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GROUND STATES OF A COVARIANT SEMIGROUP C^*-ALGEBRA

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Abstract. Let $\mathrm{P} \rtimes \mathbb{N}^\times$ be a semidirect product of an additive semigroup $\mathrm{P} = \{0, 2, 3, \cdots\}$ by a multiplicative positive natural numbers semigroup $\mathbb{N}^\times$. We consider a covariant semigroup C^*-algebra $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ of the semigroup $\mathrm{P} \rtimes \mathbb{N}^\times$. We obtain the condition that a state on $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ can be a ground state of the natural C^*-dynamical system $(\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times), \mathbb{R}, \sigma)$.

1. Introduction

The C^*-dynamical system is a mathematical model of a quantumn system. In the basic idea behind modeling quantum systems using a C^*-algebra, the observables in the quantumn system correspond to self adjoint elements of a C^*-algebra $\mathcal{A}$ and the states of the quantum system correspond to states on $\mathcal{A}$. When the system is in a state ϕ, the expected value of an observable a is given as $\phi(a)$. The time evolution of a quantum system is explained by an action $\alpha : \mathbb{R} \to Aut(\mathcal{A})$, in the sense that if the system is in state ϕ at initial time t_o, then at time t_o $+\, t$ the system is in state $\phi \circ \alpha_t$.

The theory of KMS states gives a mathematical formalism for describing the state ϕ of the system when it is in equilibrium [3], [4]. The KMS condition was originated as a characterization of Gibbs equilibrium states in quantum statistical mechanics and the condition for finite systems thoroughly characterizes the Gibbs states. This coincidence between KMS and Gibbs states appears to persist in many models after the thermodynamic limit. Thus if a thermodynamic system is described by a C^*-dynamical system $(\mathcal{A}, \tau)$ and the Gibbs formalism is considered,

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then it is natural to interpret the set K_β of (τ,β)-KMS states as the set of equilibrium states at the inverse temperature β.

Recently there are very interesting results on the KMS states of C^*-dynamical systems of C^*-algebras generated by isometries [7], [8], [9], [12], [14], [16], and [17]. Sometimes the uniqueness property of C^*-algebras generated by isometries is a strong tool in computing KMS states on them. The uniqueness property of C^*-algebras generated by isometries is the generalization of Coburn's well known theorem, which asserted that the C^*-algebra generated by a non-unitary isometry on a separable infinite dimensional Hilbert space does not depend on the particular choice of the isometry. There are lots of significant results on it [5], [6], [7], [18], [19], and [20]. It is known that the Toeplitz-Cuntz algebra $\mathcal{TO}_n$ has KMS states at every inverse temperature $\beta \geq \log n$, but only the one with $\beta = \log n$ factors through a state of $\mathcal{O}_n$. Cuntz introduced a C^*-algebra $\mathcal{Q}_\mathbb{N}$ generated by an isometric representation of the semidirect product $\mathbb{N} \rtimes \mathbb{N}^\times$ of the additive semigroup $\mathbb{N}$ by the natural action of the multiplicative semigroup $\mathbb{N}^\times$ [7]. He proved that $\mathcal{Q}_\mathbb{N}$ is simple and there exists a unique KMS state at the inverse temperature 1. He also showed that $\mathcal{Q}_\mathbb{N}$ is closely related to other very interesting C^*-algebras, such as the Bunce-Deddens algebras and Hecke C^*-algebra of Bost and Connes [2]. In [16], [17] Laca and Raeburn investigated the structure of the universal semigroup C^*-algebra $C_c^*(\mathbb{N} \rtimes \mathbb{N}^\times)$ of covariant isometric representations on $\mathbb{N} \rtimes \mathbb{N}^\times$. They showed that the semigroup C^*-algebra $C_c^*(\mathbb{N} \rtimes \mathbb{N}^\times)$ have interesting properties in the virtue of [10], [11], and [15]. In particular they showed that KMS states for the natural dynamic of $C_c^*(\mathbb{N} \rtimes \mathbb{N}^\times)$ have phase transitions [17]. If the inverse temperature is ∞, we have a specific state called by a ground state. Ground states can be described as the zero temperature limits of KMS states.

