JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **33**, No. 3, August 2020
http://dx.doi.org/10.14403/jcms.2020.33.3.365

INFINITESIMAL HOLONOMY ISOMETRIES AND THE CONTINUITY OF HOLONOMY DISPLACEMENTS

Taechang Byun*

Abstract. Given a noncompact semisimple Lie group G and its maximal compact Lie subgroup K such that the right multiplication of each element in K gives an isometry on G, consider a principal bundle $G \to G/K$, which is a Riemannian submersion. We study the infinitesimal holonomy isometries. Given a closed curve at eK in the base space G/K, consider the holonomy displacement of e by the horizontal lifting of the curve. We prove that the correspondence is continuous.

1. introduction

When the structure group of a given principal bundle is an isometry group of its total space, the bundle can be viewed as a Riemannian submersion. Holonomy displacements play a key role for us to understand the structure of the bundle.

If the dimension of a fiber is zero, then the horizontal lifts of two homotopic simple closed curves give the same holonomy displacement. But, if the dimension is greater than or equal to one, it may not. Thus, we can ask if the convergency of curves in a base space preserve that of holonomy displacements, i.e., the continuity of holonomy displacements.

Given a noncompact semisimple Lie group G and its maximal compact Lie subgroup K, assume that the right multiplication of each element in K gives an isometry on G. For example, let $G = \mathrm{SO}_0(1, n)$ and

Received April 29, 2020; Accepted August 05, 2020.

2010 Mathematics Subject Classification: 52A38, 53C05, 53C12, 53C29, 53C35, 57R05.

Key words and phrases: Holonomy displacement; Riemannian submersion; Principal bundle; Iwasawa decomposition.

*The author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2018R1D1A1A02047995).

$K = \mathrm{SO}(n)$. Then we can consider a Riemannian submersion

$$K \longrightarrow G \longrightarrow G/K.$$

We will view holonomy isometries infinitesimally and show the continuity of holonomy displacements in this bundle.

This paper is a partial result of the thesis[1].

2. Infinitesimal holonomy isometries

We use the notation in [6] for general mathematical concepts.

Recall the definition of a Riemannian submersion and basic facts:

Definition 2.1 (Definition 1.2.1; [3]). *Let $\pi : M \to B$ be a submersion, where M is a Riemannian manifold. The* horizontal distribution *of π is the orthogonal complement $\mathcal{H} = \mathcal{V}^{\perp}$ of $\mathcal{V}$. If in addition B is a Riemannian manifold, then the submersion is said to be* Riemannian *if it is isometric when restritcted to the horizontal distribution; i.e. if $|\pi_* x| = |x|$ for all $x \in \mathcal{H}$.*

Theorem 2.2 (Theorem 1.3.1; [3]). *Let $\pi : M \to B$ denote a Riemannian submersion. If $c : I \to M$ is a geodesic with $\dot{c}(t_0) \in \mathcal{H}$ for some $t_0 \in I$, then $\dot{c}(t) \in \mathcal{H}$ for all $t \in I$, and $\pi \circ c$ is a geodesic in B. Such a c will be called a* horizontal geodesic *of M. Furthermore, if M is complete, then*

1. *B is complete;*
2. *π is a submetry; i.e., π maps the closure of the metric ball $B_r(p) = \{q \in M | d(p,q) < r\}$ of radius r around p onto the closure of $B_r(\pi(p))$ for any $p \in M$;*
3. *the fibers of π are equidistant; i.e., for any two fibers F_0 and F_1, and $p \in F_0$, the distance between p and F_1 equals that between F_0 and F_1;*
4. *π is a locally trivial fiber bundle; i.e., any point b in B has a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times F$, where $F = \pi^{-1}(U)$.*

Like a path lifting in an algebraic topology, we can consider a horizontal lifting induced from a curve in the base space, which gives a diffeomorphism between two fibers over the endpoints of the curve in the base space:

Definition 2.3 (Definition 1.3.1; [3]). *Let $\pi : M \to B$ denote a Riemannian submersion, and $c : [0,1] \to B$ a piecewise smooth curve in* *the base space. The* holonomy displacement *associated to c is the map $h_c : \pi^{-1}(c(0)) \to \pi^{-1}(c(1))$ between two fibers over the endpoints of c that maps a point p in the first fiber to the endpoint of the horizontal lift $\tilde{c}$ of c that starts at p, i.e.,*

$$\pi \circ \tilde{c} = c, \quad \dot{\tilde{c}}(t) \in \mathcal{H}, \forall t \in (0,1), \quad \tilde{c}(0) = p \quad \text{and} \quad h_c(p) = \tilde{c}(1).$$

We use the symbol $\mathbf{h}$ and $\mathbf{v}$ as the horizontal part and the vertical part of a given vector, respectively, i.e., $e = e^{\mathbf{h}} + e^{\mathbf{v}} \in \mathcal{H} \oplus \mathcal{V}$ for any vector e. And let $\mathfrak{X}$ denote a Lie algebra of vector fields.

