JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **33**, No. 4, November 2020
http://dx.doi.org/10.14403/jcms.2020.33.4.413

ZERO COMMUTING MATRIX WITH PASCAL MATRIX

Eunmi Choi*

Abstract. In this work we study 0-commuting matrix $C(0)$ under relationships with the Pascal matrix $C(1)$. Like kth power $C(1)^k$ is an arithmetic matrix of $(kx+y)^n$ with $yx = xy$ $(k \geq 1)$, we express $C(0)^k$ as an arithmetic matrix of certain polynomial with $yx = 0$.

1. Introduction

An arithmetic matrix of a polynomial $f(x)$ is a matrix consisting of coefficients of $f(x)$. The Pascal matrix is a famous arithmetic matrix of $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, in which the commutativity $yx = xy$ is assumed tacitly. As a generalization, with two q-commuting variables x and y satisfying $yx = qxy$ $(q \in \mathbb{Z})$, the arithmetic matrix of $(x+y)^n$ is called a q-commuting matrix denoted by $C(q)$ [1]. The $C(q)$ is composed of q-binomials $\begin{bmatrix} i \\ j \end{bmatrix}_q = \frac{(1-q^i)(1-q^{i-1})\cdots(1-q^{i-j+1})}{(1-q)(1-q^2)\cdots(1-q^j)}$ [7]. When $q = \pm 1$, $C(1)$ is the Pascal matrix and $C(-1)$ is the Pauli Pascal matrix [4]. A block matrix form $C(-1) = \begin{bmatrix} i & & & \\ i & i & & \\ i & 2i & i & \\ i & 3i & 3i & i \\ & \cdots & & \end{bmatrix}$ with a 2×2 matrix $i = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}$ looks very similar to $C(1)$ [5]. In particular if $q = 0$ then $C(0) = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$. The $n \times n$ matrix $C(0)$ was studied as an adjacency matrix of relation $\leq$ on $\{1, 2, \cdots, n\}$ and its inverse $C(0)^{-1} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & \cdots & \end{bmatrix}$ is a sum of nilpotent matrix and identity matrix ([2], p.5)

Received May 12, 2020; Accepted August 20, 2020.
2010 Mathematics Subject Classification: 11B37, 11B39.
Key words and phrases: q-commuting matrix, Pauli Pascal matrix, arithmetic matrix.

In the work we simply denote $C(0)$ by Z and the Pascal matrix $C(1)$ by P, and study Z under relationships with P. Since P^k is an arithmetic matrix of $(kx+y)^n$ with $yx = xy$ for $k \geq 1$ [6], we develop interconnections of Z^k with P as well as an arithmetic matrix of certain polynomial.

For our notations, write $P = [e_{i,j}]$ and inverse $P^{-1} = [v_{i,j}]$, so $v_{i,j} = (-1)^{i-j} e_{i,j} = (-1)^{i-j}\binom{i}{j}$ for $i \geq j \geq 0$. As usual, M_n denotes a $n \times n$ matrix M.

2. Z^k $(k = 2, 3)$ as arithmetic matrices

Note $Z = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $Z^2 = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 3 & 2 & 1 & \\ 4 & 3 & 2 & 1 \end{bmatrix}$ and $Z^3 = \begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ 6 & 3 & 1 & \\ 10 & 6 & 3 & 1 \end{bmatrix}$. For each $k \geq 1$, write $Z^k = \left[y_0^{(k)} | \, y_1^{(k)} | \, y_2^{(k)} | \cdots \right]$ where $y_j^{(k)}$ $(j \geq 0)$ are jth column vectors.

Lemma 2.1. *Let* $Y = \left[y_0^{(1)} | \, y_0^{(2)} | \, y_0^{(3)} | \, \cdots \right] = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots \\ y_{2,1} & y_{2,2} & y_{2,3} & \cdots \\ y_{3,1} & y_{3,2} & y_{3,3} & \cdots \\ & \cdots & & \end{bmatrix}$ *be a matrix composed of* 0*th column vector* $y_0^{(k)}$ *of each* Z^k $(k \geq 1)$*. Then* $y_{i,j}$ $(i, j \geq 1)$ *satisfies* $y_{i,j} = y_{i-1,j} + y_{i,j-1}$*. In the* k*th column* $y_0^{(k)}$*, the* i*th entry is* $y_{i,k} = e_{i+k-2,k-1}$*.*

Proof. Clearly $Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \end{bmatrix}$ is a symmetric type Pascal matrix satisfying $y_{i,j} = y_{i-1,j} + y_{i,j-1}$ [3]. From $y_0^{(3)} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \\ \cdots \end{bmatrix}$, $y_0^{(4)} = \begin{bmatrix} 1 \\ 4 \\ 10 \\ 20 \\ \cdots \end{bmatrix}$ and $y_0^{(5)} = \begin{bmatrix} 1 \\ 5 \\ 15 \\ 35 \\ \cdots \end{bmatrix}$, we notice $y_{i,3} = \frac{i(i+1)}{2} = \binom{i+1}{2}$, $y_{i,4} = \frac{i(i+1)(i+2)}{6} = \binom{i+2}{3}$ and $y_{i,5} = \binom{i+3}{4}$ in $y_0^{(k)}$ $(k = 3, 4, 5)$. In $y_0^{(k)}$, assume $y_{i-1,k} = \binom{i+k-3}{k-1} = e_{i+k-3,k-1}$. Then we have

$$y_{i,k} = y_{i-1,k} + y_{i,k-1} = e_{i+k-3,k-1} + e_{i+k-3,k-2} = e_{i+k-2,k-1}. \qquad \square$$

Thus Z^k is determined by $y_0^{(k)} = \begin{bmatrix} e_{k-1,k-1} \\ e_{k,k-1} \\ e_{k+1,k-1} \\ \cdots \end{bmatrix}$ by shifting one space down at each column. For Pascal matrix $P = C(1)$ of $(x+y)^n$ with $yx = xy$, the power P^k is an arithmetic matrix of $(kx+y)^n$. Indeed $P^2 = \begin{bmatrix} 1 \\ 2\,1 \\ 4\,4\;\;1 \\ 8\,12\,6\,1 \\ \cdots \end{bmatrix}$, $P^3 = \begin{bmatrix} 1 \\ 3\;\;1 \\ 9\;\;6\;\;1 \\ 27\,27\,9\,1 \\ \cdots \end{bmatrix}$ yield expansions of $(2x+y)^n$ and $(3x+y)^n$. An analogy question is that how $Z^k = C(0)^k$ is related to polynomials $(kx+y)^n$ with $yx = 0$.

For $k \geq 1$, let $A^{(k)}$ be an arithmetic matrix of $(kx+y)^n$ with $yx = 0$.

THEOREM 2.2. *Let diag*$[k^i]$ *be a diagonal matrix having diagonal entries* k^i $(i \geq 0)$. *Then* $A^{(k)} = \text{diag}[k^i]\, Z\, \text{diag}[k^i]^{-1}$ *and* $A^{(k)}Z = ZA^{(k)}$.