We consider the semidirect product $\mathrm{P} \rtimes \mathbb{N}^\times$ of the additive semigroup $\mathrm{P} = \{0, 2, 3, \cdots\}$ by the multiplicative semigroup $\mathbb{N}^\times$. The semigroup $\mathrm{P} = \{0, 2, 3, \cdots\}$ is the generating subsemigroup of the integer group $\mathbb{Z}$. Even though $(\mathbb{Z}, \mathbb{N})$ is the typical model of a quasi-lattice ordered group, the order structure of $(\mathbb{Z}, \mathrm{P})$ with the positive cone P is not a quasi-lattice ordered group. The author showed that the reduced semigroup C^*-algebra $C^*_{red}(\mathrm{P})$ is isomorphic to the classical Toeplitz algebra $\mathcal{T}(\mathbb{N})$ by using Coburn's result [13]. We see that $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathrm{P} \rtimes \mathbb{N}^\times)$ is not a quasi-lattice ordered group but we can define a covariant isometric representation on $\mathrm{P} \rtimes \mathbb{N}^\times$ where $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ is the semidirect product of the additive rationals $\mathbb{Q}$ by the multiplicative positive rationals $\mathbb{Q}_+^*$. From the structure of a covariant semigroup C^*-algebra $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$, we obtain conditions for the existence of the ground state on the natural dynamical system of a covariant semigroup C^*-algebra $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$.

2. A covariant semigroup C^*-algebra $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$

Let $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ denote the semidirect product of the additive rationals $\mathbb{Q}$ by the multiplicative positive rationals $\mathbb{Q}_+^*$, where the group operation and the inverse element are given by

$$(r,x)(s,y) = (r+xs,\ xy) \qquad \text{for } r,s \in \mathbb{Q} \text{ and } x,y \in \mathbb{Q}_+^*,$$
$$(r,x)^{-1} = (-x^{-1}r,\ x^{-1}) \qquad \text{for } r \in \mathbb{Q} \text{ and } x \in \mathbb{Q}_+^*.$$

Let $\mathrm{P} = \{0,2,3,\cdots\}$ be a semigroup of $\mathbb{Z}$. Then the semidirect product $\mathrm{P} \rtimes \mathbb{N}^\times$ is the subsemigroup of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$.

Proposition 2.1. $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathrm{P} \rtimes \mathbb{N}^\times)$ *is a partially ordered group and the generators* $(2,1),(3,1)$, *and* $\{(0,p) : p$ *is a prime number*$\}$ *satisfy the relations*

$$(0,p)(2,1) = (2,1)^p(0,p),\ (0,p)(3,1) = (3,1)^p(0,p),\ \textit{and}\ (0,p)(0,q) = (0,q)(0,p)$$

for all prime numbers p *and* q.

Proof. Since $(\mathrm{P} \rtimes \mathbb{N}^\times) \cap (\mathrm{P} \rtimes \mathbb{N}^\times)^{-1} = \{(0,1)\}$, the subsemigroup $\mathrm{P} \rtimes \mathbb{N}^\times$ induces a left-invariant partial order on $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ as follows : for (r,x) and (s,y) in $\mathbb{Q} \times \mathbb{Q}_+^*$,

$$\begin{aligned} (r,x) \le (s,y) \quad &\Leftrightarrow \quad (r,x)^{-1}(s,y) \in \mathrm{P} \rtimes \mathbb{N}^\times \\ &\Leftrightarrow \quad x^{-1}(s-r) \in \mathrm{P} \textit{ and } x^{-1}y \in \mathbb{N}^\times. \end{aligned} \tag{2.1}$$

Suppose that G is a group containing elements u and $\{v_p : p$ is a prime number$\}$ satisfying the relations

$$v_p u = u^p v_p \text{ and } v_p v_q = v_q v_p.$$

Since $\mathbb{Q}_+^*$ is the free abelian group generated by prime numbers and v_p commutes with v_q, the map $p \mapsto v_p$ extends to a homomorphism $v : \mathbb{Q}_+^* \to \mathrm{G}$. Let $\mathbb{P} = \{p :\ p \text{ is a prime number in } \mathbb{Q}_+^*\}$. Consider an inclusion map $i : \mathbb{P} \to \mathbb{Q}_+^*$ by $p \mapsto p^1$ and a map $f : \mathbb{P} \to \mathrm{G}$ defined by $p \mapsto v_p$. Define $v : \mathbb{Q}_+^* \to \mathrm{G}$ by $v(1) = e$ and $v(p_1^{\delta_1} \cdots p_n^{\delta_n}) = f(p_1)^{\delta_1} \cdots f(p_n)^{\delta_n}$ for a nonempty reduced word $p_1^{\delta_1} \cdots p_n^{\delta_n}$.

