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NOTE ON A CLASS OF INTEGRAL OPERATORS OF SZEGÖ TYPE

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Abstract. We define new integral operators on the Haydy space similar to Szegö projection. We consider a relationship between these Szegö type operators and the partial sum of Taylor series on the Hardy space.

1. Introduction

Let $\mathbf{C}^n$ denote the Euclidean space of complex dimension n. The inner product on $\mathbf{C}^n$ is given by

$$\langle z, w\rangle := z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$, and the associated norm is $|z| := \sqrt{\langle z, z\rangle}$. The unit ball in $\mathbf{C}^n$ is the set

$$\mathbf{B}_n := \{z \in \mathbf{C}^n : |z| < 1\}$$

and its boundary is the unit sphere

$$\mathbf{S}_n := \{z \in \mathbf{C}^n : |z| = 1\}.$$

In case $n = 1$, we denote $\mathbf{D}$ in place of $\mathbf{B}_1$.

Let σ_n be the normalized surface measure on $\mathbf{S}_n$.

For $0 < p < \infty$, the Hardy space $H^p(\mathbf{B}_n)$ is the space of all holomorphic function f on $\mathbf{B}_n$ for which the "norm"

$$\|f\|_{H^p} := \left\{ \sup_{0<r<1} \int_{\mathbf{S}_n} |f(r\zeta)|^p \, d\sigma_n(\zeta) \right\}^{1/p}$$

is finite. As is well-known, the space $H^p(\mathbf{B}_n)$ equipped with the norm above is a Banach space for $1 \le p < \infty$. On the other hand, it is a

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complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f, g) \mapsto \|f - g\|_{H^p}^p$.

For a function f in $H^p(\mathbf{B}_n)$, it is known that f have a radial limit function f^* almost everywhere on $\mathbf{S}_n$. Here, the radial limit function f^* of f is defined by

$$f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta)$$

provided that the limit exists for $\zeta \in \mathbf{S}_n$. Moreover the mapping $f \mapsto f^*$ is an isometry from $H^p(\mathbf{B}_n)$ into $L^p(\mathbf{S}_n, d\sigma_n)$. Consequently, each $H^p(\mathbf{B}_n)$ can be identified with a closed subspace of $L^p(\mathbf{S}_n, d\sigma_n)$.

Since $H^2(\mathbf{B}_n)$ can be identified with a closed subspace of $L^2(\mathbf{S}_n, d\sigma_n)$, there exists an orthogonal projection from $L^2(\mathbf{S}_n, d\sigma_n)$ onto $H^2(\mathbf{B}_n)$. By using a reproducing kernel function, which is called the Szegö kernel, we also obtain a function f from its radial limit function f^*. More precisely,

$$f(z) = T[f](z) := \int_{\mathbf{S}_n} \frac{f^*(\zeta)}{(1 - \langle z, \zeta \rangle)^n} \, d\sigma_n(\zeta) \tag{1.1}$$

for $f \in H^2(\mathbf{B}_n)$. We usually call this integral operator as the Szegö projection. It is well known that for $1 < p < \infty$ the Szegö projection maps $L^p(\mathbf{S}_n, d\sigma_n)$ boundedly onto $H^p(\mathbf{B}_n)$. For more details, we refer the classical text books [1, 2, 4].

For a holomorphic function f on $\mathbf{B}_n$ with Taylor series

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha,$$

we define N-th partial sum of f by

$$S_N f(z) := \sum_{|\alpha| \le N} c_\alpha z^\alpha \tag{1.2}$$

for a positive integer N. In [3], it is known that $S_N f$ converges to f in $H^p(\mathbf{B}_n)$ for $1 < p < \infty$.

In this paper we consider a class of integral operators defined by

$$T_{m,N}[f](z) := \int_{\mathbf{S}_n} \frac{\langle z, \zeta \rangle^{m+N}}{(1 - \langle z, \zeta \rangle)^m} \, f^*(\zeta) \, d\sigma_n(\zeta) \tag{1.3}$$

for $m = 1, 2, \ldots, n$ and a positive integer N. With this operators we give a relationship between S_N and $T_{m,N}$ on $H^p(\mathbf{B}_n)$. More precisely we give the following theorem.

THEOREM 1.1. *Let $1 < p < \infty$ and N be a positive integer. For f in $H^p(\mathbf{B}_n)$,*

$$f - S_N f = \sum_{m=1}^{n} c_{m,n,N} T_{m,N}[f],$$

where $c_{m,n,N} = \frac{(m+1+N)_{n-m}}{(n-m)!}$.

We note that the Szegö projection T defined in (1.1) is bounded only if $p > 1$. It is known that T is an unbounded operator on $L^1(\mathbf{S}_n, d\sigma_n)$. For $m = n$, $T_{m,N}$ has a similar growth condition with T. Thus the range of p in Theorem 1.1 is restricted.