Proof. Observe $A^{(2)} = \begin{bmatrix} 1 \\ 2\;\;1 \\ 2^2\,2\;\;1 \\ 2^3\,2^2\;2\,1 \\ \cdots \end{bmatrix}$ and $A^{(3)} = \begin{bmatrix} 1 \\ 3\;\;1 \\ 3^2\,3\;\;1 \\ 3^3\,3^2\;3\,1 \\ \cdots \end{bmatrix}$. Now for any k, assume $(kx+y)^n = k^n x^n + k^{n-1}x^{n-1}y + \cdots + kxy^{n-1} + y^n$. Then

$(kx+y)^{n+1} = (kx+y)^n(kx+y) = k^{n+1}x^{n+1} + k^n x^n y + \cdots + kxy^n + y^{n+1}$,

so $A^{(k)} = \begin{bmatrix} 1 \\ k\;\;1 \\ k^2\,k\;\;1 \\ k^3\,k^2\;k\,1 \\ \cdots \end{bmatrix} = \text{diag}[k^i]Z\text{diag}[k^i]^{-1}$. Thus $ZA^{(k)} = A^{(k)}Z$ since sum of entries of 0th column in $A_n^{(k)}$ equals that of nth row, and all entries of Z are 1. □

Let $J = \begin{bmatrix} 0 \\ 10 \\ \;\;10 \\ \;\;\;\;10 \\ \cdots \end{bmatrix} = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$. Then $J^k = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}_k \\ \hline I \end{bmatrix}$ for $k > 0$ by taking proper size matrices. Clearly J_n is a nilpotent matrix satisfying $J_n^k = 0$ for $k \geq n$, and we may consider $J^0 = I$.

THEOREM 2.3. *Let* $\widehat{A_k^{(2)}} = A_k^{(2)} + J_k^{2\,T}$ *and* $\widehat{Z_k^2} = -Z_k^2 + J_k^{2\,T}$. *Then for any* $n \geq 4$, $A_n^{(2)} = Z_n^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(2)}} & J_{(n-2)\times 2}^{n-4} \end{array}\right]$ *and* $Z_n^2 = A_n^{(2)} \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{Z_{n-2}^2} & J_{(n-2)\times 2}^{n-4} \end{array}\right]$.

Proof. The transpose matrix $J_k^{2\,T} = \begin{bmatrix} 0010\cdots0 \\ 0001\cdots0 \\ \cdots1 \\ \cdots0 \\ 0 \quad \cdots0 \\ 0 \quad \cdots0 \end{bmatrix}$ is upper triangular, so with lower triangular matrices $A_k^{(2)} = [a_{i,j}]$ and $Z_k^2 = [z_{i,j}]$, the $\widehat{A_k^{(2)}} =$

$[\hat{a}_{i,j}] = A_k^{(2)} + J_k^{2T}$ and $\widehat{Z_k^2} = [\hat{z}_{i,j}] = -Z_k^2 + J_k^{2T}$ satisfy $\hat{a}_{i,j} = a_{i,j}$, $\hat{z}_{i,j} = -z_{i,j}$ if $i \geq j$, and $\hat{a}_{i,j} = \hat{z}_{i,j} = 1$ or 0 according to $i = j+1$ or otherwise. Indeed,

$$\widehat{A_k^{(2)}} = A_k^{(2)} + J_k^{2T} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 2^2 & 2 & 1 & \\ \cdots & & & \\ 2^{k-1} & 2^{k-2} & \cdots & 1 \end{bmatrix} + \begin{bmatrix} 0010 & \\ 0001 & \\ \cdots & 0 \\ & 1 \\ 0 \;.\;.\;. & 0 \\ 0 \;.\;.\;. & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & & \\ 2 & 1 & 0 & 1 & \\ 2^2 & 2 & 1 & 01 & \\ \cdots & & & & \\ 2^{k-3} & & \cdots & & 2101 \\ 2^{k-2} & 2^{k-3} & \cdots & & 210 \\ 2^{k-1} & 2^{k-2} & \cdots & & 21 \end{bmatrix}.$$

When $n = 5$, simple multiplications of $A_5^{(2)}$ and Z_5 show that

$$Z_5^{-2} A_5^{(2)} = \begin{bmatrix} 1 \\ 01 \\ 101 \\ 2101 \\ 42101 \end{bmatrix} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_3^{(2)}} & J_{3\times 2} \end{array}\right] \text{ and } {A_5^{(2)}}^{-1} Z_5^2 = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ -1 & 0 & 1 & & \\ -2 & -1 & 0 & 1 & \\ -3 & -2 & -1 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{Z_3^2} & J_{3\times 2} \end{array}\right],$$

because $\begin{bmatrix} 101 \\ 210 \\ 421 \end{bmatrix} = A_3^{(2)} + J_3^{2T} = \widehat{A_3^{(2)}}$ and $\begin{bmatrix} -1 & 0 & 1 \\ -2 & -1 & 0 \\ -3 & -2 & -1 \end{bmatrix} = -Z_3^2 + J_3^{2T} = \widehat{Z_3^2}$.

Now we write $Z_n^2 = \left[\begin{array}{c|c} Z_{n-1}^2 & 0 \\ \hline n, n-1, \cdots, 2 & 1 \end{array}\right]$ and $A_n^{(2)} = \left[\begin{array}{c|c} A_{n-1}^{(2)} & 0 \\ \hline 2^{n-1}, \cdots 2^2, 2 & 1 \end{array}\right]$ in block matrix forms, and assume an induction hypothesis that

$$Z_n^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(2)}} & J_{(n-2)\times 2}^{n-4} \end{array}\right] = A_n^{(2)} \text{ with } J_{(n-2)\times 2}^{n-4} = \left[\begin{array}{c} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}_{n-4} \\ \hline 1\ 0 \\ 0\ 1 \end{array}\right].$$

Then Theorem 2 says

$$\begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 3 & 2 & 1 & & \\ 4 & 3 & 2 & 1 & \\ n & n-1 & \ddots & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 1 & 0 & 1 & \\ 2 & 1 & 0 & 1 \\ 2^{n-3} & 2^{n-4} & \ddots & 2, 1, 0, 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 2^2 & 2 & 1 & & \\ 2^3 & 2^2 & 2 & 1 & \\ 2^{n-1} & 2^{n-2} & \ddots & 2 & 1 \end{bmatrix}.$$

But since the $(n-1)\times(n+1)$ block matrix $\left[\begin{array}{c|c} \widehat{A_{n-1}^{(2)}} & J_{(n-1)\times 2}^{n-3} \end{array}\right]$ equals

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & \cdots & & 00 \\ 2 & 1 & 01 & \cdots & & 00 \\ & \cdots & & & & 00 \\ 2^{n-4} & \cdots & & & 2101 & 00 \\ 2^{n-3} & 2^{n-4} & \cdots & & 210 & 10 \\ 2^{n-2} & 2^{n-3} & \cdots & & 21 & 01 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & \cdots & 0 & 0 \\ 2 & 1 & 01 & \cdots & 0 & 0 \\ & \cdots & & & 0 & 0 \\ 2^{n-4} & \cdots & & & 21010 & 0 \\ 2^{n-3} & 2^{n-4} & \cdots & & 2101 & 0 \\ \hline 2^{n-2} & 2^{n-3} & \cdots & & 210 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \widehat{A_{n-2}^{(2)}} & : J_{(n-2)\times 2}^{n-4} & 0 \\ \hline 2^{n-2} \cdots 2 : & 1\ 0 & 1 \end{array}\right],$$

the block matrix multiplications show

$$Z_{n+1}^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-1}^{(2)}} & J_{(n-1)\times 2}^{n-3} \end{array}\right] = \left[\begin{array}{c|c} Z_n^2 & 0 \\ \hline n+1, n, \cdots, 2 & 1 \end{array}\right] \left[\begin{array}{cc|c} I_2 : 0 & : 0 & 0 \\ \widehat{A_{n-2}^{(2)}} & : J_{(n-2)\times 2}^{n-4} & 0 \\ \hline 2^{n-2} \cdots 2 : & 1\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array}\right],$$

where $H = 0$, $S = 1$ and $G = Z_n^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(2)}} & J_{(n-2)\times 2}^{n-4} \end{array}\right] = A_n^{(2)}$ by hypothesis.

