proof of Cohen' factorization theorem is found in [4]. Cohen's factorization theorem means that a Banach algebra A equipped a bounded approximate identity is factorized, in the sense that $A = A \cdot A$. Since $L^1(\omega)$ has a bounded approximate identity, we have

$$L^1(\omega) * L^1(\omega) = L^1(\omega).$$

Let $f \in L^1(\omega)$. If we define $f^* = \overline{f}$, where $\overline{f}$ is the complex conjugate of f. Then f^* is an involution on $L^1(\omega)$. Hence $L^1(\omega)$ is a commutative Banach star algebra.

Let A be a Banach star algebra over $\mathbb{C}$. A linear functional f on A is *positive* if

$$f(a^*a) \geq 0 \quad (a \in A).$$

The proof of next proposition is found in [2].

PROPOSITION 2.8. *Let A be a commutative Banach star algebra over $\mathbb{C}$ such that $A^2 = A$. Then every positive linear functional on A is continuous.*

Hence we have the following theorem.

THEOREM 2.9. *Every positive linear functional on Banach algebra $L^1(\omega)$ is continuous.*

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