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SYMMETRIES ON TRANS-SASAKIAN SPACE FORMS

Sibsankar Panda*, Arindam Bhattacharya**, and Kalyan Halder***

Abstract. In this article we have studied different types of symmetricness of trans-Sasakian space forms.

1. Introduction

In 1985, J.A. Oubiña [8] introduced a new class of almost contact manifold namely trans-Sasakian manifold of type (α, β). This class contains α-Sasakian, β-Kenmotsu and cosymplectic manifolds. In particular when $\alpha = 1$ and $\beta = 0$ the manifolds are Sasakian manifolds which are analogues to Kähler manifolds. A Kähler manifold of constant holomorphic curvature is called a complex space form. Sasakian space forms are analogues to complex space forms. Many geometers [7, 1] studied the symmetric properties on Sasakian space forms. On the other hand a class of almost contact Riemannian manifolds abstracted by Kenmotsu [5] which are normal but not Sasakian are called Kenmotsu manifolds. β-Kenmotsu manifolds are the generalization of Kenmotsu manifolds. A Kenmotsu manifold with constant φ-holomorphic sectional curvature is called a Kenmotsu space form. In this article we first introduced the trans-Sasakian space form, and studied the several interesting symmetric properties as semi-symmetry, Ricci-semi-symmetry, pseudo-symmetry, Ricci-generalized-pseudo-symmetry, Weyl-projective-semi-symmetry and pseudo-projective-semi-symmetry on the trans-Sasakian space form.

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* The corresponding author.

2. Preliminaries

Let M be a $(2n+1)$ dimensional manifold and φ, ξ and η be a tensor field of type (1,1), a vector field, a 1 form on M respectively. If φ, ξ and η satisfy the conditions

$$\eta(\xi) = 1 \quad \text{and} \quad \varphi^2 X = -X + \eta(X)\xi \tag{2.1}$$

for any vector field X on M, then M is said to have an **almost contact structure** (φ, ξ, η) and is called an **almost contact manifold**.

Using equation (2.1), for an almost contact structure (φ, ξ, η) one can prove the following properties:

(i) $\varphi(\xi) = 0$,
(ii) $\eta o \varphi = 0$,
(iii) $\text{rank}\varphi = 2n$.

Every almost contact manifold M admits a Riemannian metric tensor field g such that

$$\eta(X) = g(X, \xi), \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

$$g(\varphi X, Y) = -g(X, \varphi Y). \tag{2.4}$$

The metric tensor field g called an **associated Riemannian metric tensor field** to the given almost contact structure (φ, ξ, η). If M admits the structure (φ, ξ, η, g), g being an associated Riemannian metric tensor field of an almost contact structure (φ, ξ, η) then M is said to have an **almost contact metric structure** (φ, ξ, η, g) and is called an **almost contact metric manifold**.

For an $(2n+1)$ dimensional almost contact manifold M with almost contact structure (φ, ξ, η), we consider a product manifold $M \times \mathbb{R}$, where $\mathbb{R}$ denotes a real line. Then a vector field on $M \times \mathbb{R}$ is given by $(X, f(d/dt))$, where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. We define a linear map J on the tangent space of $M \times \mathbb{R}$ by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}). \tag{2.5}$$

Then we have $J^2 = -I$ and hence J is an almost complex structure on $M \times \mathbb{R}$. The almost complex structure J is said to be integrable if its Nijenhuis torsion N vanishes, where

$$N(X, Y) = J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY].$$

If the almost complex structure J on $M \times \mathbb{R}$ is integrable, we say that the almost contact structure (φ, ξ, η) is **normal**. A normal almost contact metric manifold is called **Sasakian manifold** [2]. The sectional curvature of the plane section spanned by the unit tangent vector field X orthogonal to ξ and φX is called a φ-sectional curvature. If M has a constant φ-sectional curvature c, then M is called a **Sasakian space forms**.