And

$$R = (n+1, n, \cdots, 3, 2)\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ 2 & 1 & 0 & 1 & & & \\ 2^{n-4} & 2^{n-5} & \cdots & 1 & 0 & 1 & 0 \\ 2^{n-3} & 2^{n-4} & \cdots & 2 & 1 & 0 & 1 \end{bmatrix} + (2^{n-2}, \cdots, 2^2, 2, 1, 0)$$

$$= \Big(n+1+\sum_{i=0}^{n-3}(n-1-i)2^i, n+\sum_{i=0}^{n-4}(n-2-i)2^i, n-1+\sum_{i=0}^{n-5}(n-3-i)2^i,$$
$$\cdots, 4+\sum_{i=0}(2-i)2^i, 3, 2\Big) + (2^{n-2}, \cdots, 2, 1, 0)$$
$$= \Big(n+1+\sum_{i=0}^{n-2}(n-1-i)2^i, n+\sum_{i=0}^{n-3}(n-2-i)2^i, n-1+\sum_{i=0}^{n-4}(n-3-i)2^i,$$
$$\cdots, 4+\sum_{i=0}^{1}(2-i)2^i, 4, 2\Big).$$

Thus the first few entries in R from right are $2, 2^2, 2^3, 2^4, \cdots$. So as an induction hypothesis, we assume $n + \sum_{i=0}^{n-3}(n-2-i)2^i = 2^{n-1}$. Then

$$(n+1)+\sum_{i=0}^{n-2}(n-1-i)2^i = (n+1)+(n-1)+\sum_{i=1}^{n-2}(n-1-i)2^i$$
$$= 2n+2\sum_{i=0}^{n-3}(n-2-i)2^i = 2n+2(2^{n-1}-n) =$$
2^n.

So $R = (2^n, 2^{n-1}, \cdots, 2^3, 2^2, 2)$, thus $Z_{n+1}^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-1}^{(2)}} & J_{(n-1)\times 2}^{n-3} \end{array}\right] = A_{n+1}^{(2)}$.

And the second identity can be proved similarly. □

We now go on Z^3 and the arithmetic matrix $A^{(3)}$ of $(3x+y)^n$ with $yx = 0$.

THEOREM 2.4. *Let* $\widehat{A_k^{(3)}} = (3I_k - J_k)A_k^{(3)} + J_k^{2T}$ *and* $\widehat{Z_k^3} = -(3I_k - J_k)Z_k^3 + J_k^{2T}$. *Then* $A_n^{(3)} = Z_n^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(3)}} & J_{(n-2)\times 2}^{n-4} \end{array}\right]$ *and* $Z_n^3 = A_n^{(3)} \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{Z_{n-2}^{(3)}} & J_{(n-2)\times 2}^{n-4} \end{array}\right]$ *for* $n \geq 4$.

Proof. When $k = 5$, Theorem 2 shows $Z_5 A_5^{(3)} = A_5^{(3)} Z_5$ and

$$Z_5^{-3}A_5^{(3)} = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 3 & 0 & 1 & & \\ 8 & 3 & 0 & 1 & 0 \\ 24 & 8 & 3 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X & J_{3\times 2} \end{array}\right] \text{ with } X = \begin{bmatrix} 3 & 0 & 1 \\ 8 & 3 & 0 \\ 24 & 8 & 3 \end{bmatrix}.$$

Clearly $X = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 3 & 0 \\ 24 & 8 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Y + J_3^{2T}$ with $Y = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 3 & 0 \\ 24 & 8 & 3 \end{bmatrix}$. But since $YA_3^{(3)} = A_3^{(3)}Y$ and $A_3^{-(3)}Y = \begin{bmatrix} 3 & & \\ -1 & 3 & \\ & -1 & 3 \end{bmatrix} = 3I - J$, we have

$$X = Y + J_3^{2T} = A_3^{(3)}(3I-J) + J_3^{2T} = (3I-J)A_3^{(3)} + J_3^{2T} = \widehat{A_3^{(3)}},$$

so $Z_5^{-3}A_5^{(3)} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_3^{(3)}} & J_{3\times 2} \end{array}\right]$. For some $n > 0$, we assume

$$A_n^{(3)} = Z_n^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(3)}} & J_{(n-2)\times 2}^{n-4} \end{array}\right] = \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 6 & 3 & 1 & & \\ 10 & 6 & 3 & 1 & \\ \binom{n+1}{2} & \binom{n}{2} & \ddots & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 3 & 0 & 1 & & \\ 8 & 3 & 0 & 1 & \\ 2^3 3^{n-4} & 2^3 3^{n-5} & \ddots & 2^3\, 3 & 0\, 1 \end{bmatrix}.$$

We note that

$$\widehat{A_n^{(3)}} = (3I-J)A_n^{(3)} + J_n^{2T}$$

$$= (3I-J)\begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ 3^2 & 3 & 1 & \\ & \cdots & & \\ 3^{n-1} & 3^{n-2} & & 3\,1 \end{bmatrix} + \begin{bmatrix} 0010\cdots0 \\ 0001\cdots0 \\ \cdots 0 \\ \cdots 1 \\ \cdots 0 \\ \cdots 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 0 & \\ 8 & 3 & 0 & 1 & \\ \cdots & & & & 0\,0 \\ 2^3 3^{n-3} & 2^3 3^{n-4} & \cdots 3 & & 0\,1\,0 \\ 2^3 3^{n-2} & 2^3 3^{n-3} & \cdots 2^3 & & 3\,0\,1 \end{bmatrix}.$$

so the block matrix $\left[\widehat{A_{n-1}^{(3)}} \middle| J_{(n-1)\times 2}^{n-3}\right]$ equals

$$\left[\begin{array}{ccccc|c} 3 & 0 & 1 & 0 & & 0\,0 \\ 8 & 3 & 0 & 1 & & 0\,0 \\ \cdots & & & & & \\ 2^3 3^{n-4}, & 2^3 3^{n-5}, & \cdots, 3, & 0, 1, 0 & & 1\,0 \\ 2^3 3^{n-3}, & 2^3 3^{n-4}, & \cdots, 2^3, 3, & 0, 1 & & 0\,1 \end{array}\right] = \left[\begin{array}{ccccc|c} 3 & 0 & 1 & 0 & 0 & 0 \\ 8 & 3 & 0 & 1 & 0 & 0 \\ \cdots & & & & & \\ 2^3 3^{n-4}, & 2^3 3^{n-5}, & \cdots, 3, & 0, 1, 0 & 1 & 0 \\ \hline 2^3 3^{n-3}, & 2^3 3^{n-4}, & \cdots, 2^3, 3, & 0, 1, 0 & & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \widehat{A_{n-2}^{(3)}} \quad : J_{(n-2)\times 2}^{n-4} & 0 \\ \hline 2^3 3^{n-3}, \cdots, 2^3, 3, 0 : 1, 0 & 1 \end{array}\right].$$

