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ASYMPTOTIC STABILIZATION FOR A DISPERSIVE-DISSIPATIVE EQUATION WITH TIME-DEPENDENT DAMPING TERMS

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Abstract. A long-time behavior of global solutions for a dispersive-dissipative equation with time-dependent damping terms is investigated under null Dirichlet boundary condition. By virtue of an appropriate new Lyapunov function and the Łojasiewicz-Simon inequality, we show that any global bounded solution converges to a steady state and get the rate of convergence as well, when damping coefficients are integrally positive and positive-negative, respectively. Moreover, under the assumptions on on-off or sign-changing damping, we derive an asymptotic stability of solutions.

1. Introduction and main results

Consider the dispersive-dissipative equation with time-dependent damping terms

$$u_{tt}-\Delta u_{tt}-\Delta u+h_1(t)g(u_t)-h_2(t)\Delta u_t = f(u), \quad (x,t) \in \Omega\times[0,\infty), \tag{1.1}$$

under the null Dirichlet boundary and initial conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, the function $u_0, u_1 : \Omega \to R$ are given initial data, and the nonlinear damping function g satisfies the condition

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(G): *g is a C^1-function in R with $g(0) = 0$ and there exist positive constants α_1 and α_2 such that*

$$\alpha_1 \leq g'(v) \leq \alpha_2, \quad \forall v \in R.$$

The damping coefficients h_i $(i = 1, 2)$ and the nonlinearity f will be specified later.

Nonlinear evolution equations with the main part $u_{tt} - \Delta u_{tt} - \Delta u$ and different nonlinear terms arose in the study of the spread of longitudinal strain waves in the nonlinear elastic rods (cf. [2, 3]) and the weakly nonlinear ion acoustic and space-charge waves, see [22]. For example, in one-dimensional spaces, Hayes and Saccomandi [15] derived the nonlinear wave equation with strong damping $-u_{xxt}$ in the framework of the Mooney-Rivlin viscoelastic solids of second grade, when the propagation of transverse homogeneous waves was studied. Chree deduced the wave equation with dispersive term u_t only, in the study of the longitudinal vibration of a bar (cf. [19, 21 (p.428)]). In this framework, the dispersive term represents the lateral inertia of the bar, and the weak damping term may be introduced to model the contact of the bar with a rough substrate or a viscous external medium, see [24]. Hence, it is interesting to consider the global well-posedness and qualitative properties of solutions for the dispersive-dissipative wave model (1.1).

In the past decades, there have been many researchers dealing with the existence, asymptotic behavior and blow-up of solutions to the dispersive-dissipative equations with nonlinearity, refer to [4, 7, 18, 26, 27] for the equations with constant coefficients. Especially, one can refer to [7, 18, 27] for the existence of local solution and global solution, [18, 27] for the estimate of exponentially decay rate for global solutions with positive definite energy, and [7, 26] for blow-up property of solutions with arbitrarily positive initial energy.

In this paper, we investigate a long-time behavior of global solutions to the initial boundary problem of nonlinear dispersive-dissipative wave equation with time-dependent damping and, especially, the convergence to steady state of all global bounded solutions, and the asymptotic stability of the energy. For the topic on convergence to a steady state of solutions, there have been many studies, one can refer to [9, 10, 16] for linear damping with constant coefficient and [12] for nonlinear damping with constant coefficient and so on. Jiao [17] investigated the following wave equation with time-dependent damping and analytic nonlinearity:

$$u_{tt} - \Delta u + h(t)u_t = f(u), \quad (x, t) \in \Omega \times [0, \infty).$$

Under Dirichlet boundary condition, Jiao showed that global solutions converge to a steady state when time tends to infinite by the generalized Łojasiewicz-Simon inequality. Furthermore, Jiao considered two classes of general cases: **(i)** Initial value problem of abstract damped wave equation with analytic nonlinearity

$$\ddot{u} + h(t)B\dot{u} + Au = f(u), \quad t \in (0, \infty),$$

where $A : H \to H$ is a second order strongly elliptic operator on H with dense domain, $H = L^2(\Omega)$ is the usual Hilbert space and $B : H \to H$ is a bounded linear operator satisfying the coerciveness condition, and **(ii)** Dirichlet initial boundary value problem of a class of wave equations with nonlinear interior damping and analytic nonlinear source term

$$u_{tt} - \Delta u + h(t)g(u_t) = f(u), \quad (x, t) \in \Omega \times [0, \infty),$$

where g is a function such that **(G1)** $g \in C^1(R)$ and g is a monotone increasing function with $0 < m_1 \le g'(s) \le m_2 < \infty$, $\forall s \in R$ and **(G2)** $g(0) = 0$. The key point is that all the papers on the problems above used an inequality, so-called the Łojasiewicz-Simon inequality, to obtain the results. However, it is required that the nonlinearity $f(s)$ is analytic with respect to s.

On the topic of asymptotic stability of the problems with intermittent time-dependent damping, originated from the theory of ordinary differential equations, one can refer to [1, 13, 14, 20, 23]. Inspired by the works above, Haraux et al. [11] investigated the linear wave equations with on-off damping and the authors presented some sufficient conditions for asymptotic stability, which is an improvement of the previous result of Smith [23] concerning ordinary differential equations. They further established a stability result for the case of a positive-negative damping by employing the same method as in [11]. Later, taking the influence of the forcing term into consideration, Fragnelli and Mugnai [5] concerned with some classes of nonlinear abstract damped wave equations, whose prototype is the usual wave equation

$$u_{tt} - \Delta u + h(t)u_t = f(u), \quad (x, t) \in \Omega \times [0, \infty).$$

Under the null Dirichlet boundary condition, the authors obtained some sufficient conditions for the asymptotic stability of the solutions to the problem with nonnegative damping which may be on-off or integrally positive type. Also, Fragnelli and Mugnai [6] investigated the same problem with sign-changing damping and the authors introduced some sufficient conditions for asymptotic stability. Wu [25] studied the Dirichet initial boundary value problem for nonlinear wave equations of Kirchhoff type with an intermittent damping

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + h(t)g(u_t) + f(u) = 0, \quad (x,t) \in \Omega \times [0,\infty).$$

The author established an asymptotic stability under some conditions, and in that study, it was not necessary for f to be analytic, but only the condition that f is a C^1-function was enough.

Motivated by these works, our aim is to study the asymptotic behavior of global solutions to the dispersive-dissipative wave model (1.1)-(1.3) with intermittent time-dependent damping; that is, global bounded solutions to (1.1)-(1.3) converge to a steady state as time tends to infinity, when time-dependent coefficients are on-off and positive-negative types, and the energy of (1.1)-(1.3) is asymptotically stable, when time-dependent coefficients are on-off and positive-negative types. Our main difficulties are to construct an appropriate new Lyapunov function that is available to use the Łojasiewicz-Simon inequality and to derive the decay property of the energy on a short time closed interval.