We can see v is a well-defined homomorphism such that $v \circ i = f$. Moreover v is unique because if $g : \mathbb{Q}_+^* \to \mathrm{G}$ is a homomorphism such that $g \circ i = f$, then

$$\begin{aligned} g(p_1^{\delta_1} \cdots p_n^{\delta_n}) &= g \circ i(p_1)^{\delta_1} \cdots g \circ i(p_n)^{\delta_n} \\ &= f(p_1)^{\delta_1} \cdots f(p_n)^{\delta_n} \\ &= v(p_1^{\delta_1} \cdots p_n^{\delta_n}). \end{aligned}$$

So we have that $g = v$. Since $\mathbb{Z}$ is free abelain, for each $n \in \mathbb{N}^\times$ $(\frac{1}{n}\mathbb{Z}, +)$ is a free abelian group. So we define a homomorphism $\phi_n : \frac{1}{n}\mathbb{Z} \to \mathrm{G}$ by $k \mapsto v_n^{-1} u^k v_n$ for each $n \in \mathbb{N}^\times$ and these combine to give a well-define homomorphism $\phi : \mathbb{Q} = \cup_n \frac{1}{n}\mathbb{Z} \to \mathrm{G}$ by $\frac{k}{n} \mapsto \phi_n(k)$ for $\frac{k}{n}, \frac{l}{m} \in \mathbb{Q}$ where $m, n \in \mathbb{N}^\times$, because

$$\begin{aligned} \phi(\frac{k}{n} + \frac{l}{m}) &= v_{nm}^{-1} u^{km+nl} v_{nm} \\ &= (v_{nm}^{-1} u^{km} v_{nm})(v_{nm}^{-1} u^{nl} v_{nm}) \\ &= \phi_{nm}(km)\phi_{nm}(nl) \\ &= \phi(\frac{k}{n})\phi(\frac{l}{m}). \end{aligned}$$

Now the first relation extends to $v_r u^k = u^{rk} v_r$ and it follows easily that v and ϕ combine to give a homomorphism $\mathrm{F} : \mathbb{Q} \rtimes \mathbb{Q}_+^* \to \mathrm{G}$ of the semidirect product $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ into G by $(\frac{k}{n}, x) \mapsto \phi(\frac{k}{n}) v_x$. In more detail for $(\frac{k}{n}, x) = (\frac{k}{n}, 1)(0, x) \in \mathbb{Q} \rtimes \mathbb{Q}_+^*$,

$$\mathrm{F}(\frac{k}{n}, x) = \mathrm{F}(\frac{k}{n}, 1)\mathrm{F}(0, x) = v_n^{-1} u^k v_n v_x \quad \text{and}$$

$$\begin{aligned} \mathrm{F}((\frac{k_1}{n_1}, x_1)(\frac{k_2}{n_2}, x_2)) &= v_{n_1 n_2}^{-1} u^{k_1 n_2} u^{n_1 x_1 k_2} v_{n_1 n_2} v_{x_1 x_2} \\ &= v_{n_1}^{-1}(v_{n_2}^{-1} u^{k_1 n_2})(u^{(n_1 x_1) k_2} v_{n_1 x_1}) v_{n_2 x_2} \\ &= v_{n_1}^{-1}(u^{k_1} v_{n_2}^{-1})(v_{n_1 x_1} u^{k_2}) v_{n_2 x_2} \\ &= (v_{n_1}^{-1} u^{k_1} v_{n_1}) v_{x_1} (v_{n_2}^{-1} u^{k_2} v_{n_2}) v_{x_2} \\ &= \mathrm{F}(\frac{k_1}{n_1}, x_1)\mathrm{F}(\frac{k_2}{n_2}, x_2). \end{aligned}$$