Let (M, g) be an n dimensional Riemannian manifold $n > 2$, its curvature tensor defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Let T be $(0,k)$ tensor, define a $(0, 2+k)$ tensor field $R \cdot T$ by

$$\begin{aligned}(R{\cdot}T)(X_1, X_2, ...X_k, X, Y) &= R(X,Y)(T(X_1, X_2, ...X_k)) \\ &= -T(R(X,Y)X_1, X_2, ...X_k) \\ &\quad -T(X_1, R(X,Y)X_2, ...X_k) - ... \\ &\quad -T(X_1, X_2, ..., R(X,Y)X_k)\end{aligned}$$

one has

$$R(X,Y) \cdot T = \nabla_X(\nabla_Y T) - \nabla_Y(\nabla_X T) - \nabla_{[X,Y]}T.$$

When $T = R$, then we have a $(0,6)$ tensor $R \cdot R$.

The manifold (M, g) is called **semi-symmetric space** if

$$R \cdot R = 0.$$

and called **Ricci semi-symmetric space** if

$$R \cdot S = 0,$$

where S is the Ricci curvature tensor.

Also, we can determine a $(0, k+2)$ tensor field $Q(A, T)$, associated with any $(0,k)$ tensor field T and any symmetric $(0,2)$ tensor field A by

$$\begin{aligned}Q(A,T)(X_1, X_2, ...X_k, X, Y) &= ((X \wedge_A Y) \cdot T)(X_1, X_2, ...X_k)) \\ &= -T((X \wedge_A Y)X_1, X_2, ...X_k) \\ &\quad -T(X_1, (X \wedge_A Y)X_2, ...X_k) - ... \\ &\quad -T(X_1, X_2, ..., (X \wedge_A Y)X_k)\end{aligned} \tag{2.6}$$

where $(X \wedge_A Y)$ is the endomorphism given by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{2.7}$$

Particulary, if we put $A = g$ we get

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{2.8}$$

From now we will write $(X \wedge_g Y)$ as $(X \wedge Y)$.

The **Weyl-projective curvature tensor** P on M is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2}[g(Y,Z)QX - g(X,Z)QY], \tag{2.9}$$

where Q is the Ricci operator Q defined by $S(X,Y) = g(QX,Y)$, S being the Ricci curvature tensor. Another form of Weyl-projective curvature tensor is given by

$$\begin{aligned} P(X,Y)Z &= R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - g(X,Z)Y] \\ &= R(X,Y)Z - \frac{1}{2n}(X \wedge_S Y)Z. \end{aligned} \tag{2.10}$$

The manifold (M,g) is called **projectively-semi-symmetric space** if

$$R \cdot P = 0$$

and where P is the Weyl-projective-curvature tensor.

3. Trans Sasakian Manifold and Space form

Let (M,φ,ξ,η,g) be an almost contact metric manifold. If there are smooth functions α,β on M satisfying

$$(\nabla\varphi)(X,Y) = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\varphi X,Y)\xi - \eta(Y)\varphi X]$$

for all $X,Y \in \mathfrak{X}(M)$. Then the structure $(\varphi,\xi,\eta,g,\alpha,\beta)$ is said to be a **trans-Sasakian structure** and the manifold $(M,\varphi,\xi,\eta,g,\alpha,\beta)$ is said to be a **trans-Sasakian manifold** of type (α,β). Trans-Sasakian manifolds of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are called cosymplectic, α-Sasakian, and β-Kenmotsu manifolds respectively. Sasakian manifolds appear as examples of α-Sasakian manifolds, with $\alpha = 1$ and $\beta = 0$ and Kenmotsu manifolds appear when $\alpha = 0$ and $\beta = 1$. Marrero [6] has shown that a trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic manifold, or α-Sasakian manifold, or β-Kenmotsu manifold.