Throughout this paper, we use the following notations:

• We denote the inner products and norms on the spaces $H_0^1(\Omega)$, $H^{-1}(\Omega)$, and $L^2(\Omega)$ by $(\cdot,\cdot)_{H_0^1(\Omega)}$, $(\cdot,\cdot)_*$, and $(\cdot,\cdot)_2$ ($\|\cdot\|_{H_0^1(\Omega)}$, $\|\cdot\|_*$, and $\|\cdot\|_2$), respectively, and the norm on $L^p(\Omega)$ is denoted by $\|\cdot\|_p$.

• Let C [somewhere C_i, $(i \in N)$] denote a generic constant, not necessarily the same at different occurrences, which depend on μ, β and the measure of Ω, but it can be chosen as independent of $t \in R^+$.

We define the energy function as

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(u)dx, \tag{1.4}$$

where $F(u) = \displaystyle\int_0^u f(s)\, ds$. Multiplying (1.1) by u_t and integrating the result over Ω, and using Green's formula, one can see that

$$\frac{d}{dt}E(t) = -h_1(t)\int_\Omega g(u_t)u_t\, dx - h_2(t)\|\nabla u_t\|_2^2. \tag{1.5}$$

We now present a result on existence and uniqueness of global weak solution which can be established by the Faedo-Galerkin method as in [26].

• Suppose that the nonlinear function g satisfies condition (G) and that the damping coefficients h_i $(i = 1, 2)$ and nonlinearity f meet some conditions which will be given in Sections 2 and 3. Then for given initial data $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$, problem (1.1)-(1.3) admits a unique global solution u such that

$$u \in C(R^+; H_0^1(\Omega)) \text{ and } u_t \in C(R^+; H_0^1(\Omega)). \tag{1.6}$$

The main results of this paper can be summarized as follows:

• Let $h_1(t)$ and $h_2(t)$ be the type of integrally positive or positive-negative. If g and f satisfy (G) and (F1)-(F3) given in Section 2, respectively, then all global bounded solutions of problem (1.1)-(1.3) converge to a steady state.

• Let $h_1(t)$ and $h_2(t)$ be the type of on-off or positive-negative. If g and f satisfy (G) and (F5) given in Section 3, respectively, then the corresponding energy of problem (1.1)-(1.3) is asymptotically stable.

Remark 1.1. The definitions of on-off, integrally positive, and positive-negative damping coefficients will be given in Sections 2 and 3 in detail. However, the definitions of positive-negative in Subsections 2.2 and 3.2 are quite different from each other.

2. Convergence to a steady state

In this section, we present a convergence result on global solutions to problem (1.1)-(1.3), when damping coefficients $h_1(t)$ and $h_2(t)$ satisfy appropriate conditions. We first give the following reasonable assumptions on the nonlinearity f.

(F1): *The function f is analytic in s,*

(F2): $sf(s) \le 0$, $\forall s \in R$,

(F3): *$f(s)$ and $f'(s)$ are bounded in $(-c, c)$ for all $c > 0$ if $N = 1, 2$, and $f(s)$ is bounded in $(-c, c)$ for all $c > 0$ and there exist constants $\rho_0 \ge 0$ and $\mu > 0$ such that $(N-2)\mu < 4$ and*

$$|f'(s)| \le \rho_0(1 + |s|^\mu) \ a.e.\ s \in (-\infty, \infty),$$

if $N \ge 3$.

Remark 2.1. It follows from (F2) that

$$F(s) = \int_0^s f(\tau)d\tau \le 0, \quad \forall s \in R.$$

The proof of our convergence depends on an appropriate new Lyapunov function, compactness properties, and the Łojasiewicz-Simon inequality for the energy functional $e_u : H_0^1(\Omega) \to R$ given by

$$e_u(u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(u)dx. \tag{2.1}$$

Proposition 2.2. ([9]) *Suppose the assumptions (F1)-(F3) on f hold. Then the energy function $e_u \in C^2(H_0^1(\Omega))$ satisfies the Łojasiewicz-Simon inequality near every equilibrium point $\phi \in H_0^1(\Omega)$; that is, for every $\phi \in \mathcal{S}$, where*

$$\mathcal{S} = \{\phi \in H^2(\Omega) \cap H_0^1(\Omega) : -\Delta\phi + f(\phi) = 0\},$$

there exist constants $\beta_\phi, \sigma_\phi > 0$ and $0 < \theta_\phi \le \frac{1}{2}$ such that

$$|e_u(\phi) - e_u(\psi)|^{1-\theta_\phi} \le \beta_\phi \|\Delta\psi + f(\psi)\|_*$$

for all $\psi \in H_0^1(\Omega)$ with $\|\phi - \psi\|_{H_0^1(\Omega)} < \sigma_\phi$. The number θ_ϕ is called the Łojasiewicz exponent of e_u at ϕ.

We will try to prove convergence to equilibrium of any solution having relatively compact range in the energy space. The following assumption ensures the boundedness of any global solution for problem (1.1)-(1.3):

(F4): *There exist constants λ, μ, and λ_1 such that $\lambda < \mu\lambda_1$, $C > 0$, and*

$$F(u) \le \frac{\lambda u^2}{2} + C \text{ for all } u \in R,$$

where $\lambda_1 > 0$ is the optimal constant of the Poincaré inequality

$$\lambda_1 \|u\|_2^2 \le \|\nabla u\|_2^2, \quad u \in H_0^1(\Omega).$$

Proposition 2.3. *Assume that u is a global solution of problem (1.1)-(1.3) and (F4) holds. Then (u, u_t) is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$.*

We now present a lemma that plays a key role in the estimation of convergence rate.

Lemma 2.4. ([10]) Suppose that $v \in H_0^1(\Omega)$, $v \ge 0$ in $[0, T]$ and

$$v'(t) \le -C[v(t)]^\gamma \text{ a.e. in } [0, T],$$

for all $T > 0$, where γ is a constant. Then

- if $\gamma > 1$, we have the inequality

$$v(t) \le Ct^{-\chi}, \quad t \in [0, T],$$

where $\chi = \frac{1}{\gamma - 1}$, and

- if $\gamma = 1$, we have the inequality

$$v(t) \le v(0)e^{-Ct}, \quad t \in [0, T].$$

2.1. The integrally positive case

We begin with the definition of integrally positive.

Definition 2.5. A function $h : [0,+\infty) \to [0,+\infty)$ is said to be integrally positive, if for every $\varepsilon > 0$, there exists a constant $\eta > 0$ such that

$$\int_t^{t+\varepsilon} h(s)\, ds \geq \eta, \quad \forall t \geq 0.$$

Remark 2.6. From the definition above, it can be seen that the function h may vanish somewhere, but not on any interval. Furthermore, it is clear that there exists a constant $\kappa > 0$ such that $h(t) > \kappa$ a.e. in R.