We see that the group $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ is generated by elements $(1, 1)$ and $\{(0, p) :$ p is a prime number$\}$ which satisfy the relations

$$(0, p)(1, 1) = (1, 1)^p(0, p) \text{ and } (0, p)(0, q) = (0, q)(0, p)$$

for all prime numbers p, q and this is a presentation of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ in [17]. We shall consider the unital subsemigroup $\mathrm{P} \rtimes \mathbb{N}^\times$ of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ interpreted in the category of monoids where $\mathrm{P} = \{0, 2, 3, 4, 5, \cdots\}$. Since $(2,1)^{-1}(3,1) = (1,1)$, $\mathrm{P} \rtimes \mathbb{N}^\times$ can generate $\mathbb{Q} \rtimes \mathbb{Q}_+^*$. Since the semigroup P is generated by the elements 2 and 3, $\mathrm{P} \rtimes \mathbb{N}^\times$is generated by the elements $(2,1)$, $(3,1)$, and $\{(0,p) : p$ is a prime number$\}$ which satisfy the relations $(0,p)(2,1) = (2,1)^p(0,p)$, $(0,p)(3,1) = (3,1)^p(0,p)$, and $(0,p)(0,q) = (0,q)(0,p)$ for all prime numbers p, q. $\square$

If $(r,x), (s,y) \in \mathrm{P} \rtimes \mathbb{N}^\times$ have common upper bounds in $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathrm{P} \rtimes \mathbb{N}^\times)$, we will denote the smallest one among common upper bounds of (r,x) and (s,y) by $(r,x) \Cup (s,y)$ in the usual order in $\mathbb{Q} \rtimes \mathbb{Q}_+^*$. Even though the semigroup $\mathrm{P} \rtimes \mathbb{N}^\times$ gives a partial order on the semi-direct product group $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ by (2.1), but ($\mathbb{Q} \rtimes \mathbb{Q}_+^*$, $\mathrm{P} \rtimes \mathbb{N}^\times$) is not a quasi-lattice ordered group. However we define a covariant isometric representation on $\mathrm{P} \rtimes \mathbb{N}^\times$ in the similiar way of Nica's covariant isometric representation [1].

Now we construct the C^*-algebra $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ generated by an isometric representation of $\mathrm{P} \rtimes \mathbb{N}^\times$. First, we introduce the isometric representation of a discrete semigroup M. Let M denote a semigroup with unit e and $\mathcal{B}$ be a unital C^*-algebra. A map $\mathrm{W} : M \to \mathcal{B}, x \mapsto \mathrm{W}_x$ is called an *isometric homomorphism* if $\mathrm{W}_e = 1$, W_x is an isometry and $\mathrm{W}_{xy} = \mathrm{W}_x \mathrm{W}_y$ for all $x, y \in M$. If $\mathcal{B}$ is the $*$-algebra $\mathcal{B}(H)$ of all bounded linear operators of a non-zero Hilbert space H, we call (H, W) an *isometric representation* of M.

Nica [19] introduced the covariant isometric representation of a quasi-lattice ordered group as follows: for a quasi-lattice ordered group M, an isometric representation $V : M \to \mathcal{B}(H)$ is Nica *covariant* if

$$V_x V_x^* V_y V_y^* = \begin{cases} 0 & if\ x \vee y = \infty, \\ V_{x \vee y} V_{x \vee y}^* & if\ x \vee y < \infty \end{cases}$$

where $x \vee y$ is the least common upper bound of x and y in M. It is known that Nica's covariance is a very suitable isometric representation to explain the uniqueness property of C^*-algebra generated by isometric representations. The motivation of the condition of the covariant isometric representation is the range projections of the left regular isometric representation of a left cancellative semigroup M. The left regular isometric representation on the discrete and left cancellative semigroup M is given by

$$\mathcal{L}_m \delta_n = \delta_{mn} \quad \text{for } m, n \in M,$$

where $\{\delta_n : n \in M\}$ is the canonical orthonormal basis of $\ell^2(M)$. Even though $(\mathbb{Q} \rtimes \mathbb{Q}^*_+,\ \mathrm{P} \rtimes \mathbb{N}^\times)$ is not a quasi-lattice ordered group, we can define the covariant isometric representaion of $\mathrm{P} \rtimes \mathbb{N}^\times$ in the sence of Nica's covariant isometric representation.