Before considering the infinitesimal version of a holonomy displacement, recall the local version of a Riemannian submersion and two tensors related to it:

Definition 2.4 (Definition 1.2.2; [3]). *A foliation on a Riemannnian manifold is said to be* metric *if $\mathcal{L}_U g^{\mathbf{h}}$ is horizontally zero or any $U \in \mathfrak{X}^{\mathbf{v}}$, or equivalently, if $\nabla^{\mathbf{v}}_X X = 0$ for all $X \in \mathfrak{X}^{\mathbf{v}}$.*

Definition 2.5 (Definition 1.4.1; [3]). *The A-tensor of a metric foliation on M is the tensor field $A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ on M given by*

$$A_X Y = \nabla^{\mathbf{v}}_X Y = \frac{1}{2}[X,Y]^{\mathbf{v}}, \quad X, Y \in \mathfrak{X}^{\mathbf{h}}.$$

Definition 2.6 (Definition 1.4.2; [3]). *The S-tensor of a metric foliation on M is the tensor field $S : \mathcal{H} \times \mathcal{V} \to \mathcal{V}$ on M given by*

$$S_X U = -\nabla^{\mathbf{v}}_U X, \quad X \in \mathfrak{X}^{\mathbf{h}},\ U \in \mathfrak{X}^{\mathbf{v}}.$$

Definition 2.7 (Definition 1.4.3; [3]). *A Jocobi field J along a horizontal geodesic $c : [0,a] \to M$ that is vertical at 0 and satisfies $J'(0) = -A^*_{\dot{c}(0)} J(0) - S_{\dot{c}(0)} J(0)$ is called* a holonomy field.

Notice that for $t \in [0,a]$, a holonomy field J satisfies

$$J'(t) = -A^*_{\dot{c}(t)} J(t) - S_{\dot{c}(t)} J(t), \quad t \in [0,a].$$

It is known that a holonomy field can be constructed as follows:

Lemma 2.8 (Lemma 1.4.2; [3]). *Let $\pi : M \to B$ denote a Riemannian submersion, and $h : F_0 \to F_1$ the holonomy diffeomorphism induced by the geodesic $c : [0,1] \to B$, where $c(0) = \pi(F_0)$, $c(1) = \pi(F_1)$. Given $p \in F_0$, let c_p denote the horizontal lift of c starting at p. Then, for $u \in T_p(F_0)$,*

$$h_* u = J(1),$$

where J is the holonomy field along c_p with $J(0) = u$.

For the case of Lie groups, let G be a Lie group, K a closed subgroup of G, and consider a left-invariant metric on G that is right-invariant under K. In a principal bundle $\pi : G \longrightarrow G/K$, the following facts are already known:

1. For $X \in \mathfrak{k}^{\perp}$, $t \mapsto g\cdot \exp\,(tX) : (-\infty, \infty) \to G$ is a horizontal geodesic for any $g \in G$ [Theorem 1.3.1; [3]].
2. Each fiber $gK, g \in G$, is totally geodesic. More precisely, for $U \in \mathfrak{k}$, $t \mapsto g\cdot \exp(tU) : (-\infty, \infty) \to gK \subset G$ is a vertical geodesic for any $g \in G$. Especially, if $g \in K$, then its image lies on $K = eK$. Furthermore, for any piecewise smooth curve $c : [a, b] \to G/K$, its induced holonomy $h_c : \pi^{-1}(c(a)) \to \pi^{-1}(c(b))$ is an isometry [Theorem 2.4.1; Lemma 1.4.3; [3]].
3. For any $k \in K$, the right translation $R_k : G \to G$ by $k, R_k(g) = gk$, is an isometry. Or, equivalently, $Ad_k : \mathfrak{g} \to \mathfrak{g}$ is a linear isometry for any $k \in K$ [Proposition 2.4.1; [3]].

Consider the following Proposition, which is the explanation of the holonomy isometry h_c in terms of vector fields.