A trans-Sasakian manifold M^{2n+1} of constant φ-sectional curvature c is called a **trans-Sasakian space form** denoted by $M^{2n+1}(c)$ and its curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= \frac{\alpha(c+3)+\beta(c-3)}{4}[g(Y,Z)X - g(X,Z)Y] \\ &+ \frac{\alpha(c-1)+\beta(c+1)}{4}\{[\eta(X)Y - \eta(Y)X]\eta(Z) + [g(X,Z)\eta(Y) \end{aligned} \tag{3.1}$$

$$-g(Y,Z)\eta(X)]\xi + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y + 2g(X,\varphi Y)\varphi Z\}.$$

It can also be written by (2.2) and (2.8), as

$$R(X,Y)Z = (\alpha-\beta)(X\wedge Y)Z + \frac{\alpha(c-1)+\beta(c+1)}{4}\{(\varphi^2X\wedge\varphi^2Y)Z + (\phi X\wedge\varphi Y)Z + 2g(X,\varphi Y)\varphi Z\}. \tag{3.2}$$

The Ricci tensor on trans-Sasakian space form defined by

$$S(X,Y) = \frac{1}{2}[c(n+1)(\alpha+\beta)+(3n-1)(\alpha-\beta)]\,g(X,Y) - \frac{n+1}{2}[c(\alpha+\beta)-(\alpha-\beta)]\,\eta(X)\eta(Y). \tag{3.3}$$

It can also be written by (2.2), as

$$S(X,Y) = 2ng(X,Y) + \frac{n+1}{2}[c(\alpha+\beta)-(\alpha-\beta)]\,g(\varphi X,\varphi Y). \tag{3.4}$$

LEMMA 3.1. *Let $M^{2n+1}(c)$ be a trans-Sasakian space form and $X,Y\in\mathfrak{X}(M)$, then the following properties hold :*

(a) $\varphi\cdot S = 0$.
(b) $(X\wedge Y)\cdot S = 0$ *iff* $c(\alpha+\beta) = \alpha-\beta$.
(c) $(\varphi X\wedge\varphi Y)\cdot S = 0$.
(d) $(\varphi^2X\wedge\varphi^2Y)\cdot S = 0$.

Proof. (a) Since φ is a tensor field, we have

$$(\varphi\cdot S)(U,V) = -S(\varphi U,V) - S(U,\varphi V)$$
$$= -\frac{1}{2}[c(n+1)(\alpha+\beta)+(3n-1)(\alpha-\beta)]\{g(\varphi U,V)+g(U,\varphi V)\}$$

[Using the property (ii)]

$$= -\frac{1}{2}[c(n+1)(\alpha+\beta)+(3n-1)(\alpha-\beta)]\{g(\varphi U,V)-g(\varphi U,V)\} = 0$$

[by (2.4)]

Thus $(\varphi\cdot S)(U,V) = 0$ for any $U,V\in\mathfrak{X}(M)$.

(b) For any $U,V\in\mathfrak{X}(M)$, we have

$$((X\wedge Y).S)(U,V) = -S((X\wedge Y)U,V) - S(U,(X\wedge Y)V)$$
$$= -g(Y,U)S(X,V) + g(X,U)S(Y,V) - g(Y,V)S(U,X) + g(X,V)S(U,Y) \quad \text{[by (2.8)]}$$
$$= -\frac{n+1}{2}[c(\alpha+\beta)-(\alpha-\beta)]\{-g(Y,U)\eta(X)\eta(V) + g(X,U)\eta(Y)\eta(V) - g(Y,V)\eta(U)\eta(X) + g(X,V)\eta(U)\eta(Y)\} \quad \text{[by (3.3)]}$$

Since, $\{-g(Y,U)\eta(X)\eta(V)+g(X,U)\eta(Y)\eta(V)-g(Y,V)\eta(U)\eta(X) + g(X,V)\eta(U)\eta(Y)\} \neq 0$ always and α, β are nonzero functions, therefore

$$((X \wedge Y).S)(U,V) = 0 \text{ iff } c(\alpha+\beta) - (\alpha-\beta) = 0.$$

(c) For any $U, V \in \mathfrak{X}(M)$, we have

$$\begin{aligned} ((\varphi X\wedge\varphi Y).S)(U,V) &= -S((\varphi X\wedge\varphi Y)U,V) - S(U,(\varphi X\wedge\varphi Y)V) \\ &= -g(\varphi Y,U)S(\varphi X,V) + g(\varphi X,U)S(\varphi Y,V) \\ &\quad -g(\varphi Y,V)S(U,\varphi X) + g(\varphi X,V)S(U,\varphi Y) \\ &= \frac{1}{2}[c(n+1)(\alpha+\beta) + (3n-1)(\alpha-\beta)]\{-g(\varphi^2 Y,U)g(\varphi^2 X,V) \\ &\quad +g(\varphi^2 X,U)g(\varphi^2 Y,V) - g(\varphi^2 Y,V)g(U,\varphi^2 X) \\ &\quad +g(\varphi^2 X,V)g(U,\varphi^2 Y)\} \ [\text{using (2.8) and property (ii)}] \\ &= 0. \end{aligned}$$

(d) Proof is similar to (c).