A isometric representation $\mathrm{W} : \mathrm{P} \rtimes \mathbb{N}^\times \to \mathcal{B}(\mathrm{H})$ of $\mathrm{P} \rtimes \mathbb{N}^\times$ on a Hilbert space H is *covariant* if it satisfies

$$\begin{aligned} &\mathrm{W}_{(m,a)}\mathrm{W}^*_{(m,a)}\mathrm{W}_{(n,b)}\mathrm{W}^*_{(n,b)} \\ &= \begin{cases} 0 & if\ (m + a\mathrm{P}) \cap (n + b\mathrm{P}) = \emptyset, \\ \mathrm{W}_{(m,a)\Cup(n,b)}\mathrm{W}^*_{(m,a)\Cup(n,b)} & if\ (m + a\mathrm{P}) \cap (n + b\mathrm{P}) \neq \emptyset. \end{cases} \end{aligned}$$

We use the notation $\mathrm{W}_\infty = 0$ when $(m,a) \Cup (n,b) = \infty$, thus we can always have

$$\mathrm{W}_{(m,a)}\mathrm{W}^*_{(m,a)}\mathrm{W}_{(n,b)}\mathrm{W}^*_{(n,b)} = \mathrm{W}_{(m,a)\Cup(n,b)}\mathrm{W}^*_{(m,a)\Cup(n,b)}$$

for all $(m,a),(n,b) \in \mathrm{P} \rtimes \mathbb{N}^\times$. The covariant condition leads us to the useful following equation

$$\mathrm{W}^*_{(m,a)}\mathrm{W}_{(n,b)} = \mathrm{W}_{(m,a)^{-1}\sigma}\mathrm{W}^*_{(n,b)^{-1}\sigma} \tag{2.2}$$

for all $(m,a),(n,b) \in \mathrm{P} \rtimes \mathbb{N}^\times$ where $\sigma = (m,a) \Cup (n,b)$.

We can have a semigroup C^*-algebra generated by a covariant isometric representation of $\mathrm{P} \rtimes \mathbb{N}^\times$ by a similar way in [16].

Definition 2.2. *The universal C^*-algebra for covariant isometric representations of* $\mathrm{P} \rtimes \mathbb{N}^\times$, *denoted by* $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$, *is the C^*-algebra generated by the canonical covariant isometric representation* $\mathrm{W} : \mathrm{P} \rtimes \mathbb{N}^\times \to \mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ *with the following proprety :*

if X *is a covariant isometric representation of* $\mathrm{P} \rtimes \mathbb{N}^\times$, *then there is a homomorphism* $\pi : \mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times) \to C^*(\{\mathrm{X}_{(m,a)} : (m,a) \in \mathrm{P} \rtimes \mathbb{N}^\times\})$ *such that* $\pi(\mathrm{W}_{(m,a)}) = \mathrm{X}_{(m,a)}$. *We call* $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ *the covariant semigroup C^*- algebra of* $\mathrm{P} \rtimes \mathbb{N}^\times$.

Theorem 2.3. *Let $\mathcal{A}$ be the universal C^*-algebra generated by isometries s, t, and $\{\upsilon_p : p \text{ is a prime number}\}$ satisfying relations*
(R1) $t^2 = s^3$,
(R2) $ts = st$, $s^*t = ts^*$, *and* $t^*s = st^*$,
(T1) $\upsilon_p s = s^p \upsilon_p$, $\upsilon_p t = t^p \upsilon_p$, $\upsilon_p s^* = s^{*p}\upsilon_p$, *and* $\upsilon_p t^* = t^{*p}\upsilon_p$,
(T2) $\upsilon_p \upsilon_q = \upsilon_q \upsilon_p$,
(T3) $\upsilon_p^* \upsilon_q = \upsilon_q \upsilon_p^*$ *when* $p \neq q$,
(T4) $s^* \upsilon_p = s^{p-1}\upsilon_p s^*$,
(T5) $\upsilon_p^* s^{k_1} t^{k_2} \upsilon_p = 0$ *for* $1 \le 2k_1 + 3k_2 < p$,

(T6) $\upsilon_p^* s^k \upsilon_p = 0$ *for* $1 \leq k < p$ *when* $p \neq 2$, *and* $\upsilon_2^* s \upsilon_2 = ts^*$, $\upsilon_p^* t^k \upsilon_p = 0$ *for* $1 \leq k < p$ *when* $p \neq 3$, $\upsilon_3^* t \upsilon_3 = ts^*$, $\upsilon_3^* t^2 \upsilon_3 = s$, *and* $\upsilon_p^* t^k s^{*k} \upsilon_p = 0$ *for* $1 \leq k < p$.