Proposition 2.9. *Let G be a Lie group with a left invariant metric and K be its closed subgroup such that the right multiplication by each element of K gives an isometry on G. Consider a principal bundle $\pi : G \longrightarrow G/K$, which is also a Riemannian submersion. Then, for any $U \in \mathfrak{k}$ and for any horizontal geodesic $\tilde{c} : [a, b] \to G$, $U \circ \tilde{c}$ is a holonomy field along $\tilde{c}$.*

Proof. Consider a vertical geodesic $\gamma : (-\epsilon, \epsilon) \to \tilde{c}(a)K$ given by

$$\gamma(s) = \tilde{c}(a) \cdot \exp(sU),$$

and a variation $V(t, s) : [a, b] \times (-\epsilon, \epsilon) \to G$ defined by

$$V(t, s) = \tilde{c}(t) \cdot \exp(sU).$$

Then, for inclusions maps $i_s : [a, b] \to [a, b] \times (-\epsilon, \epsilon)$ and $j_t : (-\epsilon, \epsilon) \to [a, b] \times (-\epsilon, \epsilon)$ with $i_s(t) = (t, s) = j_t(s)$,

$$V \circ j_0 = \gamma, \quad V \circ i_0 = \tilde{c}$$

and

$$V \circ i_s \text{ is a horizontal geodesic with } \pi \circ V \circ i_s = \pi \circ \tilde{c}$$

since

$$\exp(sU) \in K, \qquad V \circ i_s(t) = V(t, s) = R_{\exp(sU)}\big(\tilde{c}(t)\big)$$

and

$$\text{the right multiplication } R_{\exp(sU)} \text{ is an isometry for each } s \in (-\epsilon, \epsilon).$$

So from

$$V(t,s) = \tilde{c}(t)\cdot \exp(s_0 U)\cdot \exp\big((s-s_0)U\big) = V(t,s_0)\cdot \exp\big((s-s_0)U\big),$$

we get

$$V_* D_2 \circ i_{s_0}(t) = L_{V(t,s_0)_*} U_e = U_{V(t,s_0)} = U \circ (V \circ i_{s_0})(t)$$

and that U is a holonomy field along $V \circ i_{s_0}$ from Lemma 2.8. Especially, $U \circ \tilde{c}$ is a holonomy field along a horizontal geodesic $\tilde{c} = V \circ i_0$. □

3. The Iwasawa decomposition

In this section, we recall the basic properties of a symmetric space of the noncompact type [Section 3.7; Section 9.2; [5]]: let $\mathfrak{g}$ be a noncompact semisimple Lie algebra over $\mathbb{R}$ and $B : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ be its Killing form. If an involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$, $\theta^2 = Id_{\mathfrak{g}}$ (and $\theta \neq Id_{\mathfrak{g}}$), induces a strictly postive definite bilinear form $B_\theta : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ given by $B_\theta(X,Y) := -B(X,\theta Y)$, then it is called a *Cartan involution.* Consider a Lie subalgebra $\mathfrak{k}$ which is the fixed point set of θ and a vector subspace $\mathfrak{p}(\subset \mathfrak{g})$ consisting of all elements $X \in \mathfrak{g}$ saitsfying $\theta(X) = -X$. Then, $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, which is called a *Cartan decomposition* of $\mathfrak{g}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{m}$ denote the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then the simultaneous diagonalization of the $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$ induces the *(restricted) root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} g_\lambda, \quad \mathfrak{g}_0 = \mathfrak{a}\oplus\mathfrak{m},$$

where each λ is a nontrivial element in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \,|\, [H,X] = \lambda(H)X \text{ for } H \in \mathfrak{a}\} \neq \{0\}.$$

Let Σ^+ be the set of positive elements in Σ and $\mathfrak{n}$ the subalgebra

$$\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda,$$

which is a nilpotent subalgebra of $\mathfrak{g}$. We have the following one, called the *Iwasawa decomposition*:

THEOREM 3.1 (Chpater 9, Theorem 1.3; [5]). *Let G be any connected noncompact semisimple Lie group with Lie algebra $\mathfrak{g}$. Then,*

$$\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k},$$
$$G = NAK,$$

that is, the mappping $(n, a, k) \mapsto nak : N \times A \times K \to G$ is a diffeomorphism, where N, A and K are analytics subgroups of G with Lie algebra $\mathfrak{n}, \mathfrak{a}$ and $\mathfrak{k}$.

4. Example

This section is based on [2].