□

THEOREM 3.2. *A trans-Sasakian space form $M^{2n+1}(c)$ of type (α,β) is Ricci-semi-symmetric if and only if $c(\alpha+\beta) = \alpha-\beta$.*

Proof. The curvature tensor is of the form

$$R(X,Y) = (\alpha-\beta)(X\wedge Y) + \frac{\alpha(c-1)+\beta(c+1)}{4}\{(\varphi^2 X\wedge\varphi^2 Y) + (\varphi X\wedge\varphi Y) + 2g(X,\varphi Y)\varphi\}$$

So,

$$R(X,Y)\cdot S = (\alpha-\beta)(X\wedge Y)\cdot S + \frac{\alpha(c-1)+\beta(c+1)}{4}\{(\varphi^2 X\wedge\varphi^2 Y)\cdot S + (\varphi X\wedge\varphi Y)\cdot S + 2g(X,\varphi Y)\varphi\cdot S\}$$

By the lemma-3.1, we have

$$R.S = 0 \text{ if and only if } c(\alpha+\beta) - (\alpha-\beta) = 0.$$

□

LEMMA 3.3. *Let $M^{2n+1}(c)$ be a trans-Sasakian space form of type (α,β) and $X, Y \in \mathfrak{X}(M)$, then the following properties hold :*

(a) $\varphi\cdot R = 0$.
(b) $(\varphi X\wedge\varphi Y)\cdot R = -(X\wedge Y)\cdot R$
(c) $(X\wedge_S Y).R = 2n(X\wedge Y)\cdot R$

Proof. (a) For any $X, Y, U, V \in \mathfrak{X}(M)$

$$\begin{aligned}(\varphi\cdot R)(X,Y,U,V) &= -R(\varphi X,Y,U,V)-R(X,\varphi Y,U,V)\\ &\quad -R(X,Y,\varphi U,V)-R(X,Y,U,\varphi V)\\ &= -g(R(\varphi X,Y)U,V)-g(R(X,\varphi Y)U,V)\\ &\quad -g(R(X,Y)\varphi U,V)-g(R(X,Y)U,\varphi V)\end{aligned}$$

Using property (ii) and after a long and straightforward computation we get

$$\begin{aligned}&(\varphi\cdot R)(X,Y,U,V)=\\ &-\frac{\alpha(c-1)+\beta(c+1)}{4}[-g(\varphi Y,U)g(\varphi X,\varphi V)+g(\varphi X,\varphi U)g(\varphi Y,V)\\ &-g(\varphi Y,\varphi U)g(\varphi X,V)+g(\varphi X,U)g(\varphi Y,\varphi V)+g(\varphi Y,\varphi U)g(\varphi X,V)\\ &-g(\varphi X,\varphi U)g(\varphi Y,V)+g(\varphi Y,U)g(\varphi X,\varphi V)-g(\varphi X,U)g(\varphi Y,\varphi V)]=0\end{aligned}$$

(b) For any $X, Y, Z, U, V, W \in \mathfrak{X}(M)$,

$$\begin{aligned}((\varphi X\wedge\varphi Y)\cdot R)(Z,U,V,W) &= -R((\varphi X\wedge\varphi Y)Z,U,V,W)\\ &\quad -R(Z,(\varphi X\wedge\varphi Y)U,V,W)\\ &\quad -R(Z,U,(\varphi X\wedge\varphi Y)V,W)\\ &\quad -R(Z,U,V,(\varphi X\wedge\varphi Y)W)\\ &= -g(R[(\varphi X\wedge\varphi Y)Z,U]V,W)\\ &\quad -g(R[Z,(\varphi X\wedge\varphi Y)U]V,W)\\ &\quad -g(R[Z,U](\varphi X\wedge\varphi Y)V,W)\\ &\quad -g(R[Z,U]V,(\varphi X\wedge\varphi Y)W)\end{aligned}$$

Using property (ii) and (2.8) and after a long and straightforward computation we get the result.

