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FUZZY ε-SUBALGEBRAS (IDEALS) IN BCI-ALGEBRAS

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Abstract. Based on a sub-BCK-algebra K of a BCI-algebra X, the notions of fuzzy (K,ε)-subalgebras, fuzzy (K,ε)-ideals and fuzzy commutative (K,ε)-ideals are introduced, and their relations/properties are investigated. Conditions for a fuzzy subalgebra/ideal to be a fuzzy (K,ε)-subalgebra/ideal are provided.

1. Introduction

It is well-known that certain information processing, especially inferences based on certain information, is based on classical two-valued logic. Due to strict and complete logical foundation (classical logic), making inference levels. Thus, it is natural and necessary to attempt to establish some rational logic system as the logical foundation for uncertain information processing. It is evident that this kind of logic cannot be two-valued logic itself but might form a certain extension of two-valued logic. Various kinds of non-classical logic systems have therefore been extensively researched in order to construct natural and efficient inference systems to deal with uncertainty. Logic appears in a 'sacred' form (resp., a 'profane') which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold-as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic and fuzzy logic takes the advantage of the classical logic to handle information with various facets of uncertainty [1], such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Fuzziness and incomparability are two kinds of uncertainties often associated with

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human's intelligent activities in the real world, and they exist not only in the processed object itself, but also in the course of the object being dealt with. After the introduction of fuzzy sets by Zadeh [2], there have been a number of generalizations of this fundamental concept. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki (see [3, 4]) and have been extensively investigated by many researchers. In this paper, we deal with a new type of fuzzy subalgebras/ideals in BCI-algebras. Based on a sub-BCK-algebra K of a BCI-algebra X, we introduce the notions of fuzzy (K,ε)-subalgebras, fuzzy (K,ε)-ideals and fuzzy commutative (K,ε)-ideals, and then we investigate their relations/properties. We provide conditions for a fuzzy subalgebra/ideal to be a fuzzy (K,ε)-subalgebra/ideal.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X;*,0)$ of type $(2,0)$ satisfying the axioms:

(a1) $(\forall x,y,z\in X)\,(((x*y)*(x*z))*(z*y)=0)$,
(a2) $(\forall x,y\in X)\,((x*(x*y))*y=0)$,
(a3) $(\forall x\in X)\,(x*x=0)$,
(a4) $(\forall x,y\in X)\,(x*y=y*x=0\;\Rightarrow\;x=y)$.

We can define a partial ordering $\leq$ by $x\leq y$ if and only if $x*y=0$. If a BCI-algebra X satisfies $0*x=0$ for all $x\in X$, then we say that X is a *BCK-algebra*.

In a BCK/BCI-algebra X, the following hold:

(b1) $(\forall x\in X)\,(x*0=x)$,
(b2) $(\forall x,y,z\in X)\,((x*y)*z=(x*z)*y)$,

A BCK-algebra X is said to be *commutative* if $x\vec{\wedge}y=y\vec{\wedge}x$ for all $x,y\in X$ where $x\vec{\wedge}y=x*(x*y)$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y\in S$ for all $x,y\in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(b3) $0\in A$,
(b4) $(\forall x,y\in X)\,\big(x*y\in A,\;y\in A\;\Rightarrow\;x\in A\big)$.

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [5]) if it satisfies (b3) and

(b5) $(\forall x,y,z\in X)\,\big((x*y)*z\in A,\;z\in A\;\Rightarrow\;x*(y\vec{\wedge}x)\in A\big)$.

TABLE 1. $*$-operation

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	b	a
a	a	a	a	0	0
b	b	b	a	2	0

We refer the reader to the books [5] and [6] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if it satisfies

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y)\}). \tag{2.1}$$

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy ideal* of X if it satisfies:

(b6) $(\forall x \in X)$ $(\mu(0) \geq \mu(x))$,
(b7) $(\forall x, y \in X)$ $\big(\mu(x) \geq \min\{\mu(x * y), \mu(y)\}\big)$.