Then there is an isomorphism π *of* $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ *onto* $\mathcal{A}$ *such that* $\pi(\mathrm{W}_{(2,1)}) = s$, $\pi(\mathrm{W}_{(3,1)}) = t$, *and* $\pi(\mathrm{W}_{(0,p)}) = \upsilon_p$ *for every prime* p.

Proof. We can see that the formular $\mathrm{X}_{(m,a)} := s^x t^y \upsilon_a$ where $m = 2x + 3y$ some x, y in $\mathbb{N}$ defines Nica-covariant isometric representation $\mathrm{X} = \mathrm{X}_{s,t,\upsilon}$ on $\mathrm{P} \rtimes \mathbb{N}^\times$ into $\mathcal{A}$ in [1]. Since $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ is the universal C^*-algebra for covariant isometric representations of $\mathrm{P} \rtimes \mathbb{N}^\times$, it induces a homomorphism $\pi_{s,t,\upsilon} : \mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times) \to \mathcal{A}$ such that $\pi_{s,t,\upsilon}(\mathrm{W}_{(m,a)}) = \mathrm{X}_{(m,a)}$. It is not hard to see that $\pi_{s,t,\upsilon}$ is an isomorphism of $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ onto $\mathcal{A}$ in [1]. □

3. Ground states on $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$

Let $\mathcal{B}$ be a C^*-algebra. A C^*-algebra dynamical system is a pair $(\mathcal{B}, \mathbb{R}, \alpha)$ consisting of a C^*-algebra $\mathcal{B}$ and a strongly continuous action $\alpha : \mathbb{R} \to Aut(\mathcal{B})$. By strong continuity we mean that the map $t \mapsto \alpha_t(a)$ is a continuous map from $\mathbb{R}$ into $\mathcal{B}$ for each fixed $a \in \mathcal{B}$. An element $a \in \mathcal{B}$ is $\alpha-$analytic if the function $\mathbb{R} \to \mathcal{B}$ given by $t \mapsto \alpha_t(a)$ extends to an entire function $\mathbb{C} \to \mathcal{B}$ given by $z \mapsto \alpha_z(a)$. For the inverse temperature $\beta > 0$ a state ϕ is a KMS_β state if $\phi(ab) = \phi(b\, \alpha_{i\beta}(a))$ for all α-analytic elements $a, b \in \mathcal{B}$. A ground state ϕ is satisfies $z \mapsto \phi(b\, \alpha_z(a))$ is bounded on the upper-half plane for all α-analytic elements $a, b \in \mathcal{B}$.

Let us consider the unitary representation $u : \mathbb{R} \to \mathcal{U}(\ell^2(\mathrm{P} \rtimes \mathbb{N}^\times))$ defined by

$$u_r \delta_{(m,a)} := a^{ir} \delta_{(m,a)} \qquad (r \in \mathbb{R})$$

where $\{\delta_{(m,a)} : (m,a) \in \mathrm{P} \rtimes \mathbb{N}\}$ is the canonical orthonormal basis of $\ell^2(\mathrm{P} \rtimes \mathbb{N}^\times)$ and $\mathcal{U}(\ell^2(\mathrm{P} \rtimes \mathbb{N}^\times))$ is the group of unitary operators in $B(\ell^2(\mathrm{P} \rtimes \mathbb{N}^\times))$. Let $\mathcal{L} : \mathrm{P} \rtimes \mathbb{N}^\times \to B(\ell^2(\mathrm{P} \rtimes \mathbb{N}^\times))$ be the left regular isometric representation on $\mathrm{P} \rtimes \mathbb{N}^\times$ defined by

$$\mathcal{L}_{(m,a)} \delta_{(n,b)} = \delta_{(m,a)(n,b)} \quad \text{for } (m,a), (n,b) \in \mathrm{P} \rtimes \mathbb{N}^\times.$$