Our first main result, which is on the convergence of solutions to problem (1.1)-(1.3) when damping coefficients $h_1(t)$ and $h_2(t)$ are integrally positive, can be given as follows:

Theorem 2.7. *Suppose that $h_1(t)$ and $h_2(t)$ are integrally positive functions satisfying (2.2) and that the nonlinear functions g and f satisfy (G) and (F1)-(F3), respectively. Let u be a global solution of problem (1.1)-(1.3) and assume also that*

(T1): *(u, u_t) is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$,*

(T2): *$\{u(t) : t \geq 0\}$ is relatively compact in $H_0^1(\Omega)$.*

Then there exists a function $\phi \in \mathcal{S}$ such that

$$\|u_t(t)\|_{H_0^1(\Omega)} + \|u(t) - \phi\|_{H_0^1(\Omega)} \to 0,$$

as $t \to \infty$. Furthermore, let $\theta = \theta_\phi$ be the Łojasiewicz exponent of E_μ at ϕ. Then the following assertions hold:

(i) *If $0 < \theta < \frac{1}{2}$, we have*

$$\|u(t) - \phi\|_{H_0^1(\Omega)} = o(t^{-\frac{\theta}{1-2\theta}}), \quad t \to \infty,$$

(ii) *If $\theta = \frac{1}{2}$, we have*

$$\|u(t) - \phi\|_{H_0^1(\Omega)} = o(e^{-\zeta t}), \quad t \to \infty,$$

where $\zeta > 0$.

To prove Theorem 2.7, we need to prove the following useful result:

Lemma 2.8. *Assume that $h_1(t)$ and $h_2(t)$ are integrally positive functions satisfying*

$$h_2(t) \geq \alpha_1 h_1(t), \quad \forall t > 0, \tag{2.2}$$

where α_1 is a constant given in (G), and that the functions g and f satisfy (G) and (F1)-(F3), respectively. If u is a solution of problem (1.1)-(1.3), then

$$\lim_{t\to\infty} \|u_t\|_2^2 = \lim_{t\to\infty} \|\nabla u_t\|_2^2 = 0. \tag{2.3}$$

Proof. It follows from (1.4) and (1.5) that there exists a constant $E_\infty \geq 0$ such that

$$\lim_{t\to\infty} E(t) = E_\infty. \tag{2.4}$$

Then there exist constants L_1 and L_2 in $[0, 2E_\infty]$ such that

$$\limsup_{t\to\infty} \|u_t\|_2^2 = L_1 \text{ and } \limsup_{t\to\infty} \|\nabla u_t\|_2^2 = L_2. \tag{2.5}$$

We will show that $L_1 = L_2 = 0$ and for this, it suffices to show that $L_1 + L_2 = 0$. Assume that $L_1 + L_2 > 0$.

We consider the following two cases:

Case1. $\|u_t\|_2^2 + \|\nabla u_t\|_2^2 = L_1 + L_2$, $\forall t > 0$.

From (1.5), (2.2) and (G), one can easily see that

$$\begin{aligned} 0 < E_\infty &= E(0) + \int_0^\infty E'(\tau)\, d\tau \\ &= E(0) - \int_0^\infty \left\{ h_1(\tau) \int_\Omega g(u_\tau) u_\tau dx + h_2(\tau) \|\nabla u_\tau\|_2^2 \right\} d\tau \\ &\leq E(0) - \int_0^\infty \alpha_1 h_1(\tau) (\|u_\tau\|_2^2 + \|\nabla u_\tau\|_2^2)\, d\tau. \end{aligned} \tag{2.6}$$

Then, it follows from (2.6) that

$$0 < E_\infty \leq E(0) - (L_1 + L_2)\alpha_1 \int_0^\infty h_1(\tau)\, d\tau. \tag{2.7}$$

Since h_1 is integrally positive, there exists a constant $\eta > 0$ such that

$$\int_n^{n+1} h_1(\tau) d\tau \geq \eta, \quad \forall n \in N. \tag{2.8}$$

Hence, combining (2.7) and (2.8), we have

$$0 < E_\infty \leq E(0) - (L_1 + L_2)\alpha_1 \sum_{n=1}^{\infty} \eta = -\infty,$$

which is a contradiction.

Case2. $\|u_t\|_2^2 + \|\nabla u_t\|_2^2 \neq L_1 + L_2$.

We set

$$\liminf_{t\to\infty} (\|u_t\|_2^2 + \|\nabla u_t\|_2^2) = l \in [0, L_1 + L_2). \tag{2.9}$$

Since $u \in C^1([0,T]; H_0^1(\Omega))$, there exist two sequences $\{t_n\}_{n\in N}$ and $\{\bar{t}_n\}_{n\in N}$ such that

- $t_n \to \infty,\ n \to \infty,$
- $0 < t_n < \bar{t}_n < t_{n+1}, \quad \forall n \in N,$
- $\dfrac{L_1+L_2+l}{2} = \|u_t(t_n)\|_2^2 + \|\nabla u_t(t_n)\|_2^2$
  $\leq \|u_t(\bar{t}_n)\|_2^2 + \|\nabla u_t(\bar{t}_n)\|_2^2 = \dfrac{3(L_1+L_2)+l}{4}, \quad \forall n \in N,$
- $\dfrac{L_1+L_2+l}{2} \leq \|u_t\|_2^2 + \|\nabla u_t\|_2^2 \leq \dfrac{3(L_1+L_2)+l}{4}, \quad \forall t \in (t_n, \bar{t}_n),$

by (1.6), (2.5), and (2.9). By equation (1.1) and the definition of weak solution, it can be shown that

$$
\begin{aligned}
\frac{d}{dt}(\|u_t\|_2^2 + \|\nabla u_t\|_2^2) &= 2(u_t, u_{tt} - \Delta u_{tt})_2 \\
&= 2(u_t, \Delta u - h_1(t)g(u_t) + h_2(t)\Delta u_t + f(u))_2 \\
&\leq 2\|\nabla u_t\|_2\|\nabla u\|_2 + 2\|u_t\|_2\|f(u)\|_2 \\
&\leq K.
\end{aligned} \tag{2.10}
$$

Integrating the inequality above over $(t_n, \bar{t}_n)$, we have

$$
\begin{aligned}
K(t_n - \bar{t}_n) &\geq \int_{t_n}^{\bar{t}_n} \frac{d}{dt}(\|u_t\|_2^2 + \|\nabla u_t\|_2^2)\, dt \\
&= (\|u_t(\bar{t}_n)\|_2^2 + \|\nabla u_t(\bar{t}_n)\|_2^2) - (\|u_t(t_n)\|_2^2 + \|\nabla u_t(t_n)\|_2^2) \\
&= \frac{3(L_1+L_2)+l}{4} - \frac{L_1+L_2+l}{2} \\
&= \frac{L_1+L_2-l}{4},
\end{aligned}
$$

i.e.,

$$
t_n - \bar{t}_n \geq \frac{L_1+L_2-l}{4K}, \quad \forall n \in N. \tag{2.11}
$$

By (2.6) and (2.11), we can obtain the inequalities

$$
0 < E_\infty \leq E(0) - \int_{\bigcup (t_n, t_n + \frac{L_1+L_2-l}{4K})} h_1(\tau)\alpha_1(\|u_\tau\|_2^2 + \|\nabla u_\tau\|_2^2)\, d\tau.
$$

Since $\frac{L_1+L_2+l}{2} \leq \|u_t\|_2^2 + \|\nabla u_t\|_2^2$, $\forall t \in \left(t_n, t_n + \frac{L_1+L_2-l}{4K}\right)$ and h_1 is integrally positive, there exists a constant $\eta > 0$ such that

$$\int_{t_n}^{t_n+\frac{L_1+L_2-l}{4K}} h_1(\tau)d\tau \geq \eta, \quad \forall n \in N. \tag{2.12}$$

Therefore, we can derive the inequality

$$0 < E(0) - \frac{L_1+L_2+l}{2}\alpha_1 \sum_{n=1}^{\infty} \eta = -\infty,$$

which is a contradiction. Moreover, we have

$$\liminf_{t\to\infty}(\|u_t\|_2^2 + \|\nabla u_t\|_2^2) = L_1 + L_2, \tag{2.13}$$

which implies that

$$\lim_{t\to\infty}(\|u_t\|_2^2 + \|\nabla u_t\|_2^2) = L_1 + L_2, \tag{2.14}$$

by (2.5).