PROPOSITION 2.1. *Let μ be a fuzzy set in a BCK/BCI-algebra X. Then μ is a fuzzy subalgebra (resp. fuzzy ideal) of X if and only if the set*

$$\mu_t := \{x \in X \mid \mu(x) \geq t\}$$

is a subalgebra (resp ideal) of X for all $t \in (0, 1]$. For our convenience, the empty set $\emptyset$ is regarded as a subalgebra (resp. ideal) of X.

3. Fuzzy ε-subalgebras

DEFINITION 3.1. Let $(X; *, 0)$ be a BCI-algebra. By a *sub-BCK-algebra* of X we mean a subset K of X such that $0 \in K$ and $(K; *, 0)$ is a BCK-algebra.

EXAMPLE 3.2. *Let $X = \{0, 1, 2, a, b\}$ be a set with the $*$-operation given by Table 1. Then $(X; *, 0)$ is a BCI-algebra and $(K = \{0, 1, 2\}; *, 0)$ is a sub-BCK-algebra of X.*

DEFINITION 3.3. Let K be a sub-BCK-algebra of a BCI-algebra X and let $\varepsilon \in [0, 1]$. A fuzzy set μ in X is called a *fuzzy ε-subalgebra* of

TABLE 2. $*$-operation

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

X based on K (briefly, *fuzzy* (K,ε)*-subalgebra* of X) if it is a fuzzy subalgebra of X that satisfies the following condition:

(c1) $(\forall x \in K)\ (\forall y \in X \setminus K)\ \big(\mu(x) \geq \varepsilon \geq \mu(y)\big)$.

EXAMPLE 3.4. *Let X and K be as in Example 3.2.*

(1) A fuzzy set μ in X given by

$$\mu = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.7 & 0.6 & 0.5 & 0.3 & 0.3 \end{pmatrix}.$$

is a fuzzy (K,ε)-subalgebra of X for $\varepsilon \in [0.3, 0.5]$.

(2) Let ν be a fuzzy set in X given by

$$\nu = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.7 & 0.5 & 0.2 & 0.4 & 0.2 \end{pmatrix}.$$

Then ν is a fuzzy subalgebra of X, but it does not satisfy (c1) since $\nu(2) = 0.2 < 0.4 = \nu(a)$.

The following example shows that there exists a fuzzy set μ in a BCI-algebra X such that it satisfies the condition (c1), but it is not a fuzzy subalgebra of X.

EXAMPLE 3.5. *Let $X = \{0, 1, a, b, c\}$ be a set with the $*$-operation given by Table 2. Then $(X; *, 0)$ is a BCI-algebra and $(K = \{0, 1\}; *, 0)$ is only a sub-BCK-algebra of X. Let μ be a fuzzy set in X given by*

$$\mu = \begin{pmatrix} 0 & 1 & a & b & c \\ 0.5 & 0.6 & 0.2 & 0.4 & 0.3 \end{pmatrix}.$$

Then μ satisfies the condition (c1) for $\varepsilon \in [0.4, 0.5]$, but it is not a fuzzy subalgebra of X since $\mu(b * c) = \mu(a) = 0.2 < 0.3 = \min\{\mu(b), \mu(c)\}$.

THEOREM 3.6. *Let K be a sub-BCK-algebra of a BCI-algebra X. If a fuzzy subalgebra μ of X satisfies the following condition:*

$$(\forall x \in K)\ (\forall y \in X \setminus K)\ \big(\mu(x) \geq \mu(y)\big), \tag{3.1}$$

then μ is a fuzzy (K,ε)-subalgebra of X for all $\varepsilon \in \big[\sup_{y\in X\setminus K} \mu(y), \inf_{x\in K} \mu(x)\big]$.

Proof. Straightforward. □

Obviously, a restriction of a fuzzy (K,ε)-subalgebra of a BCI-algebra X to a sub-BCK-algebra K of X is a fuzzy subalgebra of $(K;*,0)$.

Theorem 3.7. *Let $\varepsilon \in [0,1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. Then every fuzzy (K,ε)-subalgebra μ of X satisfies the following assertions:*

(c2) $K \subseteq \mu_\varepsilon$.

(c3) $(\forall t \in (0,1])$ $(t > \varepsilon \;\Rightarrow\; \mu_t$ *is a subalgebra of* $K)$.