Thus the block matrix multiplications yield

$$Z_{n+1}^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-1}^{(3)}} & J_{(n-1)\times 2}^{n-3} \end{array}\right] = \left[\begin{array}{c|c} Z_n^3 & 0 \\ \hline \binom{n+2}{2}, \binom{n+1}{2}, \cdots, 3 & 1 \end{array}\right] \left[\begin{array}{cc|c} I_2 : 0 & : 0 & 0 \\ \widehat{A_{n-2}^{(3)}} & : J_{(n-2)\times 2}^{n-4} & 0 \\ \hline 2^3 3^{n-3}, \cdots, 2^3, 3 : 1, 0 & & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array}\right],$$

where $G = Z_n^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(3)}} & J_{(n-2)\times 2}^{n-4} \end{array}\right] = A_n^{(3)}$, $H = 0_{n\times 1}$ and $S = 1$. And for R,

$$R = (\binom{n+2}{2}, \binom{n+1}{2}, \cdots, 6, 3) \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(3)}} & J_{(n-2)\times 2}^{n-4} \end{array}\right] + (2^3 3^{n-3}, \cdots, 2^3, 3, 0, 1, 0)$$

$$= \left(\binom{n+2}{2}, \binom{n+1}{2}, \cdots, 6, 3\right) \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 3 & 0 & 1 & \\ 8 & 3 & 0 & 1 \\ & \cdots & & \\ 2^3 3^{n-4}, & 2^3 3^{n-5}, & \ddots & 3, 0, 1 \end{bmatrix} + (2^3 3^{n-3}, \cdots, 3, 0, 1, 0)$$

$= (\theta_1, \theta_2, \cdots, \theta_n, \theta_{n+1}) + (2^3 3^{n-3}, \cdots, 2^3, 3, 0, 1, 0)$,

where

$\theta_1 = \binom{n+2}{2} + 3\binom{n}{2} + 2^3\binom{n-1}{2} + 2^3 3\binom{n-2}{2} + \cdots + 2^3 3^{n-4}\binom{3}{2}$,

$\theta_2 = \binom{n+1}{2} + 3\binom{n-1}{2} + 2^3\binom{n-2}{2} + 2^3 3\binom{n-3}{2} + \cdots + 2^3 3^{n-5}\binom{3}{2}, \cdots,$

$\theta_{n-2} = \frac{6(5)}{2} + 6 \cdot 3 + 3 \cdot 2^3, \ \ \theta_{n-1} = \frac{5(4)}{2} + 3, \ \theta_n = 6, \ \ \theta_{n+1} = 3.$

So the last few entries in R are $\theta_{n+1} = 3$, $\theta_n + 3 = 3^2$, $\theta_{n-1} + 2^3 = 3^3$ and $\theta_{n-2} + 2^3 \cdot 3 = 3^4$. We now assume $3^{n-1} = \theta_2 + 2^3 3^{n-4}$. Then

$3^{n-1} = \left(\binom{n+1}{2} + 3\binom{n-1}{2} + 2^3\binom{n-2}{2} + 2^3 3\binom{n-3}{2} + \cdots + 2^3 3^{n-5}\binom{3}{2}\right) + 2^3 3^{n-4}$.

Hence

$$\begin{aligned} &\theta_1 + 2^3 3^{n-3} \\ &= \left(\binom{n+2}{2} + 3\binom{n}{2} + 2^3\binom{n-1}{2} + 2^3 3\binom{n-2}{2} + \cdots + 2^3 3^{n-4}\binom{3}{2}\right) + 2^3 3^{n-3} \\ &= \binom{n+2}{2} + 3\binom{n}{2} + 2^3\binom{n-1}{2} + 3 \cdot 2^3\left(\binom{n-2}{2} + 3\binom{n-3}{2} + \cdots + 3^{n-5}\binom{3}{2} + 3^{n-4}\right) \\ &= \binom{n+2}{2} + 3\binom{n}{2} + 2^3\binom{n-1}{2} + 3\left(3^{n-1} - \binom{n+1}{2} - 3\binom{n-1}{2}\right) \\ &= 3^n + \left(\binom{n+2}{2} + 3\binom{n}{2} + 2^3\binom{n-1}{2} - 3\binom{n+1}{2} - 3^2\binom{n-1}{2}\right) = 3^n, \end{aligned}$$

by induction hypothesis. Thus $R = (3^n, 3^{n-1}, \cdots, 3^2, 3)$, so we have

$$Z_{n+1}^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-1}^{(3)}} & J_{(n-1)\times 2}^{n-3} \end{array}\right] = \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array}\right] = \left[\begin{array}{c|c} A_n^{(3)} & 0 \\ \hline 3^n, 3^{n-1}, \cdots, 3 & 1 \end{array}\right] = A_{n+1}^{(3)}.$$

Similarly for the second identity, we have

$$A^{(3)^{-1}} Z^3 = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ -3 & 0 & 1 & & \\ -8 & -3 & 0 & 1 & \\ -15 & -8 & -3 & 0 & 1 \\ & \cdots & & & \end{bmatrix} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X & J \end{array}\right] \text{ with } X = \begin{bmatrix} -3 & 0 & 1 \\ -8 & -3 & 0 \\ -15 & -8 & -3 \\ \cdots & & \end{bmatrix}.$$

But since $X = \begin{bmatrix} -3 & 0 & 0 \\ -8 & -3 & 0 \\ -15 & -8 & -3 \\ \cdots & & \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdots & & \end{bmatrix} = -Y + J^{2^T}$ with $Y = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 3 & 0 \\ 15 & 8 & 3 \\ \cdots & & \end{bmatrix}$,

we have $YZ^{-3} = \begin{bmatrix} 3 & & \\ -1 & 3 & \\ & -1 & 3 \\ & \cdots & \end{bmatrix} = 3I - J$. So $Y = Z^3(3I - J) = (3I - J)Z^3$

and $X = -(3I - J)Z^3 + J^{2^T} = \widehat{Z^3}$. Thus $Z_n^3 = A_n^{(3)} \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{Z_{n-2}^3} & J_{(n-2)\times 2}^{n-4} \end{array}\right]$. □

Connections of $A^{(k)}$ and Z^k $(k=2,3)$ are further explained by J.

THEOREM 2.5. *(1)* $Z_n^{-2}A_n^{(2)} = I_n + J_n^2 A_n^{(2)}$ *and* ${A_n^{(2)}}^{-1} Z_n^2 = I_n - J_n^2 Z_n^2$.

(2) $Z_n^{-3}A_n^{(3)} = I_n + J_n^2(3I_n - J_n)A_n^{(3)}$ *and* ${A_n^{(3)}}^{-1} Z_n^3 = I_n - J_n^2(3I_n - J_n)Z_n^3$.

(3) ${A_n^{(2)}}^{-1} = Z_n^{-2} - J_n^2$ *and* ${A_n^{(3)}}^{-1} = Z_n^{-3} - J_n^2(3I_n - J_n)$.