2. Preliminary results

We use the conventional multi-index notation. For a multi-index

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

with nonnegative integers α_i, the following are common notations;

$$|\alpha| := \alpha_1 + \cdots + \alpha_n,$$

$$\alpha! := \alpha_1! \cdots \alpha_n!.$$

For $z \in \mathbf{C}^n$, the monomial is defined as

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

At first, we show that the Szegö type operators $T_{m,N}$ defined in (1.3) are actually coefficient multipliers.

PROPOSITION 2.1. *Let m, N be positive integers with $1 \le m \le n$. For a multi-index α, there exists $\lambda_\alpha = \lambda_\alpha(m, n, N, |\alpha|)$ such that*

$$T_{m,N}[\zeta^\alpha](z) = \lambda_\alpha z^\alpha.$$

Proof. From the definition of $T_{m,N}$, we have

$$T_{m,N}[\zeta^\alpha](z) = \int_{\mathbf{S}_n} \frac{\langle z, \zeta\rangle^{m+N}\ \zeta^\alpha}{(1 - \langle z, \zeta\rangle)^m}\, d\sigma_n(\zeta)$$

for a multi-index α. Note that

$$\frac{1}{(1 - \langle z, \zeta\rangle)^m} = \sum_{k=0}^{\infty} \binom{k+m-1}{k} \langle z, \zeta\rangle^k.$$

Since the monomials are orthogonal on $L^2(\mathbf{S}_n, d\sigma_n)$; see [2, Proposition 1.4.8], we have $T_{m,N}[\zeta^\alpha](z) = 0$ if $|\alpha| < m+N$. In case of $|\alpha| \geq m+N$, we have

$$\begin{aligned} T_{m,N}[\zeta^\alpha](z) &= \int_{\mathbf{S}_n} \sum_{k=0}^{\infty} \binom{k+m-1}{k} \langle z, \zeta\rangle^{k+m+N} \zeta^\alpha \ d\sigma_n(\zeta) \\ &= \binom{|\alpha|-1-N}{|\alpha|-m-N} \int_{\mathbf{S}_n} \langle z,\zeta\rangle^{|\alpha|} \ \zeta^\alpha \ d\sigma_n(\zeta). \end{aligned}$$

Expanding the term inside the above integral as

$$\langle z,\zeta\rangle^{|\alpha|} = \sum_{|\beta|=|\alpha|} \frac{|\alpha|!}{\beta!} z^\beta \overline{\zeta^\beta},$$

we obtain that

$$\begin{aligned} T_{m,N}[\zeta^\alpha](z) &= \binom{|\alpha|-1-N}{|\alpha|-m-N} \frac{|\alpha|!}{\alpha!} \ z^\alpha \int_{\mathbf{S}_n} |\zeta^\alpha|^2 \ d\sigma_n(\zeta) \\ &= \binom{|\alpha|-1-N}{|\alpha|-m-N} \frac{(n-1)!|\alpha|!}{(n-1+|\alpha|)!} z^\alpha, \end{aligned}$$

see [2, Proposition 1.4.9] for the last equality. Putting λ_α as

$$\lambda_\alpha = \lambda_\alpha(m,n,N,|\alpha|) := \begin{cases} 0 & \text{if } \ |\alpha| < m+N \\ \binom{|\alpha|-1-N}{|\alpha|-m-N} \frac{(n-1)!|\alpha|!}{(n-1+|\alpha|)!} & \text{if } \ |\alpha| \geq m+N, \end{cases}$$

we conclude the lemma. □

For the proof of the main Theorem 1.1, we prove the following lemma.

LEMMA 2.2. *For $a \in \mathbf{D}$ and positive integers m and M, we have*

$$\frac{1}{(1-a)^m} = \sum_{k=0}^{M} \binom{k+m-1}{k} a^k + \sum_{k=1}^{m} \frac{(k+M+1)_{m-k}}{(m-k)!} \frac{a^{k+M}}{(1-a)^k}.$$

Here $(M)_k := M(M+1)\cdots(M+k-1)$ denotes the usual Pochhammer symbol for a positive integer k.

Proof. By elementary calculation, we have

$$\frac{1}{(1-a)^m} = \sum_{k=0}^{M} \binom{k+m-1}{k} a^k + \frac{(M+1)_m}{(m-1)!} a^{M+1} \int_0^1 \frac{(1-t)^M}{(1-at)^{m+M+1}} \ dt.$$

So we can prove the lemma by showing that

$$\int_0^1 \frac{(a-at)^M}{(1-at)^{m+M+1}} \ dt = \sum_{k=1}^{m} \frac{(m-1)! a^{k+M-1}}{(M+1)_k (m-k)! (1-a)^k}. \tag{2.1}$$