(c) The Ricci curvature tensor can be written as

$$S(X,Y)=2ng(X,Y)+\frac{n+1}{2}\left[c(\alpha+\beta)-(\alpha-\beta)\right]g(\varphi X,\varphi Y).$$

So, we have

$$S(Y,Z)X=2ng(Y,Z)X+\frac{1}{2}[(n+1)(c-1)]g(\varphi Y,\varphi Z)X$$

and

$$S(X,Z)Y=2ng(X,Z)Y+\frac{n+1}{2}\left[c(\alpha+\beta)-(\alpha-\beta)\right]g(\varphi X,\varphi Z)Y.$$

Thus,

$$(X\wedge_S Y)Z=S(Y,Z)X-S(X,Z)Y=2n\{g(Y,Z)X-g(X,Z)Y\}$$

$$+\frac{n+1}{2}\left[c(\alpha+\beta)-(\alpha-\beta)\right]\{g(\varphi Y,\varphi Z)X-g(\varphi X,\varphi Z)Y\}.$$

$$= 2n(X\wedge Y)Z+\frac{n+1}{2}\left[c(\alpha+\beta)-(\alpha-\beta)\right]\{g(\varphi Y,\varphi Z)X-g(\varphi X,\varphi Z)Y\}.$$

If we put $Z = R$(the Riemann curvature tensor), then

$$(X\wedge_S Y)\cdot R = 2n(X\wedge Y)\cdot R+\frac{n+1}{2}\left[c(\alpha+\beta)-(\alpha-\beta)\right]\{g(\varphi Y,\varphi\cdot R)X$$
$$-g(\varphi X,\varphi\cdot R)Y\}.$$

From lemma-3.3(a) $\varphi\cdot R = 0$, therefore

$$(X\wedge_S Y)\cdot R = 2n(X\wedge Y)\cdot R.$$

□

DEFINITION 3.4. A Riemannian manifold (M,g), $\dim M \geq 3$, is said to be **pseudo-symmetric** (in the sense of R. Deszcz) if the $(0,6)$ tensor field $R\cdot R$ and $Q(g,R)$ on M are linearly dependent, i.e., if there exists a function $\mathcal{L}_R : M \to \mathbb{R}$ such that

$$R\cdot R = \mathcal{L}_R Q(g,R)$$

holds on $\mathcal{U}_R = \{x \in M | R-(\tau/n(n-1))G \neq 0\}$, where τ is the scalar curvature of M and G is the $(0,4)$ tensor field of M defined by

$$G(X_1,X_2,X_3,X_4) = g((X_1\wedge X_2)X_3,X_4)$$

see [3].

THEOREM 3.5. *Every trans-Sasakian space forms $M^{2n+1}(c)$ is pseudo-symmetric, more precisely for every trans-Sasakian space forms:*

$$R(X,Y)\cdot R = (\alpha-\beta)Q(g,R) = (\alpha-\beta)(X\wedge Y)\cdot R.$$

Proof. The curvature tensor is of the form

$$R(X,Y) = (\alpha-\beta)(X\wedge Y)+\frac{\alpha(c-1)+\beta(c+1)}{4}\{(\varphi^2X\wedge\varphi^2Y)$$
$$+(\varphi X\wedge\varphi Y)+2g(X,\varphi Y)\varphi\}$$

So,

$$R(X,Y)\cdot R = (\alpha-\beta)(X\wedge Y)\cdot R+\frac{\alpha(c-1)+\beta(c+1)}{4}\{(\varphi^2X\wedge\varphi^2Y)\cdot S$$
$$+(\varphi X\wedge\varphi Y)\cdot R+2g(X,\varphi Y)\varphi\cdot R\}$$

By the lemma-3.3 (a) and (b), we have

$$R(X,Y)\cdot R = (\alpha-\beta)(X\wedge Y)\cdot R = (\alpha-\beta)Q(g,R).$$

□

Corollary 3.6. A trans-Sasakian space forms $M^{2n+1}(c)$ can not be semi-symmetric.