Proof. With $\widehat{A_k^{(2)}} = A_k^{(2)} + {J_k^2}^T$, Theorem 3 shows

$$Z_n^{-2}A_n^{(2)} = \left[\begin{array}{c|c} \widehat{I_2 : 0} & 0 \\ \hline A_{n-2}^{(2)} & J_{(n-2)\times 2}^{n-4} \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ \hline 1 & 0 & 1 & & \\ 2 & 1 & 0 & 1 & \\ \cdots & & & & 10 \\ 2^{n-3} & 2^{n-4} & \cdots & & 01 \end{array}\right] = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline A_{n-2}^{(2)} \end{array}\right]$$

$$= I_n + J_n^2 A_n^{(2)}.$$

Similarly with $\widehat{Z_k^2} = -Z_k^2 + {J_k^2}^T$, we also have

$$ {A_n^{(2)}}^{-1} Z_n^2 = \left[\begin{array}{c|c} \widehat{I_2 : 0} & 0 \\ \hline Z_{n-2}^2 & J_{(n-2)\times 2}^{n-4} \end{array}\right] = \left[\begin{array}{cccc|c} 1 & & 00 & & \\ 0 & & 10 & & \\ \hline -1 & & 01 & & \\ -2 & & -101 & & \\ . & & .. & & 10 \\ -e_{n-2,1} & -e_{n-3,1} & \cdots & & 01 \end{array}\right] = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline -Z_{n-2}^2 \end{array}\right]$$

$$= I_n - J_n^2 Z_n^2.$$

On the other hand, with $\widehat{A_k^{(3)}} = (3I_k - J_k)A_k^{(3)} + {J_k^2}^T$, Theorem 4 shows

$$Z_n^{-3}A_n^{(3)} = \left[\begin{array}{c|c} \widehat{I_2 : 0} & 0 \\ \hline A_{n-2}^{(3)} & J_{(n-2)\times 2}^{n-4} \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ \hline 3 & 0 & 1 & & \\ 8 & 3 & 01 & & \\ 24 & 8 & 301 & & 00 \\ \cdots & & & & 10 \\ 2^3 3^{n-4} & 2^3 2^{n-5} & \cdots & & 01 \end{array}\right] = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline (3I-J)A_{n-2}^{(3)} \end{array}\right]$$

$$= I_n + J_n^2(3I_n - J_n)A_n^{(3)}.$$

Moreover ${A_n^{(3)}}^{-1} Z_n^3 = \left[\begin{array}{c|c} \widehat{I_2 : 0} & 0 \\ \hline Z_{n-2}^3 & J_{(n-2)\times 2}^{n-4} \end{array}\right] = I_n - J_n^2(3I - J)Z_n^3$. And also $Z_n^{-2} = {A_n^{(2)}}^{-1} + J_n^2$ and $Z_n^{-3} = {A_n^{(3)}}^{-1} + J_n^2(3I_n - J_n)$ by (1) and (2). □

In fact, $Z^{-2} - J^2 = \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & & & \\ 0 & & & \\ 1 & 0 & 0 & \\ & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ & -2 & 1 & \\ & & -2 & 1 \end{bmatrix} = {A^{(2)}}^{-1}$.

3. Z^k as an arithmetic matrix

Theorem 5 says $Z_n^{-2}A_n^{(2)} = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline A_{n-1}^{(2)} \end{array}\right] = I_n + J_n^2 A_n^{(2)}$,

$$Z_n^{-3}A_n^{(3)} = I_n + \begin{bmatrix} 0 \\ 0 \\ \hline (3I_{n-2} - J_{n-2})A_{n-2}^{(3)} \end{bmatrix} = I_n + J_n^2(3I_n - J_n)A_n^{(3)},$$

$A_n^{(2)^{-1}} Z_n^2 = I_n - J_n^2 Z_n^2$ and $A_n^{(3)^{-1}} Z_n^3 = I_n - J_n^2(3I_n - J_n)Z_n^3$.

In order to generalize the identities, first consider $k = 4, 5, 6$. Then we have

$$Z_n^{-k}A_n^{(k)} = I_n + \begin{bmatrix} 0 \\ 0 \\ \hline S^{(k)} \end{bmatrix} = I_n + J_n^2 S^{(k)} \tag{1}$$

with $S^{(4)} = \begin{bmatrix} 6 & & & \\ 20 & 6 & & \\ 81 & 20 & 6 & \\ 324 & 81 & 20 & 6 \\ \cdots & & & \end{bmatrix}$, $S^{(5)} = \begin{bmatrix} 10 & & & \\ 40 & 10 & & \\ 205 & 40 & 10 & \\ 1024 & 205 & 40 & 10 \\ \cdots & & & \end{bmatrix}$ and $S^{(6)} = \begin{bmatrix} 15 & & & \\ 70 & 15 & & \\ 435 & 70 & 15 & \\ 2604 & 435 & 70 & 15 \\ \cdots & & & \end{bmatrix}$.

Also

$$A_n^{(k)^{-1}} Z_n^k = I_n - \begin{bmatrix} 0 \\ 0 \\ \hline T^{(k)} \end{bmatrix} = I_n - J_n^2 T^{(k)} \tag{2}$$

with $T^{(4)} = \begin{bmatrix} 6 & & & \\ 20 & 6 & & \\ 45 & 20 & 6 & \\ 84 & 45 & 20 & 6 \\ \cdots & & & \end{bmatrix}$, $T^{(5)} = \begin{bmatrix} 10 & & & \\ 40 & 10 & & \\ 105 & 40 & 10 & \\ 224 & 105 & 40 & 10 \\ \cdots & & & \end{bmatrix}$ and $T^{(6)} = \begin{bmatrix} 15 & & & \\ 70 & 15 & & \\ 210 & 70 & 15 & \\ 504 & 210 & 70 & 15 \\ \cdots & & & \end{bmatrix}$.

Now by means of the kth row $(v_{k,0}, \cdots, v_{k,k})$ of P^{-1}, we define a matrix

$$X^{(k)} = \sum_{j=0}^{k-2} v_{k,j} J^{k-j-2}. \tag{3}$$

THEOREM 3.1. *For $2 \le k \le 6$, $X_n^{(k)} A_n^{(k)} = X_n^{(k)} A_n^{(k)}$ and $X_n^{(k)} Z_n^k = Z_n^k X_n^{(k)}$. Moreover $Z_n^{-k} A_n^{(k)} = I_n + J_n^2 X_n^{(k)} A_n^{(k)}$ and $A_n^{(k)^{-1}} Z_n^k = I_n - J_n^2 X_n^{(k)} Z_n^k$.*

Proof. Clearly $X^{(2)} = I$ and $X^{(3)} = 3I - J$, so Theorem 5 implies

$I_n + J_n^2 X_n^{(2)} A_n^{(2)} = I_n + J_n^2 A_n^{(2)} = Z_n^{-2} A_n^{(2)}$

$I_n + J_n^2 X_n^{(3)} A_n^{(3)} = I_n + J_n^2(3I_n - J_n)A_n^{(3)} = Z_n^{-3} A_n^{(3)}$

$I_n - J_n^2 X_n^{(2)} A_n^{(2)} = A_n^{(2)^{-1}} Z_n^2$ and $I_n + J_n^2 X_n^{(3)} A_n^{(3)} = A_n^{(3)^{-1}} Z_n^3$.