Now, we are in the position to prove $L_1 + L_2 = 0$. By virtue of (2.14), there exists a constant $T > 0$ such that

$$\|u_t\|_2^2 + \|\nabla u_t\|_2^2 \geq \frac{L_1+L_2}{2}, \tag{2.15}$$

for all $t \geq T$. Combining (2.6) and (2.15), one can see that

$$0 < E_\infty \leq E(0) - \left(\frac{L_1+L_2}{2}\right)\alpha_1 \int_T^\infty h_1(\tau)d\tau.$$

Since h_1 is integrally positive, there exists a constant $\eta > 0$ such that

$$\int_n^{n+1} h_1(\tau)d\tau \geq \eta, \quad \forall n \in N,$$

and hence, we have the inequality

$$0 < E(0) - \frac{L_1+L_2}{2}\alpha_1 \sum_{n=1}^{\infty} \eta = -\infty,$$

which is a contradiction. Therefore, we have $L_1 + L_2 = 0$, which implies $L_1 = L_2 = 0$. □

Remark 2.9. In the proof of the previous result we can also obtain

$$\lim_{t\to\infty} e_u(t) = E_\infty,$$

where e_u is the energy functional given in (2.1).

The proof of Theorem 2.7

We divide our proof into 4 steps.

Step 1. Let us recall the $\omega-$limit set of the global solution $u : R^+ \to H_0^1(\Omega)$ to problem (1.1)-(1.3), which is defined as

$$\omega(u) = \left\{\phi \in H_0^1(\Omega) : \exists t_n \to +\infty \text{ such that } \lim_{n\to\infty} \|u(t_n) - \phi\|_{H_0^1(\Omega)} = 0\right\}.$$

It has been shown that

- $\omega(u)$ is a non-empty, compact and connected subset of $H_0^1(\Omega)$,
- For $\forall \phi \in \omega(u)$, we have $-\Delta\phi = f(\phi)$, i.e., $\omega(u) \subseteq \mathcal{S}$,
- $e_u(t)$ is a constant over $\omega(u)$,

see [8].

Step 2. Without loss of generality, we assume that $h(t) > \kappa$ for all $t \in R$, and define the Lyapunov functional as

$$H(t) = E(t) - \varepsilon(\Delta u + f(u), u_t - \Delta u_t)_*,$$

where $E(t)$ is the energy function given in (1.4) and $\varepsilon > 0$ is a constant which will be specified later.

We first estimate $H'(t)$.

By (1.5) and a direct calculation, one can see that

$$\begin{aligned}
H'(t) \le & - h_1(t)\int_\Omega g(u_t)u_t\, dx \\
& - h_2(t)\|\nabla u_t\|_2^2 - \varepsilon(\Delta u_t + f'(u)u_t, u_t - \Delta u_t)_* \\
& - \varepsilon(\Delta u + f(u), \Delta u + f(u) - h_1(t)g(u_t) + h_2(t)\Delta u_t)_* \\
\le & - \alpha_1 h_1(t)\|u_t\|_2^2 - h_2(t)\|\nabla u_t\|_2^2 \\
& - \varepsilon(\Delta u_t + f'(u)u_t, u_t - \Delta u_t)_* \\
& - \frac{\varepsilon}{2}\|\Delta u + f(u)\|_*^2 + \frac{\varepsilon}{2}\| - h_1(t)g(u_t) + h_2(t)\Delta u_t\|_*^2.
\end{aligned} \tag{2.16}$$

For $N \ge 3$ and $0 < \mu < \frac{4}{N-2}$, it can be seen that

$$\begin{aligned}
& \|f'(u)u_t\|_* \\
\le & C \sup_{\|\varphi\|_{H_0^1(\Omega)}\le 1} \left(\int_\Omega |u_t\varphi| dx + \int_\Omega |u|^\mu |u_t||\varphi|\, dx\right) \\
\le & C \sup_{\|\varphi\|_{H_0^1(\Omega)}\le 1} \left(\|u_t\|_2\|\varphi\|_2 + \|u_t\|_{\frac{2N}{N-2}}\|\varphi\|_{\frac{2N}{N-2}}\|u^\mu\|_{\frac{N}{2}}\right) \\
\le & C\|\nabla u_t\|_2,
\end{aligned} \tag{2.17}$$

by using (F3) and the boundedness of u in $H_0^1(\Omega)$.

For $N = 1, 2$, we can also derive the estimate above by setting $\mu = 1$ in (F3). Moreover, a direct calculation yields the following estimate:

$$
\begin{aligned}
\|\Delta u_t\|_* \le & C \sup_{\|\varphi\|_{H_0^1(\Omega)} \le 1} \int_\Omega |\Delta u_t \varphi| \, dx \\
\le & C \sup_{\|\varphi\|_{H_0^1(\Omega)} \le 1} \int_\Omega |\nabla u_t \cdot \nabla \varphi| \, dx \\
\le & C \sup_{\|\varphi\|_{H_0^1(\Omega)} \le 1} \|\nabla u_t\|_2 \|\nabla \varphi\|_2 \\
\le & C \|\nabla u_t\|_2.
\end{aligned} \tag{2.18}
$$

Similarly, we can have the inequality

$$\|\Delta u\|_* \le C\|\nabla u\|_2.$$

By virtue of the inequality above and (2.18) and using the boundedness of $f(u)$ in (F3), one can see that

$$(\Delta u + f(u), u_t - \Delta u_t)_* \le C(\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2),$$

and hence, $H(t) \ge 0$ for $\varepsilon > 0$ small enough. Combining (2.16)-(2.18) and choosing $\varepsilon > 0$ small enough, we have the inequalities

$$
\begin{aligned}
H'(t) \le & - C(\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\Delta u + f(u)\|_*^2) \\
\le & - C\{\|u_t\|_{H_0^1(\Omega)} + \|\Delta u + f(u)\|_*\}^2,
\end{aligned} \tag{2.19}
$$

for all $t \ge T_1$. Then $H(t)$ is non-negative and non-increasing on $[T_1, \infty)$, and hence, $H(t)$ has a limit at infinity. Since $\phi \in \omega(u)$, there exists a sequence $\{t_n\}_{n \ge 1}$ such that

$$\lim_{n\to\infty} t_n = \infty \text{ and } \lim_{n\to\infty} u(t_n) = \phi \text{ in } H_0^1(\Omega). \tag{2.20}$$