Proof. Assume that μ is a fuzzy (K,ε)-subalgebra of X. Obviously, $K \subseteq \mu_\varepsilon$. Let $t \in (0,1]$ be such that $t > \varepsilon$. Then $\mu_t \subseteq K$. Let $x, y \in \mu_t$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus $\mu(x*y) \geq \min\{\mu(x), \mu(y)\} \geq t$, and so $x*y \in \mu_t$. Therefore μ_t is a subalgebra of K. □

We give conditions for a fuzzy subalgebra to be a fuzzy (K,ε)-subalgebra.

Theorem 3.8. *Let $\varepsilon \in [0,1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. If μ is a fuzzy subalgebra of X satisfying two conditions* (c2) *and* (c3), *then μ is a fuzzy (K,ε)-subalgebra of X.*

Proof. Let $x \in K$ and $y \in X \setminus K$. Then $x \in \mu_\varepsilon$ by (c2), and so $\mu(x) \geq \varepsilon$. Let $\mu(y) = t$. If $t > \varepsilon$, then $y \in \mu_t \subseteq K$ by (c3). This is a contradiction, and thus $\mu(x) \geq \varepsilon \geq t = \mu(y)$. Consequently, μ is a fuzzy (K,ε)-subalgebra of X. □

4. Fuzzy (commutative) ε-ideals

Definition 4.1. Let $\varepsilon \in [0,1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. A fuzzy set μ in X is called a *fuzzy ε-ideal* of X based on K (briefly, *fuzzy (K,ε)-ideal* of X) if it satisfies:

(c4) $(\forall x \in K)$ $(\forall y \in X \setminus K)$ $\big(\mu(0) \geq \mu(x) \geq \varepsilon \geq \mu(y)\big)$.

(c5) $(\forall x, y \in K)$ $\big(\mu(x) \geq \min\{\mu(x*y), \mu(y)\}\big)$.

Example 4.2. *Let $X = \{0,1,2,a,b\}$ be a set with the $*$-operation given by Table 3. Then $(X;*,0)$ is a BCI-algebra and $(K = \{0,1,2\};*,0)$ is a sub-BCK-algebra of X. Let μ be a fuzzy set in X given by*

$$\mu = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.8 & 0.5 & 0.7 & 0.1 & 0.2 \end{pmatrix}.$$

TABLE 3. $*$-operation

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Then μ is a fuzzy (K, ε)-ideal of X. But it is not a fuzzy ideal of X since $\mu(a) = 0.1 \ngeq 0.2 = \min\{\mu(a * b), \mu(b)\}$.

THEOREM 4.3. *Let $\varepsilon \in [0, 1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. If μ is a fuzzy (K, ε)-ideal of X, then*

(c6) $K \subseteq \mu_\varepsilon$.
(c7) $(\forall t \in (0, 1])$ $(t > \varepsilon \ \Rightarrow \ \mu_t$ *is an ideal of* $K)$.

Proof. Let $x \in K$. Then $\mu(x) \geq \varepsilon$ by (c4), and so $x \in \mu_\varepsilon$. Hence $K \subseteq \mu_\varepsilon$. Let $t \in (0, 1]$ be such that $t > \varepsilon$. If $x \in \mu_t$, then $\mu(x) \geq t > \varepsilon$ and thus $x \in K$. Hence $\mu_t \subseteq K$. From (c4), we know that $\mu(0) \geq \mu(x)$ for all $x \in X$. Hence $\mu(0) \geq \mu(x) \geq t$ for $x \in \mu_t$, and so $0 \in \mu_t$. Let $x, y \in K$ be such that $x * y \in \mu_t$ and $y \in \mu_t$. Then $\mu(x * y) \geq t$ and $\mu(y) \geq t$. It follows from (c5) that

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq t$$

so that $x \in \mu_t$. Therefore μ_t is an ideal of K. $\square$

For a sub-BCK-algebra of a BCI-algebra X and $\varepsilon \in [0, 1]$, the following example shows that a fuzzy ideal μ of X may not be a fuzzy (K, ε)-ideal of X.