Now from $Z_n^{-k} A_n^{(k)} = I_n + J_n^2 S^{(k)}$ in (1), we have

$$A^{(4)^{-1}} S^{(4)} = \begin{bmatrix} 6 & & & \\ -4 & 6 & & \\ 1 & -4 & 6 & \\ & 1 & -4 & 6 \\ & & \cdots & \end{bmatrix} = 6I - 4J + J^2 = S^{(4)} A^{(4)^{-1}}$$

$$A^{(5)^{-1}} S^{(5)} = \begin{bmatrix} 10 & & & & \\ -10 & 10 & & & \\ 5 & -10 & 10 & & \\ -1 & 5 & -10 & 10 & \\ & -1 & 5 & -10 & 10 \\ & & & \cdots & \end{bmatrix} = 10I - 10J + 5J^2 - J^3 = S^{(5)} A^{(5)^{-1}}$$

$$A^{(6)^{-1}}S^{(6)} = \begin{bmatrix} 15 & & & \\ -20 & 15 & & \\ 15 & -20 & 15 & \\ -6 & 15 & -20 & 15 \\ & & \cdots & \end{bmatrix} = 15I - 20J + 15J^2 - 6J^3 + J^4.$$

Since each coefficients in expressions of $A^{(k)^{-1}}S^{(k)}$ correspond to elements in kth row of P^{-1}, we have $A^{(k)^{-1}}S^{(k)} = S^{(k)}A^{(k)^{-1}} = X^{(k)}$, so (1) gives

$Z_n^{-k}A_n^{(k)} = I_n + J_n^2 S^{(k)} = I_n + J_n^2 X_n^{(k)} A_n^{(k)}$.

Similarly each $T^{(k)}$ in (2) satisfies $Z^{-k}T^{(k)} = T^{(k)}Z^{-k} = A^{(k)^{-1}}S^{(k)} = X^{(k)}$. So we have $A_n^{-(k)}Z_n^k = I_n - J_n^2 T^{(k)} = I_n - J_n^2 X_n^{(k)} Z_n^k$. □

THEOREM 3.2. *Let* $\mu = \sum_{j=0}^{k-2} k^j v_{k,j}$ *for* $k \geq 2$. *Then* $X^{(k)}A^{(k)}$ *is equal to*

$$\begin{bmatrix} v_{k,k-2} & & & \\ v_{k,k-3} + kv_{k,k-2}, v_{k,k-2} & & & \\ \cdots & & & \\ \sum_{j=1}^{k-2} k^{j-1}v_{k,j}, & \sum_{j=2}^{k-2} k^{j-2}v_{k,j}, & \sum_{j=3}^{k-2} k^{j-3}v_{k,j}, \cdots & \\ \mu & \sum_{j=1}^{k-2} k^{j-2}v_{k,j}, & \sum_{j=2}^{k-2} k^{j-2}v_{k,j}, \cdots & \\ k\mu & \mu & \sum_{j=1}^{k-2} k^{j-1}v_{k,j}, \cdots & \\ k^2\mu & k\mu & \mu & \cdots \\ \cdots & & & \\ k^{n-k+1}\mu & k^{n-k}\mu & \cdots, \ \mu, \ \cdots, v_{k,k-3} + kv_{k,k-2}, v_{k,k-2} & \end{bmatrix}$$

Proof. We note that the first few $X^{(k)}$ in (3) are

$$X^{(4)} = v_{4,0}J^2 + v_{4,1}J + v_{4,2}I = \begin{bmatrix} 0 & & & \\ 0 & & & \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & \cdots & & \end{bmatrix} - 4\begin{bmatrix} 0 & & \\ 1 & & \\ & 1 & \\ & & 1 \\ \cdots & & \end{bmatrix} + 6\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ \cdots & & 1 \end{bmatrix} = \begin{bmatrix} 6 & & & \\ -4 & 6 & & \\ 1 & -4 & 6 & \\ & 1 & -4 & 6 \\ & & \cdots & \end{bmatrix}$$

$$X^{(5)} = v_{5,0}J^3 + v_{5,1}J^2 + v_{5,2}J + v_{5,3}I = \begin{bmatrix} 10 & & & & & \\ -10 & 10 & & & & \\ 5 & -10 & 10 & & & \\ -1 & 5 & -10 & 10 & & \\ & -1 & 5 & -10 & 10 & \\ & & -1 & 5 & -10 & \\ & & & & \cdots & \end{bmatrix} \text{ and also,}$$

$$X^{(6)} = \begin{bmatrix} 15 & & & & \\ -20 & 15 & & & \\ 15 & -20 & 15 & & \\ -6 & 15 & -20 & 15 & \\ 1 & -6 & 15 & -20 & 15 \\ & 1 & -6 & 15 & -20 \\ & & & \cdots & \end{bmatrix}. \text{ So } X^{(k)} = \begin{bmatrix} v_{k,k-2} & & & & & \\ v_{k,k-3} & v_{k,k-2} & & & & \\ & \cdots & & & & \\ v_{k,1} & v_{k,2} & \cdots & v_{k,k-2} & & \\ v_{k,0} & v_{k,1} & \cdots & & & \\ & v_{k,0} & v_{k,1} & v_{k,2} & \cdots & v_{k,k-2} \\ & & v_{k,0} & v_{k,1} & \cdots & v_{k,k-3} \\ & & & & \cdots & \end{bmatrix}$$

for any k. Then since $A^{(3)} = \begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ 3^2 & 3 & 1 & \\ 3^3 & 3^2 & 3 & 1 \\ & \cdots & & \end{bmatrix}$ and $\mu = v_{3,0} + 3v_{3,1}$, we have

$$X^{(3)}A_n^{(3)} = \begin{bmatrix} v_{3,1} \\ v_{3,0}+3v_{3,1} & v_{3,1} \\ 3v_{3,0}+3^2v_{3,1} & v_{3,0}+3v_{3,1} & v_{3,1} \\ 3^2v_{3,0}+3v_{3,1} & \cdots & & v_{3,1} \\ \cdots \end{bmatrix} = \begin{bmatrix} v_{3,1} \\ \mu & v_{3,1} \\ 3\mu & \mu & v_{3,1} \\ 3^2\mu & 3\mu & \mu & v_{3,1} \\ \cdots \\ 3^{n-2}\mu, & 3^{n-3}\mu, & \cdots & \mu, & v_{3,1} \end{bmatrix}.$$

Similarly $A^{(4)}$ and $\mu = \sum_{j=0}^{2} 4^j v_{4,j}$ imply

$$X^{(4)}A^{(4)} = \begin{bmatrix} v_{4,2} \\ v_{4,1}+4v_{4,2}, v_{4,2} \\ \mu & v_{4,1}+4v_{4,2}, v_{4,2} \\ 4\mu & \mu & v_{4,1}+4v_{4,2}, v_{4,2} \\ 4^2\mu & 4\mu & \mu & v_{4,1}+4v_{4,2}, v_{4,2} \\ \cdots \\ 4^{n-3}\mu & 4^{n-4}\mu & 4^{n-5}\mu & \cdots & \mu & v_{4,1}+4v_{4,2}, v_{4,2} \end{bmatrix}.$$

Thus for any $k > 1$, it follows that

$$X^{(k)}A^{(k)} = (v_{k,0}J^{k-2} + v_{k,1}J^{k-3} + \cdots + v_{k,k-3}J + v_{k,k-2}I)A^{(k)}$$

$$= v_{k,0}\begin{bmatrix} \left.\begin{matrix}0\\ \vdots \\ 0\end{matrix}\right\}_{k-2} \\ \hline A^{(k)} \end{bmatrix} + v_{k,1}\begin{bmatrix} \left.\begin{matrix}0\\ \vdots \\ 0\end{matrix}\right\}_{k-3} \\ \hline A^{(k)} \end{bmatrix} + \cdots + v_{k,k-3}\begin{bmatrix} 0 \\ \hline A^{(k)} \end{bmatrix} + v_{k,k-2}A^{(k)}.$$

So with $\mu = \sum_{j=0}^{k-2} k^j v_{k,j}$, $X^{(k)}A^{(k)}$ holds the required form in Theorem 7. □

We are ready to generalize Theorem 6 to Z^k and $A^{(k)}$ for all $k \geq 2$.