And we can also have

$$\lim_{n\to\infty} e_u(t_n) = e_u(\phi). \tag{2.21}$$

We now estimate $[H(t) - e_u(\phi)]^{1-\theta}$.
It follows from Young's inequality that

$$
\begin{aligned}
[H(t) - e_u(\phi)]^{1-\theta} \le & \|u_t\|_2^{2(1-\theta)} + \|\nabla u_t\|_2^{2(1-\theta)} + |e_u(u) - e_u(\phi)|^{1-\theta} \\
& + \|\Delta u + f(u)\|_* + \|u_t - \Delta u_t\|_*^{\frac{1-\theta}{\theta}}.
\end{aligned}
$$

Noting that $2(1-\theta) > 1$, $\frac{1-\theta}{\theta} > 1$ and using (2.3), one can see that there exists a constant $T_2 > T_1$ such that for all $t > T_2$

$$\begin{aligned}(2.22)\qquad &[H(t) - e_u(\phi)]^{1-\theta} \\ &\leq C\{\|u_t\|_{H_0^1(\Omega)} + |e_u(u) - e_u(\phi)|^{1-\theta} + \|\Delta u + f(u)\|_*\}.\end{aligned}$$

Step 3. It has been shown that $H(t)$ has a limit at infinity and, by means of (2.19), we have for all $\delta > 0$ with $\delta \ll \sigma_\phi$, there exists an N such that $t_N > T_2$,

$$\|u(t_N) - \phi\|_{H_0^1(\Omega)} < \frac{\delta}{2}, \tag{2.23}$$

$$\frac{C}{\theta}\{[H(t_N) - e_u(\phi)]^\theta - [H(t) - e_u(\phi)]^\theta\} < \frac{\delta}{2}, \tag{2.24}$$

and

$$H(t) \geq e_u(\phi), \tag{2.25}$$

for all $t \geq t_N$.

Let

$$\bar{t} = \sup\{t \geq t_N : \|u(s) - \phi\|_{H_0^1(\Omega)} < \sigma_\phi,\ \forall s \in [t_N, t]\}.$$

By Proposition 2.2 and (2.22), we have the inequality

$$[H(t) - e_u(\phi)]^{1-\theta} \leq 2C\{\|u_t\|_{H_0^1(\Omega)} + \|\Delta u + f(u)\|_*\}, \tag{2.26}$$

for all $t \in [t_N, \bar{t})$. Moreover, by a direct calculation, we can derive the equation

$$-\frac{d}{dt}[H(t) - E_\mu(\phi)]^\theta = -\theta[H(t) - E_\mu(\phi)]^{\theta-1}H'(t). \tag{2.27}$$

Combining (2.19), (2.26) and (2.27), it can be shown that

$$-\frac{d}{dt}[H(t) - e_u(\phi)]^\theta \geq \theta C\{\|u_t\|_{H_0^1(\Omega)} + \|\mu\Delta u + f(u)\|_*\}. \tag{2.28}$$

Integrating (2.28) over $[t_N, \bar{t})$, one can have the inequalities

$$\begin{aligned}(2.29)\qquad \int_{t_N}^{\bar{t}} \|u_t\|_{H_0^1(\Omega)} dt &\leq \int_{t_N}^{\bar{t}} \{\|u_t\|_{H_0^1(\Omega)} + \|\mu\Delta u + f(u)\|_*\} \\ &\leq \frac{C}{\theta}\{[H(t_N) - e_u(\phi)]^\theta - [H(t) - e_u(\phi)]^\theta\}.\end{aligned}$$

Assuming $\bar{t} < \infty$, we get the inequality

$$\|u(\bar{t}) - \phi\|_{H_0^1(\Omega)} \leq \int_{t_N}^{\bar{t}} \|u_t\|_{H_0^1(\Omega)} dt + \|u(t_N) - \phi\|_{H_0^1(\Omega)} \leq \delta,$$

by (2.14), (2.23) and (2.29), which contradicts the definition of $\bar{t}$. Therefore, $\bar{t} = \infty$. Then it follows from (2.29) that

$$\int_{t_N}^{\infty} \|u_t\|_{H_0^1(\Omega)} dt < \infty,$$

which implies the integrability of u in $H_0^1(\Omega)$. By the compactness of the range of u, we have

$$\lim_{t\to\infty} \|u(t) - \phi\|_{H_0^1(\Omega)} = 0.$$

Step4. By (2.19) and (2.26), there exists a constant $C > 0$ such that

$$\frac{d}{dt}[H(t) - e_u(\phi)] + C[H(t) - e_u(\phi)]^{2(1-\theta)} \le 0, \tag{2.30}$$

for all $t \ge T = t_N$. We need to consider the following two cases:
Case1. $0 < \theta < \frac{1}{2} \Rightarrow 1 < 2(1-\theta) < 2$.

By Lemma 2.4, one can see that for all $t \ge T$

$$H(t) - e_u(\phi) \le Ct^{-\frac{1}{1-2\theta}}.$$

Integrating (2.28) over (t, ∞), $t \ge T$, we have the inequalities

$$\begin{aligned}&\int_t^{\infty} \{\|u_\tau\|_{H_0^1(\Omega)} + \|\Delta u + f(u)\|_*\} d\tau \\ \le& \frac{C}{\theta}\{[H(t_N) - e_u(\phi)]^\theta - [H(t) - e_u(\phi)]^\theta\} \le Ct^{-\frac{\theta}{1-2\theta}}.\end{aligned}$$

It then follows that

$$\|u(t) - \phi\|_{H_0^1(\Omega)} \le \int_t^{\infty} \|u_\tau\|_{H_0^1(\Omega)}\, d\tau \le Ct^{-\frac{\theta}{1-2\theta}}.$$

Case2. $\theta = \frac{1}{2} \Rightarrow 2(1-\theta) = 1$.

By Lemma 2.4, it can be seen that for all $t \ge T$

$$H(t) - e_u(\phi) \le Ce^{-Ct}.$$

Integrating (2.28) over (t, ∞) for $t \ge T$, we have the inequalities

$$\begin{aligned}&\int_t^{\infty} \{\|u_\tau\|_{H_0^1(\Omega)} + \|\Delta u + f(u)\|_*\}\, d\tau \\ &\le \frac{C}{\theta}\{[H(t_N) - e_u(\phi)]^\theta - [H(t) - e_u(\phi)]^\theta\} \\ &\le Ce^{-Ct}.\end{aligned}$$

Then we obtain the inequalities

$$\|u(t) - \phi\|_{H_0^1(\Omega)} \le \int_t^{\infty} \|u_\tau\|_{H_0^1(\Omega)} d\tau \le Ce^{-Ct},$$

which completes the proof.

2.2. The case of positive-negative

We begin with the definition of positive-negative.

Definition 2.10. Let $\{I_n\}_{n\in N}$ be a sequence of disjoint intervals in $(0,\infty)$, where $I_n=(a_n,b_n)$, $a_1=0$, $b_n=a_{n+1}$, and $a_n\to\infty$ as $n\to\infty$. We say that a function $h:[0,+\infty)\to R$ is in the positive-negative case, if for all $n\in N$, there exist constants m_n and M_n such that

$$0<m_n\le M_n<\infty \text{ and } m_n\le h(t)\le M_n,$$

for all $t\in I_n$.