Definition 3.7. In a Riemannian manifold (M,g), $\dim M \geq 3$, if the $(0,6)$ tensor field $R \cdot R$ and $Q(S,R)$ are linearly dependent, then the manifold is called **Ricci-generalized-pseudo-symmetric** [4]. That is equivalent to

$$R \cdot R = \mathcal{L}_S Q(S,R)$$

holding on $\mathcal{U}_S = \{x \in M \mid Q(S,R) \neq 0\}$, where $\mathcal{L}_S$ is a function on $\mathcal{U}_S$.

Theorem 3.8. *A trans-Sasakian space forms $M^{2n+1}(c)$ of type (α,β) is Ricci-generalized-pseudo-symmetric.*

Proof. By lemma-3.3 (c) and the result of Theorem 3.5

$$R(X,Y) \cdot R = (\alpha - \beta)(X \wedge Y) \cdot R = \frac{\alpha - \beta}{2n}(X \wedge_S Y) \cdot R = \mathcal{L}_S Q(S,R)$$

where $\mathcal{L}_S = \frac{\alpha-\beta}{2n}$ is a function on $M \supseteq \mathcal{U}_S$. Hence the result. □

Lemma 3.9. *In a trans-Sasakian space form $M^{2n+1}(c)$ the following are hold:*

(a) $\varphi \cdot P = 0$,
(b) $(X \wedge Y) \cdot P = (X \wedge Y) \cdot R$

Proof. (a)

$$\begin{aligned}(\varphi \cdot P)(X,Y,U,V) &= -P(\varphi X,Y,U,V) - P(X,\varphi Y,U,V)\\ &\quad -P(X,Y,\varphi U,V) - P(X,Y,U,\varphi V)\\ &= -g(P(\varphi X,Y)U,V) - g(P(X,\varphi Y)U,V)\\ &\quad -g(P(X,Y)\varphi U,V) - g(P(X,Y)U,\varphi V)\\ &= \varphi \cdot R \quad [\text{by } (2.4), (3.1) \text{ and } (3.3)]\\ &= 0. \quad [\text{by lemma-3.3}(a)]\end{aligned}$$

(b)

$$\begin{aligned}((X\wedge Y)\cdot P)(Z,U,V,W) &= -P((X\wedge Y)Z,U,V,W) - P(Z,(X\wedge Y)U,V,W)\\ &\quad -P(Z,U,(X\wedge Y)V,W) - P(Z,U,V,(X\wedge Y)W)\\ &\qquad [\text{by } (2.8) \text{ and } (2.10)]\\ &= -R((X\wedge Y)Z,U,V,W) - R(Z,(X\wedge Y)U,V,W)\\ &\quad -R(Z,U,(X\wedge Y)V,W) - R(Z,U,V,(X\wedge Y)W)\\ &= ((X\wedge Y)\cdot R)(Z,U,V,W) \quad [\text{by } (2.6)]\end{aligned}$$

□

THEOREM 3.10. *A trans-Sasakian space forms $M^{2n+1}(c)$ of type (α,β) is not projectively semi-symmetric.*

Proof. The curvature tensor of the form (3.2) is

$$R(X,Y) = (\alpha-\beta)(X\wedge Y) + \frac{\alpha(c-1)+\beta(c+1)}{4}\{(\varphi^2X\wedge\varphi^2Y) + (\varphi X\wedge\varphi Y) + 2g(X,\varphi Y)\varphi\}.$$