THEOREM 3.3. *$Z^{-k}A^{(k)} = I + J^2X^{(k)}A^{(k)}$ and ${A^{(k)}}^{-1}Z^k = I - J^2X^{(k)}Z^k$.*

Proof. It is Theorem 6 if $2 \leq k \leq 6$. For any k, consider $A_n^{(k)}$ and $X_n^{(k)} = v_{k,0}J_n^{k-2} + \cdots + v_{k,k-4}J_n^2 + v_{k,k-3}J_n + v_{k,k-2}I_n$.

Let $n = 3$. Since the nilpotent matrix $J_n^t = 0$ for $t \geq n$, we have

$$X_3^{(k)} = v_{k,k-4}J_n^2 + v_{k,k-3}J_n + v_{k,k-2}I = \begin{bmatrix} e_{k,k-2} & 0 & 0 \\ -e_{k,k-3} & e_{k,k-2} & 1 \\ e_{k,k-4} & -e_{k,k-3} & e_{k,k-2} \end{bmatrix}.$$

So with $A_3^{(k)} = \begin{bmatrix} 1 \\ k & 1 \\ k^2 & k & 1 \end{bmatrix}$ and $Z_3^k = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ e_{k+1,k-1} & k & 1 \end{bmatrix}$ in Lemma 1, we have

$$I_3 + J_3^2X_3^{(k)}A_3^{(k)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e_{k,k-2} & 0 & 1 \end{bmatrix} \text{ and } Z_3^k(I_3 + J_3^2X_3^{(k)}A_3^{(k)}) = \begin{bmatrix} 1 \\ k & 1 \\ k^2 & k & 1 \end{bmatrix} = A_3^{(k)}.$$

Now for some n, as an induction hypothesis, we assume

$$A_n^{(k)} = Z_n^k(I_n + J_n^2X_n^{(k)}A_n^{(k)}) = Z_n^k\left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X_{n-2}^{(k)}A_{n-2}^{(k)} + J_{n-2}^2 & {J^{n-4}_{(n-2)\times 2}}^T \end{array}\right].$$

By Lemma 1 we note that

$$Z_n^k = \begin{bmatrix} 1 & & & \\ e_{k,k-1} & 1 & & \\ e_{k+1,k-1} & e_{k,k-1} & 1 & \\ & \cdots & & \\ e_{k+(n-2),k-1} & \cdots & e_{k,k-1} & 1 \end{bmatrix} = \left[\begin{array}{c|c} Z_{n-1}^k & 0 \\ \hline e_{k+(n-2),k-1}, \cdots, e_{k,k-1} & 1 \end{array}\right] \quad (4)$$

Then

$$Z_{n+1}^k(I_{n+1} + J_{n+1}^2 X_{n+1}^{(k)} A_{n+1}^{(k)}) = Z_{n+1}^k \left[\begin{array}{c|c} I_2 \ : \ 0 & 0 \\ \hline X_{n-1}^{(k)} A_{n-1}^{(k)} + J_{n-1}^2{}^T & J_{(n-1)\times 2}^{n-3} \end{array}\right]$$

$$= \left[\begin{array}{c|c} Z_n^k & 0 \\ \hline Z' & 1 \end{array}\right] \left[\begin{array}{c|c} \begin{array}{cc} I_2 & : 0 \\ X_{n-2}^{(k)} A_{n-2}^{(k)} + J_{n-2}^2{}^T & : J_{(n-2)\times 2}^{n-4} \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline X' & 1 \end{array}\right] \quad (5)$$

where bottom rows Z' and X' are from (4) and Theorem 7 that

$$Z' = (e_{k+(n-1),k-1}, e_{k+(n-2),k-1}, \cdots, e_{k+2,k-1}, e_{k+1,k-1}, e_{k,k-1}),$$

$$X' = (k^{n-k}\mu, k^{n-k-1}\mu, \cdots, k\mu, \mu, \sum_{j=1}^{k-2} k^{j-1} v_{k,j}, \sum_{j=2}^{k-2} k^{j-2} v_{k,j},$$

$$\cdots, \sum_{j=k-3}^{k-2} k^{j-(k-3)} v_{k,j},\ v_{k,k-2},\ 0). \quad (6)$$

Therefore the block matrix multiplications in (5) yield

$$Z_{n+1}^k(I_{n+1} + J_{n+1}^2 X_{n+1}^{(k)} A_{n+1}^{(k)}) = \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array}\right]$$

with $H = 0$, $S = 1$ and $G = Z_n^k \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X_{n-2}^{(k)} A_{n-2}^{(k)} + J_{n-2}^2{}^T & J_{(n-2)\times 2}^{n-4} \end{array}\right] = A_n^{(k)}$ by induction hypothesis. Now we claim $R = (k^n, k^{n-1}, \cdots, k)$. If so, we get

$$Z_{n+1}^k(I_{n+1} + J_{n+1}^2 X_{n+1}^{(k)} A_{n+1}^{(k)}) = \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array}\right] = \left[\begin{array}{c|c} A_n^{(k)} & 0 \\ \hline k^n, k^{n-1}, \cdots, k & 1 \end{array}\right] = A_{n+1}^{(k)},$$

as required. Now write $R = Z' \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X_{n-2}^{(k)} A_{n-2}^{(k)} + J_{n-2}^2{}^T & J_{(n-2)\times 2}^{n-4} \end{array}\right] + X'$ by

$$R = (\theta_1, \theta_2, \cdots, \theta_n) + X' = (\chi_1, \chi_2, \cdots, \chi_n), \quad (7)$$

with some $\theta_i, \chi_i \in \mathbb{Z}$. Due to Theorem 7, the multiplication

$$(\theta_1,\cdots,\theta_n)=Z'\begin{bmatrix}1 &&&&&&\\ 0 & 1 &&&&&\\ v_{k,k-2} & 0 & 1 &&&&\\ v_{k,k-3}+kv_{k,k-2} & v_{k,k-2} & 0 & 1 &&&\\ \cdots &&&&&&\\ \sum_{j=1}^{k-2}k^{j-1}v_{k,j} & \sum_{j=2}^{k-2}k^jv_{k,j}\cdots &&&&&\\ \mu & \sum_{j=1}^{k-2}k^jv_{k,j} & \sum_{j=2}^{k-2}k^jv_{k,j}\cdots 0 & 1 &&&\\ k\mu & \mu & \sum_{j=1}^{k-2}k^jv_{k,j}\cdots && 0 & 1 &\\ k^2\mu & k\mu & \mu & \cdots & 0 & 1 &\\ \cdots &&&&&&\\ k^{n-k-1}\mu & k^{n-k-2}\mu & \cdots\mu & \cdots v_{k,k-2} & 0 & & 1\end{bmatrix}$$

with Z' in (6) satisfies

$$\theta_1=e_{k+(n-1),k-1}+e_{k+(n-3),k-1}v_{k,k-2}+e_{k+(n-4),k-1}(v_{k,k-3}+kv_{k,k-2})$$
$$+e_{k+(n-5),k-1}\sum_{j=k-4}^{k-2}k^{j-(k-4)}v_{k,j}+\cdots+e_{n,k-1}\sum_{j=1}^{k-2}k^{j-1}v_{k,j}$$
$$+e_{n-1,k-1}\mu+e_{n-2,k-1}k\mu+\cdots+e_{k+1,k-1}k^{n-k-2}\mu+e_{k,k-1}k^{n-k-1}\mu,$$
$$\theta_2=e_{k+(n-2),k-1}+e_{k+(n-4),k-1}v_{k,k-2}+e_{k+(n-5),k-1}(v_{k,k-3}+kv_{k,k-2})$$
$$+e_{k+(n-6),k-1}\sum_{j=k-4}^{k-2}k^{j-(k-4)}v_{k,j}+\cdots+e_{n-1,k-1}\sum_{j=1}^{k-2}k^{j-1}v_{k,j}$$
$$+e_{n-2,k-1}\mu+e_{n-3,k-1}k\mu+\cdots+e_{k,k-1}k^{n-k-2}\mu,\quad\cdots,$$
$$\theta_{n-2}=e_{k+2,k-1}+e_{k,k-1}v_{k,k-2},\ \ \theta_{n-1}=e_{k+1,k-1},\text{ and }\theta_n=e_{k,k-1}.$$