Remark 2.11. This kind of intermitting damping may change sign at the discontinuous points. If $h(b_n)=0$ at all the discontinuous points, we say this damping is in on-off case.

One can obtain the following theorem on the convergence to equilibrium, when $h_1(t)$ and $h_2(t)$ are positive-negative by using the same argument as in the proof of Theorem 2.7, and hence, we omit the proof.

Theorem 2.12. *Suppose that $h_1(t)$ and $h_2(t)$ are positive-negative functions satisfying (2.2), and g and f satisfy (G) and (F1)-(F3), respectively. Let u be a global solution of problem (1.1)-(1.3), and assume that*

(T1): *(u,u_t) is bounded in $H_0^1(\Omega)\times H_0^1(\Omega)$,*

(T2): *$\{u(t):t\ge 0\}$ is relatively compact in $H_0^1(\Omega)$.*

Then there exists a function $\phi\in\mathcal{S}$ such that

$$\|u_t(t)\|_{H_0^1(\Omega)}+\|u(t)-\phi\|_{H_0^1(\Omega)}\to 0,$$

as $t\to\infty$. Furthermore, let $\theta=\theta_\phi$ be the Łojasiewicz exponent of E_μ at ϕ. Then the following assertions hold:

(i) *If $0<\theta<\frac{1}{2}$, we have*

$$\|u(t)-\phi\|_{H_0^1(\Omega)}=o(t^{-\frac{\theta}{1-2\theta}}),\quad t\to\infty,$$

(ii) *If $\theta=\frac{1}{2}$, we have*

$$\|u(t)-\phi\|_{H_0^1(\Omega)}=o(e^{-\zeta t}),\quad t\to\infty,$$

where ζ is a positive constant.

2.3. Boundedness of global solutions

In this subsection, we present a boundedness for global solutions to problem (1.1)-(1.3) under the assumption (F4). We give a proof of Proposition 2.3 as follows:

Proof. The energy function E given in (1.4) is nonincreasing by (1.5). Based on the condition (F3), we can have the inequality

$$\left|\int_\Omega F(u)\,dx\right| \le C\left(1+\|u_0\|^{\mu+2}_{H_0^1(\Omega)}\right),$$

where $C \ge 0$ is a constant depending on the constant in (F3), the measure of Ω, and the constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{\mu+2}(\Omega)$. According to the inequality above and the definition of E, there exists a constant $C_1 \ge 0$ such that

$$E(0) \le C_1\left(1+\|\nabla u_0\|_2^2+\|\nabla u_1\|_2^2+\|u_0\|^{\mu+2}_{H_0^1(\Omega)}\right). \tag{2.31}$$

On the other hand, it follows from the definition of E and the condition (F4) that there exist positive constants C_2 and C_3 such that

$$\|\nabla u(t)\|_2^2+\|\nabla u_t(t)\|_2^2 \le C_2E(t)+C_3. \tag{2.32}$$

Combining (2.31) and (2.32), and using the nonincreasing property of E, one can obtain the result. □

3. Asymptotic stability

In this section, we investigate the asymptotic stability of energy for problem (1.1)-(1.3). We first give the following reasonable condition on the nonlinearity f.

(F5): *f is a C^1-function on R such that*

$$sf(s) \le F(s) \le 0, \quad \forall s \in R,$$

where $F(s) = \int_0^s f(\tau)\,d\tau$.

In order to establish a result related with the estimate of energy decay on a short closed time interval, we make the following assumption on damping coefficients:

(H): *Suppose that a and b are constants with $0 \le a < b$ and that there exist constants m_1 and M_1 with $0 < m_1 \le M_1$ such that $h_1(t), h_2(t) > 0$ and*

$$m_1 \le h_1(t)+h_2(t) \le M_1, \quad \forall t \in\ [a,b].$$

Remark 3.1. It follows from (G) and (H) that for all $v \in L^2(\Omega)$

$$m_1\alpha_1\|v\|_2^2 \leq \int_\Omega h_1(t)g(v)v\,dx, \quad \forall t \in [a,b], \tag{3.1}$$

$$\int_\Omega [h_1(t)g(v)]^2\,dx \leq M_1\alpha_2 \int_\Omega h_1(t)g(v)v\,dx, \quad \forall t \in [a,b]. \tag{3.2}$$

We present the following proposition, which plays a key role in the proof of the result on asymptotic stability:

Proposition 3.2. *Suppose that g and f satisfy (G) and (F5), respectively, and that $h_1(t)$ and $h_2(t)$ admit to (H). If $(u_0,u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$, then the solution u of problem (1.1)-(1.3) satisfies the inequality*

$$E(b) \leq \frac{1}{1 + \frac{16\alpha_1}{15(B^2+\alpha_1)} \cdot \frac{m_1(b-a)^3}{256 + \left\{\frac{3(1+\alpha_1)}{B^2+\alpha_1} + \frac{4\alpha_1(1+\alpha_2 B^2)}{B^2+\alpha_1}M_1 m_1\right\}(b-a)^2}} E(a).$$

Proof. Setting $\theta(t) = (t-a)^2(b-t)^2$, $\forall t \in [a,b]$, we have

$$|\theta'(t)| \leq 2T\theta^{\frac{1}{2}}(t), \ \max_{t\in[a,b]} \theta(t) = \frac{T^4}{16}, \text{ and } \int_a^b \theta(t)dt = \frac{T^5}{30}, \tag{3.3}$$

where $T = b-a$. Multiplying (1.1) by θu and integrating the result over $[a,b] \times \Omega$ by using integration by parts, one can see that

$$\begin{aligned}
&\int_a^b \theta\|\nabla u\|_2^2\,dt \\
&= \int_a^b \theta\|u_t\|_2^2\,dt + \int_a^b \theta\|\nabla u_t\|_2^2 dt + \int_a^b\int_\Omega \theta' u u_t\,dxdt \\
&\quad + \int_a^b\int_\Omega \theta'\nabla u \cdot \nabla u_t\,dxdt - \int_a^b\int_\Omega h_1(t)\theta u g(u_t)\,dxdt \\
&\quad - \int_a^b\int_\Omega h_2(t)\theta\nabla u \cdot \nabla u_t\,dxdt + \int_a^b\int_\Omega \theta u f(u)\,dxdt.
\end{aligned} \tag{3.4}$$