If $a = 0$, it is trivial. Suppose that $a \neq 0$, then

$$\begin{aligned}
&\int_0^1 \frac{(a-at)^M}{(1-at)^{m+M+1}}\ dt \\
&= \frac{1}{a(1-a)^m}\int_0^1 \left(1-\frac{1-a}{1-at}\right)^M \left(\frac{1-a}{1-at}\right)^{m-1} \frac{a(1-a)}{(1-at)^2}\ dt. \\
&= \frac{1}{a(1-a)^m}\int_0^a z^M(1-z)^{m-1}\ dz,
\end{aligned}$$

where we used the change of variables by $z = 1-(1-a)/(1-at)$. Define

$$\varphi(i,j) := \int_0^a z^i(1-z)^j\ dz$$

for nonnegative integers i and j. By integration by parts we obtain

$$\begin{aligned}
\varphi(i,j) &= \int_0^a z^i(1-z)^j dz \\
&= \left.\frac{z^{i+1}(1-z)^j}{i+1}\right|_{z=0}^{a} + \frac{j}{i+1}\int_0^a z^{i+1}(1-z)^{j-1}\ dz \\
&= \frac{a^{i+1}(1-a)^j}{i+1} + \frac{j}{i+1}\varphi(i+1,j-1),
\end{aligned}$$

with

$$\varphi(i,0) = \frac{a^{i+1}}{i+1}.$$

By solving $\varphi(i,j)$ defined inductively, we get

$$\varphi(i,j) = \sum_{k=1}^{j+1} \frac{a^{i+k}(1-a)^{j-k+1}j!}{(i+1)_k(j-k+1)!}.$$

Thus we have

$$\int_0^1 \frac{(a-at)^M}{(1-at)^{m+M+1}}\ dt = \sum_{k=1}^{m} \frac{(m-1)!a^{k+M-1}}{(M+1)_k(m-k)!(1-a)^k}.$$

□

3. Proof

Now we prove the Theorem 1.1.

THEOREM 3.1. *Let $1 < p < \infty$ and N be a positive integer. For f in $H^p(\mathbf{B}_n)$,*

$$f - S_N f = \sum_{m=1}^{n} c_{m,n,N} T_{m,N}[f],$$

where $c_{m,n,N} = \frac{(m+1+N)_{n-m}}{(n-m)!}$.

Proof. Let $f \in H^p(\mathbf{B}_n)$ with $1 \leq p < 2$, then we have

$$f(z) = \int_{\mathbf{S}_n} \frac{f^*(\zeta)}{(1 - \langle z, \zeta \rangle)^n} \, d\sigma_n(\zeta)$$

where f^* is a radial limit function of f. So we have

$$f(z) - S_N f(z) = \int_{\mathbf{S}_n} \frac{f^*(\zeta) - S_N f^*(\zeta)}{(1 - \langle z, \zeta \rangle)^n} \, d\sigma_n(\zeta),$$

for a positive integer N. Here S_N is N-th partial sum defined in (1.2). By Lemma 2.2 we have

$$f(z) - S_N f(z) = \sum_{j=0}^{N} \binom{n-1+j}{j} \int_{\mathbf{S}_n} \langle z, \zeta \rangle^j (f^*(\zeta) - (S_N f)^*(\zeta)) \, d\sigma_n(\zeta)$$

$$+ \frac{(N+1)_n}{(n-1)!} \int_{\mathbf{S}_n} \langle z, \zeta \rangle^{N+1} (f^*(\zeta) - (S_N f)^*(\zeta)) \int_0^1 \frac{(1-t)^N \, dt \, d\sigma_n(\zeta)}{(1 - t\langle z, \zeta \rangle)^{N+n+1}}.$$

Since $f^* - (S_N f)^*$ have polynomials with order of greater than N, the first term equals to zero by orthogonality. Similarly the integration with $(S_N f)^*$ in the second term is also zero. Combining (2.1) in Lemma 2.2, we get

$$\begin{aligned} &f(z) - S_N f(z) \\ &= \frac{(N+1)_n}{(n-1)!} \int_{\mathbf{S}_n} \int_0^1 \frac{(1-t)^N \langle z, \zeta \rangle^{N+1}}{(1 - t\langle z, \zeta \rangle)^{N+n+1}} \, f^*(\zeta) \, dt \, d\sigma_n(\zeta) \\ &= (N+1)_n \int_{\mathbf{S}_n} \sum_{m=1}^{n} \frac{\langle z, \zeta \rangle^{N+m} f^*(\zeta)}{(N+1)_m (n-m)! (1 - \langle z, \zeta \rangle)^m} \, d\sigma_n(\zeta) \\ &= \sum_{m=1}^{n} \frac{(m+N+1)_{n-m}}{(n-m)!} \int_{\mathbf{S}_n} \frac{\langle z, \zeta \rangle^{N+k} f^*(\zeta)}{(1 - \langle z, \zeta \rangle)^k} \, d\sigma_n(\zeta). \end{aligned}$$

Thus we have the relationship between the Szegö type operators and the partial sums of Taylor series as

$$f - S_N f = \sum_{m=1}^{n} c_{m,n,N} T_{m,N}[f],$$

where $c_{m,n,N} = \frac{(m+1+N)_{n-m}}{(n-m)!}$. □

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