EXAMPLE 4.4. *Let X and K be as in Example 4.2. Define a fuzzy set μ in X by*

$$\mu = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.8 & 0.3 & 0.7 & 0.5 & 0.3 \end{pmatrix}.$$

Then

$$\mu_t = \begin{cases} \emptyset & \textit{if } t \in (0.8, 1], \\ \{0\} & \textit{if } t \in (0.7, 0.8], \\ \{0, 2\} & \textit{if } t \in (0.5, 0.7], \\ \{0, 2, a\} & \textit{if } t \in (0.3, 0.5], \\ X & \textit{if } t \in (0, 0.3], \end{cases}$$

and so μ_t is an ideal of X for all $t \in (0,1]$ with $\mu_t \neq \emptyset$. Hence μ is a fuzzy ideal of X by Proposition 2.1. But μ is not a fuzzy (K,ε)-ideal of X for $\varepsilon \in (0.3, 0.5)$ because if $\varepsilon \in (0.3, 0.5)$ then $\mu(1) = 0.3 < \varepsilon < 0.5 = \mu(a)$.

We provide conditions for a fuzzy ideal to be a fuzzy (K,ε)-ideal.

THEOREM 4.5. *Let $\varepsilon \in [0,1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. If a fuzzy ideal μ of X satisfies conditions* (c6) *and* (c7)*, then μ is a fuzzy (K,ε)-ideal of X.*

Proof. Let $x \in K$ and $y \in X \setminus K$. Then $x \in \mu_\varepsilon$ by (c6), which implies $\mu(x) \geq \varepsilon$. If $\mu(y) > \varepsilon$, then $y \in \mu_{\mu(y)} \subseteq K$ by (c7). This is a contradiction, and so $\mu(y) \leq \varepsilon$. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, it follows that $\mu(0) \geq \mu(x) \geq \varepsilon \geq \mu(y)$ so that (c4) is valid. Since μ is a fuzzy ideal of X, the condition (c5) is obvious. Therefore μ is a fuzzy (K,ε)-ideal of X. □

The following example shows that a fuzzy (K,ε)-subalgebra may not be a fuzzy (K,ε)-ideal, and vice versa.

EXAMPLE 4.6. *(1) Let X and K be as in Example 3.2. Let ν be a fuzzy set in X given by*

$$\nu = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.8 & 0.5 & 0.7 & 0.2 & 0.2 \end{pmatrix}.$$

Then ν is a fuzzy (K,ε)-subalgebra of X for $\varepsilon \in [0.2, 0.5]$. But μ is not a fuzzy (K,ε)-ideal of X for $\varepsilon \in [0.2, 0.5]$ since

$$\mu(1) = 0.5 \ngeq 0.7 = \min\{\mu(1 * 2), \mu(2)\}.$$

(2) Let X and K be as in Example 3.2. Define a fuzzy set μ in X by

$$\mu = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.7 & 0.6 & 0.5 & 0.4 & 0.2 \end{pmatrix}.$$

Then μ is a fuzzy (K,ε)-ideal of X for $\varepsilon \in [0.4, 0.5]$. Since

$$\mu(2 * a) = \mu(b) = 0.2 \ngeq 0.4 = \min\{\mu(2), \mu(a)\},$$

μ is not a fuzzy subalgebra of X. Therefore μ is not a fuzzy (K,ε)-subalgebra of X for $\varepsilon \in [0.4, 0.5]$.

DEFINITION 4.7. Let $\varepsilon \in [0,1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. A fuzzy set μ in X is called a *fuzzy commutative ε-ideal* of X based on K (briefly, *fuzzy commutative (K,ε)-ideal* of X) if it satisfies (c4) and

(c8) $(\forall x, y, z \in K)\ \big(\mu(x * (y \barwedge x)) \geq \min\{\mu((x * y) * z), \mu(z)\}\big)$.