Let $\mathrm{O}(1, n) = \{A \in \mathrm{GL}(n{+}1; \mathbb{R}) \mid A^t S A = S\}$, where $S = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{I}_n \end{pmatrix}$.

Let $\mathrm{SO}_0(1, n)$ be the identity component of $\mathrm{O}(1, n)$, which is also the identity component of $\mathrm{SO}(1, n)$, and consider a subgroup of $\mathrm{SO}_0(1, n)$ consisting of all matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$, where $B \in \mathrm{SO}(n)$. Denote the embedded subgroup by $\mathrm{SO}(n)$ for the simplicity of the notation.

Note the Lie algebra $\mathfrak{o}(1, n)$ is given by

$$\mathfrak{o}(1, n) = \{X \in \mathfrak{gl}(n + 1; \mathbb{R}) | X^t S + SX = 0\}.$$

Define a left-invariant metric on $\mathrm{SO}_0(1, n)$ from an inner product $\langle \cdot\, , \cdot \rangle$ on the Lie algebra $\mathfrak{o}(1, n)$, given by

$$\langle A, B \rangle = \tfrac{1}{2}\ \mathrm{trace}(A^t B), \quad A, B \in \mathfrak{o}(1, n).$$

If ϕ is a Killing-Cartan form, then

$$\phi(X, Y) = 2(n - 1) \langle X, Y \rangle \qquad \text{for } X, Y \in \mathfrak{o}(n)^\perp \subset \mathfrak{o}(1, n)\,.$$

The right action of $\mathrm{SO}(n)$ becomes an isometry and $\mathrm{SO}_0(1, n)/\mathrm{SO}(n)$ becomes isometric to $\mathbb{H}^n$. Under this metric, we have a principal bundle structure

$$\mathrm{SO}(n) \longrightarrow \mathrm{SO}_0(1, n) \xrightarrow{\ \pi\ } \mathbb{H}^n,$$

where $\pi : \mathrm{SO}_0(1, n) \to \mathbb{H}^n$ is a Riemannian submersion.

Let $G = \mathrm{SO}_0(1, n)$, $K = \mathrm{SO}(n)$, and $\mathfrak{g}$ and $\mathfrak{k}$ be their Lie algebras, respectively.

Let

$$E_{ij} = \epsilon_{ij} e_{ij} + e_{ji}, \quad 1 \le i < j \le n + 1$$

for the matrix e_{ij} whose (i, j)-entry is 1 and 0 elsewhere and for ϵ_{ij} whose value is -1 if $j < n + 1$ and is 1 if $j = n + 1$. It is well-known([Section 4.2; [2]]) that the subgroup NA of $\mathrm{SO}_0(1, n)$ has the structure

$$N \cong \mathbb{R}^{n-1}, \quad A \cong \mathbb{R}^+$$

as Lie groups. The subgroup NA with the Riemannian metric induced form that of $\mathrm{SO}_0(1,n)$ has an orthonormal basis

$$\{\tfrac{1}{\sqrt{2}}N_1, \cdots \tfrac{1}{\sqrt{2}}N_{n-1}, A_1\}$$

at the identity while the quotient $\mathrm{SO}_0(1,n)/SO(n)$ is isometric to the Lie group NA with a new left-invariant metrc coming from the orthonormal basis

$$\{N_1, \cdots N_{n-1}, A_1\},$$

where

$$N_i = E_{i\,n} + E_{i\,n+1} \qquad \text{for } i = 1, \cdots, n-1$$

For $n = 2$, $N,\ A$ and K are

$$N = \begin{pmatrix} 1 & -t & t \\ t & \frac{1}{2}(2-t^2) & \frac{1}{2}t^2 \\ t & -\frac{1}{2}t^2 & \frac{1}{2}(2+t^2) \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix},$$

$$K = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider our Riemannian submersion

$$\mathrm{SO}(n) \longrightarrow \mathrm{SO}_0(1,n) \longrightarrow \mathrm{SO}_0(1,n)/\mathrm{SO}(n).$$

This bundle has a global cross section $s : \mathbb{H} \to NA \subset G = \mathrm{SO}_0(1,n)$, which comes from the Iwasawa decomposition NAK, where $K = \mathrm{SO}(n)$. That is, every element of G is uniquely written as nak, and the projection maps this to $naK \in \mathbb{H}$ [Chapter 0; Chapter 1; Section 4.1; [2]].