With Hölder's and Young's inequalities and (3.3), one can obtain the inequalities

$$\begin{aligned}
\int_a^b\int_\Omega \theta' u u_t\,dxdt &\leq \varepsilon_1\int_a^b (\theta')^2\|u\|_2^2\,dt + \frac{1}{4\varepsilon_1}\int_a^b \|u_t\|_2^2\,dt \\
&\leq 4T^2B^2\varepsilon_1\int_a^b \theta\|\nabla u\|_2^2\,dt + 4\varepsilon_1\int_a^b \|u_t\|_2^2\,dt,
\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&\int_a^b \int_\Omega \theta' \nabla u \cdot \nabla u_t \, dxdt \\
&\le \varepsilon_2 \int_a^b (\theta')^2 \|\nabla u\|_2^2 \, dt + \frac{1}{4\varepsilon_2} \int_a^b \|\nabla u_t\|_2^2 \, dt \\
&\le 4T^2 \varepsilon_2 \int_a^b \theta \|\nabla u\|_2^2 \, dt + \frac{1}{4\varepsilon_2} \int_a^b \|\nabla u_t\|_2^2 \, dt,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&\left| -\int_a^b \int_\Omega h_1(t) \theta u g(u_t) \, dxdt \right| \\
&\le \varepsilon_3 \int_a^b \theta \|u\|_2^2 \, dt + \frac{1}{4\varepsilon_3} \int_a^b \int_\Omega \theta [h_1(t) g(u_t)]^2 \, dxdt \\
&\le \varepsilon_3 B^2 \int_a^b \theta \|\nabla u\|_2^2 \, dt + \frac{1}{4\varepsilon_3} \int_a^b \int_\Omega \theta [h_1(t) g(u_t)]^2 \, dxdt,
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
&\left| -\int_a^b \int_\Omega \theta h_2(t) \nabla u \cdot \nabla u_t \, dxdt \right| \\
&\le \varepsilon_4 \int_a^b \theta \|\nabla u\|_2^2 dt + \frac{1}{4\varepsilon_4} \int_a^b \theta [h_2(t)]^2 \|\nabla u_t\|_2^2 \, dt,
\end{aligned} \tag{3.8}
$$

where $\varepsilon_i > 0$, $(i = 1, 2, 3, 4)$ are constants which will be specified later. It can be shown that

$$
\begin{aligned}
&[1 - 4T^2 B^2 \varepsilon_1 - 4T^2 \varepsilon_2 - B^2 \varepsilon_3 - \varepsilon_4] \int_a^b \theta \|\nabla u\|_2^2 \, dt \\
&\le \left(\frac{T^4}{16} + \frac{1}{4\varepsilon_1} \right) \int_a^b \|u_t\|_2^2 \, dt + \left(\frac{T^4}{16} + \frac{1}{4\varepsilon_2} \right) \int_a^b \|\nabla u_t\|_2^2 \, dt \\
&\quad + \frac{T^4}{64\varepsilon_3} \int_a^b \int_\Omega [h_1(t) g(u_t)]^2 \, dxdt + \frac{T^4}{64\varepsilon_4} \int_a^b \theta [h_2(t)]^2 \|\nabla u_t\|_2^2 \, dt \\
&\quad + \int_a^b \int_\Omega \theta u f(u) \, dxdt,
\end{aligned} \tag{3.9}
$$

by substituting (3.5)-(3.8) into (3.4) and using (3.3). By setting $4T^2B^2\varepsilon_1 = 4T^2\varepsilon_2 = B^2\varepsilon_3 = \varepsilon_4 = \frac{1}{8}$ in (3.9), we obtain the inequality

$$
\begin{aligned}
&\frac{1}{2}\int_a^b \theta\|\nabla u\|_2^2\,dt \\
&\leq \left(\frac{T^4}{16} + 8B^2T^2\right)\int_a^b \|u_t\|_2^2\,dt + \left(\frac{T^4}{16} + 8T^2\right)\int_a^b \|\nabla u_t\|_2^2\,dt \\
&\quad + \frac{B^2T^4}{8}\int_a^b\int_\Omega [h_1(t)g(u_t)]^2\,dxdt + \frac{T^4}{8}\int_a^b [h_2(t)]^2\|\nabla u_t\|_2^2\,dt \\
&\quad + \int_a^b\int_\Omega \theta u f(u)\,dxdt.
\end{aligned} \tag{3.10}
$$

On the other hand, multiplying (1.4) by θ, integrating the result over $[a,b]$, noting that $E(t)$ is nonincreasing on $[a,b]$ by (1.5), and then employing (3.3), one can have the inequality

$$
\begin{aligned}
\frac{T^5}{30}E(b) \leq \frac{1}{2}\int_a^b \theta\|\nabla u\|_2^2\,dt + \frac{T^4}{32}\int_a^b \|u_t\|_2^2\,dt \\
+ \frac{T^4}{32}\int_a^b \|\nabla u_t\|_2^2\,dt - \int_a^b\int_\Omega \theta F(u)\,dxdt.
\end{aligned} \tag{3.11}
$$

Hence, combining (3.10) and (3.11), we have the inequalities

$$
\begin{aligned}
&\frac{T^5}{30}E(b) \leq \left(\frac{T^4}{32} + \frac{T^4}{16} + 8B^2T^2\right)\int_a^b \|u_t\|_2^2\,dt \\
&+ \left(\frac{T^4}{32} + \frac{T^4}{16} + 8T^2\right)\int_a^b \|\nabla u_t\|_2^2\,dt + \frac{B^2T^4}{8}\int_a^b\int_\Omega [h_1(t)g(u_t)]^2\,dxdt \\
&+ \frac{T^4}{8}\int_a^b [h_2(t)]^2\|\nabla u_t\|_2^2\,dt \; - \int_a^b\int_\Omega \theta[uf(u) - F(u)]\,dxdt \\
&\leq \left[\frac{1}{m_1\alpha_1}\left(\frac{3T^4}{32} + 8B^2T^2\right) + \frac{M_1B^2\alpha_2T^4}{8}\right]\int_a^b\int_\Omega h_1(t)g(u_t)u_t\,dxdt \\
&+ \left[\frac{1}{m_1}\left(\frac{3T^4}{32} + 8T^2\right) + \frac{M_1T^4}{8}\right]\int_a^b [h_2(t)]^2\|\nabla u_t\|_2^2\,dt \\
&\leq \left\{\left[\frac{3(1+\alpha_1)}{32m_1\alpha_1} + \frac{M_1(1+\alpha_2B^2)}{8}\right]T^4 + \frac{8(B^2+\alpha_1)}{m_1\alpha_1}T^2\right\}[E(a) - E(b)],
\end{aligned}
$$

from which one can easily obtain the result by a simple calculation. □

3.1. The case of on-off

We prove the stability result in this subsection, when damping coefficients are on-off. We begin with the definition of on-off case.

Definition 3.3. Let $\{I_n\}_{n\in N}$ be a sequence of disjoint intervals in $(0,\infty)$ such that $I_n = (a_n, b_n)$, $a_1 = 0$, $b_n \le a_{n+1}$, and $a_n \to \infty$ as $n \to \infty$. Then we say that a function $h : [0, +\infty) \to R$ is in on-off case, if for all $n \in N$, there exist constants m_n and M_n such that

$$0 < m_n \le M_n < \infty \ \text{ and } \ m_n \le h(t) \le M_n,$$

for all $t \in I_n$.