TABLE 4. BCK-operation

$*_X$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

EXAMPLE 4.8. *Let $(X; *_X, 0)$ be the BCK-algebra whose elements are $X = \{0, a, b, c\}$ and whose BCK-operation is given by the Cayley table (see Table 4). For any group G with identity e, let $Y = (G \setminus \{e\}) \cup X$. We define the operation $*$ on Y by the following way:*

(1) *for $x, y \in G \setminus \{e\}$, we put*

$$x * y = \begin{cases} xy \in G \setminus \{e\} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

(2) *for $x, y \in X$, we put $x * y = x *_X y$ in X,*

(3) *for $x \in G \setminus \{e\}$ and $y \in X$, we put $x * y = y * x = x$.*

Then $(Y; *, 0)$ is a BCI-algebra (see [7]) and $(X; *, 0)$ is a sub-BCK-algebra of Y. Let μ be a fuzzy set in Y defined by $\mu(0) = 0.9$, $\mu(a) = \mu(b) = 0.6$, $\mu(c) = 0.8$ and $\mu(x) = 0.4$ for all $x \in G \setminus \{e\}$. Then μ is a fuzzy commutative (K, ε)-ideal of Y for $\varepsilon \in [0.4, 0.6]$ where $K = X$.

THEOREM 4.9. *Let $\varepsilon \in [0, 1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. Then every fuzzy commutative (K, ε)-ideal is a fuzzy (K, ε)-ideal.*

Proof. Let μ be a fuzzy commutative (K, ε)-ideal of X. Taking $y = 0$ in (c8) induces (c5), and so μ is a fuzzy (K, ε)-ideal of X. $\square$

The following example shows that the converse of Theorem 4.9 may not be true.

EXAMPLE 4.10. *Consider the BCK-algebra $K = \{0, a, b, c\}$ with the operation $*_K$ which is given by the Table 5. For any group G with identity e, let $Y = (G \setminus \{e\}) \cup X$ and define the operation $*$ on Y by the similar way to Example 4.8. Then $(Y; *, 0)$ is a BCI-algebra and $(K; *, 0)$ is a sub-BCK-algebra of Y. Let μ be a fuzzy set in Y defined by $\mu(0) = 0.8$, $\mu(a) = 0.5$, $\mu(b) = 0.4$, $\mu(c) = 0.6$ and $\mu(x) = 0.3$ for all $x \in G \setminus \{e\}$. Then μ is a fuzzy (K, ε)-ideal of Y for $\varepsilon \in [0.3, 0.4]$, but μ is not a fuzzy commutative (K, ε)-ideal of Y since $\mu(a * (b \vec{\barwedge} a)) = 0.5 \ngeq 0.6 = \min\{\mu((a * b) * c), \mu(c)\}$.*

TABLE 5. BCK-operation

$*_K$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

PROPOSITION 4.11. *Let $\varepsilon \in [0,1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. Then every fuzzy commutative (K,ε)-ideal μ of X satisfies the following inequality:*

$$(\forall x, y \in K)\ \big(\mu(x * (y \vec{\wedge} x)) \geq \mu(x * y)\big).$$

Proof. Let μ be a fuzzy commutative (K,ε)-ideal of X. Taking $z = 0$ in (c8) and using (c4) and (b1) induces the desired result. □

LEMMA 4.12. [5] *An ideal A of a BCK-algebra X is commutative if and only if it satisfies:*

$$(\forall x, y \in X)\ (x * y \in A \ \Rightarrow\ x * (y \vec{\wedge} x) \in A).$$

THEOREM 4.13. *Let $\varepsilon \in [0,1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. If μ is a fuzzy commutative (K,ε)-ideal of X, then $K \subseteq \mu_\varepsilon$ and μ_t is a commutative ideal of K for all $t \in (0,1]$ with $t > \varepsilon$.*

Proof. If μ is a fuzzy commutative (K,ε)-ideal of X, then μ is a fuzzy (K,ε)-ideal of X by Theorem 4.9. It follows from Theorem 4.3 that $K \subseteq \mu_\varepsilon$ and μ_t is an ideal of K for all $t \in (0,1]$ with $t > \varepsilon$. Let $x, y \in K$ be such that $x * y \in \mu_t$. Then $\mu(x * y) \geq t$, and so $\mu(x * (y \vec{\wedge} x)) \geq \mu(x * y) \geq t$ by Proposition 4.11. Hence $x * (y \vec{\wedge} x) \in \mu_t$. It follows from Lemma 4.12 that μ_t is a commutative ideal of K for all $t \in (0,1]$ with $t > \varepsilon$. □

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