5. The continuity of holonomy displacements

Consider a noncompact semisimple Lie group G and its Iwasawa decomposition NAK. Then the bundle

$$K \longrightarrow G \longrightarrow G/K$$

has a global cross section $s : G/K \to NA \subset G$, which comes from the Iwasawa decomposition NAK. That is, every element of G is uniquely written as nak, and the projection maps this to $naK \in G/K$.

The cross section s provides us with a one-to-one correspondence between the space of all continuous piecewise C^k-curves in G/K and in

NA, with initial points $\bar{e}$ and e, by the correspondence of a curve h in G/K to another one $s \circ h$ in NA. By abusing notations, express $s \circ h$ by h. For a curve $h : [0,1] \to \mathbb{H}^n$, the unique horizontal lift $\tilde{h} : [0,1] \to G$ is given by $h(t) \cdot a(t) = \tilde{h}(t)$ for a unique curve $a(t)$ in K. Under the identification of the tangent space of G at the identity e with its Lie algebra $\mathfrak{g}$, such an $a(t)$ is obtained from the solution of the differential equation

$$\langle h^{-1}h' + a'a^{-1},\ V\rangle = 0 \tag{5.1}$$

for every $V \in \mathfrak{k}$, where both h' and a' are tangent vectors. Note that the first entry $h^{-1}h' + a'\,a^{-1}$ is an element of the Lie algebra $\mathfrak{g}$. The equation (5.1) can be obtained as follows. The curve $\tilde{h}(t)$ being horizontal implies that the following equalities should hold on the tangent space at $h(t)a(t)$:

$$\begin{aligned} 0 &= \langle (h(t)a(t))', (h(t)a(t))V\rangle \\ &= \langle (h(t)a(t))\left(a(t)^{-1}h(t)^{-1}h'(t)a(t) + a(t)^{-1}a'(t)\right), (h(t)a(t))V\rangle \end{aligned}$$

for every $V \in \mathfrak{k}$. Thus, we get

$$0 = \langle a(t)^{-1}h(t)^{-1}h'(t)a(t) + a(t)^{-1}a'(t), V\rangle, \qquad V \in \mathfrak{k},$$

Since this holds for all $V \in \mathfrak{k}$ and the multiplication by any element in K, especially $a(t)^{-1} \in K$, on the right-hand side is also an isometry, the conjugation by $a(t)$ produces the equivalence of the above equlaity to (5.1).

We examine the equality (5.1) more closely. It holds for every $V \in \mathfrak{k}$, so $h(t)^{-1}h'(t) + a'(t)a^{-1}(t)$ does not have any vertical component. That is, $-a'(t)a^{-1}(t)$ is the vertical component of $h(t)^{-1}h'(t)$ so that

$$h(t)^{-1}h'(t) = -a'(t)a^{-1}(t) + X_1 \in \mathfrak{k} \oplus \mathfrak{k}^{\perp}$$

is a vertical and horizontal splitting.

Let $g(t)$ be another path with a unique horizontal lift $\tilde{g}(t) = g(t)b(t)$, satisfying

$$0 = \langle g^{-1}g' + b'b^{-1}, V\rangle, \tag{5.2}$$

for every $V \in \mathfrak{k}$. Again, we have a splitting

$$g(t)^{-1}g'(t) = -b'(t)b^{-1}(t) + X_2 \in \mathfrak{k} \oplus \mathfrak{k}^{\perp}.$$

From $||h(t)^{-1}h'(t) - g(t)^{-1}g'(t)||^2 = ||a'(t)a^{-1}(t) - b'(t)b^{-1}(t)||^2 + ||X_1 - X_2||^2$, we get

$$||a'(t)a^{-1}(t) - b'(t)b^{-1}(t)|| \le ||h(t)^{-1}h'(t) - g(t)^{-1}g'(t)||. \tag{5.3}$$

These are norms on the Lie algebra $\mathfrak{g}$.

On the space of continuous piecewise C^k-curves ($k \geq 1$) in G with initial point e, we define a distance function by

$$\rho(h, g) = \int_0^1 ||h(t)^{-1} \cdot h'(t) - g(t)^{-1} \cdot g'(t)|| \, dt.$$

Note that $h(t)^{-1} \cdot h'(t) \in \mathfrak{g}$ and $||.||$ is the norm there. We argue that this is a metric. Suppose $\rho(h, g) = 0$. Then, by continuity (on each proper subinterval of $[0, 1]$ if needed), $h(t)^{-1} \cdot h'(t) = g(t)^{-1} \cdot g'(t)$ for every t. Now we apply the similar statement of the following Lemma to the C^1-curves piece by piece to conclude $h(t) = g(t)$ for all $t \in [0, 1]$ from the continuity of h and g and from translation by right multiplication if needed, see [[4], vol 1, p69]. In fact, for $\tilde{h}(t) := h(t_0)^{-1}h(t_0 + t)$, $t \in [0, t_1 - t_0]$, and for $s = t_0 + t \in [t_0, t_1]$, we get both $h(s) = h(t_0)\tilde{h}(s - t_0)$ and $\tilde{h}(t)^{-1}\tilde{h}'(t) = h(s)^{-1}h'(s)$ from $\tilde{h}'(t) = h(t_0)^{-1}h'(t_0 + t)$.