So,

$$\begin{aligned} R(X,Y)\cdot P &= (\alpha-\beta)(X\wedge Y)\cdot P + \frac{\alpha(c-1)+\beta(c+1)}{4}\{(\varphi^2X\wedge\varphi^2Y)\cdot P \\ &\quad + (\varphi X\wedge\varphi Y)\cdot P + 2g(X,\varphi Y)\varphi\cdot P\} \\ &= (\alpha-\beta)(X\wedge Y)\cdot R + \frac{\alpha(c-1)+\beta(c+1)}{4}\{(\varphi^2X\wedge\varphi^2Y)\cdot R \\ &\quad + (\varphi X\wedge\varphi Y)\cdot R\} \quad \text{[by lemma-3.9]} \\ &= (\alpha-\beta)(X\wedge Y)\cdot R + \frac{\alpha(c-1)+\beta(c+1)}{4}\{-(\varphi X\wedge\varphi Y)\cdot R \\ &\quad + (\varphi X\wedge\varphi Y)\cdot R\} \quad \text{[by lemma-3.3}(b)] \\ &= (\alpha-\beta)(X\wedge Y)\cdot R \neq 0 \qquad [\because \alpha\neq\beta]. \end{aligned}$$

□

DEFINITION 3.11. The **Pseudo projective curvature tensor** $\overline{P}$ on a Riemannian manifold (M^{2n+1},g) is defined as:

$$\overline{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{\tau}{2n+1}\left[\frac{a}{2n}+b\right][g(Y,Z)X - g(X,Z)Y]. \tag{3.5}$$

By (2.8), we can write

$$\overline{P}(X,Y)Z = aR(X,Y)Z + b(X\wedge_S Y)Z - \frac{\tau}{2n+1}\left[\frac{a}{2n}+b\right](X\wedge Y)Z \tag{3.6}$$

where a and b are non-zero constants and τ is the scalar curvature.

If $a=1$ and $b=-\frac{1}{2n}$, then (3.5) and (3.6) take the form

$$\overline{P}(X,Y)Z = P(X,Y)Z$$

where P is Projective curvature tensor. A Riemannian manifold is pseudo-projectively semi-symmetric if

$$R\cdot\overline{P} = 0.$$

LEMMA 3.12. (a) $\varphi.\overline{P} = 0$

(b) $(X\wedge Y)\cdot\overline{P} = a(X\wedge Y)\cdot R$

Proof is similar to lemma-3.9.

THEOREM 3.13. *A trans-Sasakian space forms $M^{2n+1}(c)$ of (α, β) is not pseudo-projectively semi-symmetric.*

Proof. The curvature tensor of the form (3.2) is

$$R(X,Y) = (\alpha - \beta)(X \wedge Y) + \frac{\alpha(c-1) + \beta(c+1)}{4}\{(\varphi^2 X \wedge \varphi^2 Y) + (\varphi X \wedge \varphi Y) + 2g(X, \varphi Y)\varphi\}.$$

Now,

$$\begin{aligned} R(X,Y) \cdot \overline{P} &= (\alpha - \beta)(X \wedge Y) \cdot \overline{P} + \frac{\alpha(c-1) + \beta(c+1)}{4}\{(\varphi^2 X \wedge \varphi^2 Y) \cdot \overline{P} \\ &\quad + (\varphi X \wedge \varphi Y) \cdot \overline{P} + 2g(X, \varphi Y)\varphi \cdot \overline{P}\} \\ &= a(\alpha - \beta)(X \wedge Y) \cdot R + \frac{\alpha(c-1) + \beta(c+1)}{4}\{a(\varphi^2 X \wedge \varphi^2 Y) \cdot R \\ &\quad + a(\varphi X \wedge \varphi Y) \cdot R\} \quad [\text{by lemma-3.12}] \\ &= a(\alpha - \beta)(X \wedge Y) \cdot R + \frac{\alpha(c-1) + \beta(c+1)}{4}\{-a(\varphi X \wedge \varphi Y) \cdot R \\ &\quad + a(\varphi X \wedge \varphi Y) \cdot R\} \quad [\text{by lemma-3.3}(b)] \\ &= a(\alpha - \beta)(X \wedge Y) \cdot R \neq 0 \qquad [\because \alpha \neq \beta \ \text{ and } \ a \neq 0]. \end{aligned}$$

□

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Department of Mathematics
Sidho-Kanho-Birsha University
E-mail: shibu.panda@gmail.com

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Department of Mathematics
Jadavpur University
E-mail: bhattachar1968@yahoo.co.in

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Department of Mathematics
Sidho-Kanho-Birsha University
E-mail: halderk@rediffmail.com