Then $(\chi_1,\cdots,\chi_n)=R=(\theta_1,\cdots,\theta_n)+X'$ in (6) and (7) satisfies

$$\chi_n=\theta_n+0=e_{k,k-1}=k,$$
$$\chi_{n-1}=\theta_{n-1}+v_{k,k-2}=e_{k+1,k-1}+e_{k,k-2}=\binom{k+1}{k-1}+\binom{k}{k-2}=k^2,$$

and $\chi_{n-2}=(e_{k+2,k-1}+e_{k,k-1}v_{k,k-2})+(v_{k,k-3}+kv_{k,k-2})=k^3$.

Assume $\chi_2=k^{n-1}$. Then the proof is complete if we show $\chi_1=k^n$.

$$\chi_1=\theta_1+k^{n-k}\mu$$
$$=(e_{k+(n-1),k-1}+e_{k+(n-3),k-1}v_{k,k-2}+e_{k+(n-4),k-1}(v_{k,k-3}+kv_{k,k-2})$$
$$+\cdots+e_{n-2,k-1}k\mu+e_{n-3,k-1}k^2\mu+\cdots+e_{k,k-1}k^{n-k-1}\mu)+k^{n-k}\mu$$
$$=A+k(e_{n-2,k-1}\mu+e_{n-3,k-1}k\mu+\cdots+e_{k,k-1}k^{n-k-2}\mu+k^{n-k-1}\mu),$$

where $A=e_{k+(n-1),k-1}+e_{k+(n-3),k-1}v_{k,k-2}+e_{k+(n-4),k-1}(v_{k,k-3}+kv_{k,k-2})+\cdots+e_{n,k-1}\sum_{j=1}^{k-2}k^{j-1}v_{k,j}+e_{n-1,k-1}\mu$.

On the other hand, the induction hypothesis shows

$$k^{n-1}=\chi_2=\theta_2+k^{n-k-1}\mu$$

$= B + (e_{n-2,k-1}\mu + e_{n-3,k-1}k\mu + \cdots + e_{k,k-1}k^{n-k-2}\mu + k^{n-k-1}\mu)$,
where $B = e_{k+(n-2),k-1} + e_{k+(n-4),k-1}v_{k,k-2} + e_{k+(n-5),k-1}(v_{k,k-3} + kv_{k,k-2}) + \cdots + e_{n,k-1}\sum_{j=2}^{k-2} k^{j-2}v_{k,j} + e_{n-1,k-1}\sum_{j=1}^{k-2} k^{j-1}v_{k,j}$.

Therefore

$$\chi_1 = A + k(e_{n-2,k-1}\mu + e_{n-3,k-1}k\mu + \cdots + e_{k,k-1}k^{n-k-2}\mu + k^{n-k-1}\mu)$$
$$= A + k(k^{n-1} - B) = k^n + (A - kB).$$

But we have

$$\begin{aligned}
A - kB &= \Big(e_{k+(n-1),k-1} + e_{k+(n-3),k-1}v_{k,k-2} + e_{k+(n-4),k-1}(v_{k,k-3} + kv_{k,k-2}) \\
&\quad + \cdots + e_{n,k-1}\sum_{j=1}^{k-2} k^{j-1}v_{k,j} + e_{n-1,k-1}\mu\Big) \\
&\quad - k\Big(e_{k+(n-2),k-1} + e_{k+(n-4),k-1}v_{k,k-2} + e_{k+(n-5),k-1}(v_{k,k-3} + kv_{k,k-2}) \\
&\quad + \cdots + e_{n,k-1}\sum_{j=2}^{k-2} k^{j-2}v_{k,j} + e_{n-1,k-1}\sum_{j=1}^{k-2} k^{j-1}v_{k,j}\Big) \\
&= e_{k+(n-1),k-1} - e_{k+(n-2),k-1}k + e_{k+(n-3),k-1}v_{k,k-2} \\
&\quad + e_{k+(n-4),k-1}((v_{k,k-3} + kv_{k,k-2}) - kv_{k,k-2}) + \cdots \\
&\quad + e_{n,k-1}(\sum_{j=1}^{k-2} k^{j-1}v_{k,j} - \sum_{j=2}^{k-2} k^{j-1}v_{k,j}) + e_{n-1,k-1}(\mu - \sum_{j=1}^{k-2} k^{j}v_{k,j}) \\
&= e_{k+(n-1),k-1} - e_{k+(n-2),k-1}k + e_{k+(n-3),k-1}v_{k,k-2} \\
&\quad + e_{k+(n-4),k-1}v_{k,k-3} + \cdots + e_{n,k-1}v_{k,1} + e_{n-1,k-1}v_{k,0} \\
&= \sum_{i=0}^{k} v_{k,k-i}e_{k+(n-i-1),k-1}.
\end{aligned}$$

It means $A - kB$ is a multiplication of kth row in P^{-1} and $(k-1)$th column in P, so $A - kB = 0$ and we have $\chi_1 = k^n$, as required. □

Therefore it follows that $Z^{-k} - {A^{(k)}}^{-1} = J^2X^{(k)} = \begin{bmatrix} 0 \\ 0 \\ \hline X^{(k)} \end{bmatrix}$.

References

[1] H. Cohn, *Projective geometry over F_1 and the gaussian binomial coefficients*, Amer. Math. Monthly, **111** (2004), 487 - 495.

[2] Hans Cuypers, Discrete Mathematics, https://www.coursehero.com/file/14299872/notes/ (2007), 1-74

[3] A. Edelman and G. Strang, *Pascal matrices*, Amer. Math. Monthely, **111** (2004), 361 - 385.

[4] M.E. Horn, *Pascal Pyramids, Pascal Hyper-Pyramids and a Bilateral Multinomial Theorem*, arXiv:math/0311035v1 (math.GM), **4** (2003).

[5] M.E. Horn, *Pauli Pascal Pyramids, Pauli Fibonacci Numbers, and Pauli Jacobsthal Numbers*, arXiv:0711.4030v1 (math.GM), **26** (2007).

[6] J. Jo, Y. Oh, and E. Choi, *Arithmetic matrix of quadratic polynomial with negative exponent by Pascal matrix*, J. of Algebra and Applied Mathematics, **17** (2019), 67-90.

[7] T.H. Koornwinder, *Special functions and q-commuting variables*, in: Special Functions, q-Series and Related Topics, M.E.H. Ismail, D.R. Masson, M. Rahman (eds.), Fields Institute Communications **14**, AMS (1997), 131 -166.

*
Department of Mathematics
Hannam University
Daejon, Republic of Korea
E-mail: emc@hnu.kr