Lemma 5.1. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra identified with $T_e(G)$. Let Y_t, $0 \leq t \leq 1$, be a continuous curve in $T_e(G)$. Then there exists in G a unique curve a_t of class C^1 such that $a_0 = e$ and $\dot{a}_t a_t^{-1} = Y_t$ for $0 \leq t \leq 1$.*

Let h be a curve in G/K (or in NA, by abuse of notation). The unique curve $a : [0, 1] \to K$ such that $h(t) \cdot a(t)$ is the horizontal lift of $h(t)$ will be called w_h.

For two curves h and g, the inequality (5.3) shows that $\rho(w_h, w_g) \leq \rho(h, g)$. Let $\mathfrak{P}$ be the space of all continuous piecewise C^k-curves on NA with the initial point e. Then, we can get the following result:

Theorem 5.2. *The map $\mathfrak{P} \longrightarrow G$ sending h to $w_h(1)$ is continuous. More precisely, let $h : [0, 1] \to NA$ be a piecewise C^k-curve. For every $\epsilon > 0$, there exists $\delta > 0$ such that, if $g \in \mathfrak{P}$ and $\rho(h, g) < \delta$, then $d(e, w_h(1)^{-1} \cdot w_g(1)) = d(w_h(1), w_g(1)) < \epsilon$.*

Proof. For simplicity, we write $w_h(t)$, $w_g(t)$ by $a(t)$, $b(t)$, respectively. Note

$$0 = (bb^{-1})' = b'b^{-1} + b(b^{-1})' \tag{5.4}$$

Then, the differentiation of $a(a^{-1}b)b^{-1} = e$ and the equality (5.4) give

$$a(a^{-1}b)'b^{-1} = -a'(a^{-1}b)b^{-1} - a(a^{-1}b)(b^{-1})' = -a'a^{-1} + b'b^{-1}.$$

Thus,

$$||a'a^{-1} - b'b^{-1}|| = ||a(a^{-1}b)'b^{-1}||.$$

Observe that $(a^{-1}b)' \in T_{a^{-1}b}(K)$. The left translation L_a and the right translation $R_{b^{-1}}$ maps this vector to a tangent vector at $T_e(K)$. However, both these translations are isometries so that they preserve the norms. We have,

$$||a'a^{-1} - b'b^{-1}|| = ||a(a^{-1}b)'b^{-1}|| = ||(a^{-1}b)'||.$$

Consequently, if $\int_0^1 ||(a^{-1}b)'||\ dt = \int_0^1 ||a'a^{-1} - b'b^{-1}||\ dt$ is small, the arc-length of the path $a(t)^{-1}b(t)$ is small. Therefore, if $a(0)$ and $b(0)$ are close (or if $a(0) = b(0)$), then $a(1)$ and $b(1)$ are close. This finishes the proof from the inequality (5.3). □

References

[1] T. Byun, *Holonomy displacement of curves in bundle* $SO(n) \to SO_0(1,n) \to \mathbb{H}^n$, Ph.D Thesis, University of Oklahoma, Norman, Oklahoma, USA, 77 pages

[2] T. Byun, K. Jo, and K. Lee, *The geometry of* $SO(n)\backslash SO_0(n,1)$, Geom. Dedicata, **161** (2012), 377-398.

[3] D. Gromoll and G. Walschap, *Metric foliations and curvature*, Progress in Mathematics, 268. Birkhäuser, Verlag, Basel, 2009.

[4] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, volume 1, 2*, Jonh Wiley and Sons, Inc., New York, 1969.

[5] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, Inc., 1979.

[6] G. Walschap, *Metric structures in differential geometry*, Graduate Texts in Mathematics, 224, Springer-Verlag, New York, 2004.

*
Department of Mathematics and statistics
Sejong University
Seoul, 05006, Republic of Korea
E-mail: tcbyun@sejong.ac.kr