Theorem 3.4. *Suppose that* $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and that* $h_1(t), h_2(t) \ge 0$ *and* $h_1(t) + h_2(t)$ *is in the on-off case, i.e., there exist constants* m_n *an* M_n *such that* $0 < m_n \le M_n$ *and*

$$m_n \le h_1(t) + h_2(t) \le M_n, \quad \forall t \in (a_n, b_n). \tag{3.12}$$

If

$$\sum_{n=1}^{\infty} m_n(b_n - a_n) \min\left\{(b_n - a_n)^2, \frac{1}{1 + M_n m_n}\right\} = \infty, \tag{3.13}$$

then the solution u *of problem (1.1)-(1.3) satisfies*

$$E(t) \to 0 \quad as \quad t \to \infty.$$

Proof. For $n \ge 0$, we can obtain the inequality

$$E(b_n) \le \frac{1}{1 + c_1 k_n} E(a_n), \tag{3.14}$$

by applying Proposition 3.2 to the interval (a_n, b_n) instead of (a, b), where $c_1 = \frac{16\alpha_1}{15(B^2+\alpha_1)}$, $k_n = \frac{m_n T_n^3}{256+d_n T_n^2}$, $d_n = \frac{3(1+\alpha_1)}{B^2+\alpha_1} + \frac{4\alpha_1(1+\alpha_2 B^2)}{B^2+\alpha_1} M_n m_n$, and $T_n = b_n - a_n$. Since $E(t)$ is nonincreasing by (1.5), we have the inequalities

$$E(a_{n+1}) \le E(b_n) \le \frac{E(a_n)}{1 + c_1 k_n} \le \prod_{i=1}^{n} \frac{E(a_0)}{1 + c_1 k_i} \le \prod_{i=1}^{n} \frac{E(0)}{1 + c_1 k_i}, \tag{3.15}$$

by inequality (3.14). Theorem 3.4 holds, if $E(a_{n+1}) \to 0$ as $n \to \infty$, and hence, it suffices to show that $\prod_{i=1}^{\infty} \frac{1}{1+c_1 k_i} = 0$ or equivalently $\sum_{i=1}^{\infty} \ln(\frac{1}{1+c_1 k_i}) = 0$. It is clear that if $k_i \to 0$ as $i \to \infty$, the result holds, and while if $k_i \to 0$, its proof reduces to showing that $\sum_{i=1}^{\infty} k_i = \infty$. In fact, $k_n \geq \frac{m_n T_n}{2} \min\left\{\frac{T_n^2}{256}, \frac{1}{d_n}\right\}$ if $d_n \geq \frac{256}{T_n^2}$, and $k_n \geq \frac{m_n T_n}{2} \min\left\{\frac{T_n^2}{256}, \frac{1}{d_n}\right\}$ if $d_n \leq \frac{256}{T_n^2}$. Thus, we have

$$\frac{m_n T_n}{2} \min\left\{\frac{T_n^2}{256}, \frac{1}{d_n}\right\} \leq k_n = \frac{m_n T_n^3}{256 + d_n T_n^2} \leq m_n T_n \min\left\{\frac{T_n^2}{256}, \frac{1}{d_n}\right\},$$

from which the desired result follows. □

3.2. The case of positive-negative

The definition of positive-negative in this subsection in quite different from the one in Subsection 2.2.

Definition 3.5. Let $\{t_n\}$ be a strictly increasing sequence on $(0, \infty)$ with $t_n \in N$ and $t_n \to \infty$ as $n \to \infty$. For all $n \in N$, let $I_{2n} = (t_{2n}, t_{2n+1})$, $I_{2n+1} = (t_{2n+1}, t_{2n+2})$, and let T_n be the length of I_n. We say that a function $h(t)$ is in the case of positive-negative, if for all $n \in N$, there exist constants m_{2n}, M_{2n}, and $M_{2n+1} > 0$ such that

$$m_{2n} \leq h(t) \leq M_{2n}, \quad \forall t \in I_{2n},$$
$$-M_{2n+1} \leq h(t) \leq 0, \quad \forall t \in I_{2n+1}.$$

Theorem 3.6. *Suppose that* $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and that for all* $n \in N$, $h_1(t)$ *and* $h_2(t)$ *have the same sign and* $h_1(t) + h_2(t)$ *is in the positive-negative case, i.e., there exist constants* m_{2n}, M_{2n}, *and* $M_{2n+1} > 0$ *such that*

$$m_{2n} \leq h_1(t) + h_2(t) \leq M_{2n}, \quad \forall t \in I_{2n}, \tag{3.16}$$

$$-M_{2n+1} \leq h_1(t) + h_2(t) \leq 0, \quad \forall t \in I_{2n+1}. \tag{3.17}$$

If

$$\sum_{n=0}^{\infty} M_{2n+1} T_{2n+1} < \infty \text{ and } \sum_{n=0}^{\infty} m_{2n} T_{2n} \min\left\{T_{2n}^2, \frac{1}{1 + M_{2n} m_{2n}}\right\} = \infty,$$

then the solution u *of problem (1.1)-(1.3) satisfies*

$$E(t) \to 0 \quad as \quad t \to \infty.$$

Remark 3.7. It follows from the assumptions of Theorem 3.6 that the energy function $E(t)$ is decreasing on I_{2n} and increasing on I_{2n+1}.

Proof. For $n \in N$, employing Theorem 3.4 on I_{2n}, one can see that

$$E(t_{2n+1}) \leq \frac{1}{1 + c_1 k_{2n}} E(t_{2n}), \tag{3.18}$$

where $k_{2n} = \frac{m_{2n}T_{2n}^3}{256+\{\frac{3(1+\alpha_1)}{B^2+\alpha_1}+\frac{4\alpha_1(1+\alpha_2 B^2)}{B^2+\alpha_1}M_{2n}m_{2n}\}}T_{2n}^2$ and $T_{2n} = t_{2n+1} - t_{2n}$.

On the other hand, applying (1.4), (1.5) and (3.17) to I_{2n+1}, we can have the inequalities

$$\begin{aligned} E'(t) \leq & - h_1(t) \int_\Omega g(u_t)u_t dx - h_2(t)\|\nabla u_t\|_2^2 \\ \leq & - h_1(t)\alpha_1\|u_t\|_2^2 - h_2(t)\|\nabla u_t\|_2^2 \\ \leq & 2M_{2n+1}\alpha_1 E(t), \end{aligned}$$

and hence, we get the inequality

$$E(t_{2n+2}) \leq E(t_{2n+1})e^{2\alpha_1 M_{2n+1}T_{2n+1}},$$

which implies that

$$E(t_{2n+2}) \leq \left[\prod_{i=0}^{n} e^{M_{2i+1}T_{2i+1}} \left(\frac{1}{1+c_1 k_{2i}}\right)\right] e^{2\alpha_1} E(0),$$

by inequality (3.18).

We can obtain the desired result if $\prod_{i=0}^n e^{M_{2i+1}T_{2i+1}}\left(\frac{1}{1+c_1k_{2i}}\right) = 0$, and this condition is equivalent to

$$\sum_{i=1}^{n}[M_{2i+1}T_{2i+1} - \ln(1+c_1k_{2i})] = -\infty. \tag{3.19}$$

In particular, if we assume

$$\sum_{i=0}^{\infty} M_{2i+1}T_{2i+1} < \infty \ \text{ and } \ \sum_{i=0}^{\infty} m_{2i}T_{2i}\min\left\{T_{2i}^2, \frac{1}{1+M_{2i}m_{2i}}\right\} = \infty,$$

the condition (3.19) holds. The proof is completed. □

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