The reduced semigroup C^*-algebra $\mathcal{C}_{red}(\mathrm{P} \rtimes \mathbb{N}^\times)$ on $\ell^2(\mathrm{P} \rtimes \mathbb{N}^\times)$ is generated by the left regular isometric representation $\mathcal{L}$ on $\mathrm{P} \rtimes \mathbb{N}^\times$. Then the unitary group $\{u_r | r \in \mathbb{R}\}$ induces the automorphism group $\tau_r(a) = u_r a u_r^* (a \in \mathcal{C}_{red}(\mathrm{P} \rtimes \mathbb{N}^\times))$ of the reduced semigroup C^*-algebra $\mathcal{C}_{red}(\mathrm{P} \rtimes$

$\mathbb{N}^\times$) on $\ell^2(\mathrm{P} \rtimes \mathbb{N}^\times)$. In fact, it follows from the definition of the left regular isometric representation that

$$\tau_r(\mathcal{L}_{(2,1)}) = \mathcal{L}_{(2,1)},\ \tau_r(\mathcal{L}_{(3,1)}) = \mathcal{L}_{(3,1)},\ \text{and } \tau_r(\mathcal{L}_{(0,p)}) = p^{ir}\mathcal{L}_{(0,p)}$$

for $p \in \mathbb{P}$ and $r \in \mathbb{R}$. Since the left regular isometric representation is covariant, by the universality of $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ there is a $*$-homomorphism Φ from $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ onto $\mathcal{C}_{red}(\mathrm{P} \rtimes \mathbb{N}^\times)$ where $\Phi(W_{(m,a)}) = \mathcal{L}_{(m,a)}$ for $(m, a) \in \mathrm{P} \rtimes \mathbb{N}^\times$. Thus we can see that there is a strongly continuous action σ of $\mathbb{R}$ on $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ such that

$$\sigma_r(s) = s,\ \sigma_r(t) = t,\ \text{and } \sigma_r(\upsilon_p) = p^{ir}\upsilon_p \text{ for } p \in \mathbb{P} \text{ and } r \in \mathbb{R}.$$

PROPOSITION 3.1. *For our system* $(\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times), \mathbb{R}, \sigma)$ *the elements* $s^{x_1}t^{y_1}\upsilon_a\upsilon_b^*t^{*y_2}s^{*x_2}$ *for* $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ *are all analytic for* $x_1, y_1, x_2, y_2 \in \mathbb{N}$ *and* $a,\ b \in \mathbb{N}^\times$.

Proof. By the definition of σ_r, we have

$$\begin{aligned}\sigma_r(s^{x_1}t^{y_1}\upsilon_a\upsilon_b^*t^{*y_2}s^{*x_2}) &= (a^{ir}s^{x_1}t^{y_1}\upsilon_a)(b^{-ir}\upsilon_b^*t^{*y_2}s^{*x_2}) \\ &= (ab^{-1})^{ir}(s^{x_1}t^{y_1}\upsilon_a\upsilon_b^*t^{*y_2}s^{*x_2}).\end{aligned}$$

Therefore the function $r \mapsto \sigma_r(s^{x_1}t^{y_1}\upsilon_a\upsilon_b^*t^{*y_2}s^{*x_2})$ is the restriction to $\mathbb{R}$ of an entire function on $\mathbb{C}$. So $s^xt^y\upsilon_a$ and $\upsilon_b^*t^{*y'}s^{*x'}$ are all analytic for $x, y, x', y' \in \mathbb{N}$ and $a,\ b \in \mathbb{N}^\times$. □

Since $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ is generated by the canonical covariant isometric representation, we can see that $span\{\mathrm{W}_{(m,a)}\mathrm{W}^*_{(n,b)} : (m, a), (n, b) \in \mathrm{P} \rtimes \mathbb{N}^\times\}$ is a dense $*$-subalgebra of $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ by the covariance. Furthermore, from Theorem 2.3 we have also that $span\{s^{x_1}t^{y_1}\upsilon_a\upsilon_b^*t^{*y_2}s^{*x_2} : (m, a), (n, b) \in \mathrm{P} \rtimes \mathbb{N}^\times,\ m = 2x_1 + 3y_1,\ n = 2x_2 + 3y_2, \text{ and } x_1, x_2, y_1, y_2 \in \mathbb{N}\}$ is a dense $*$-subalgebra of $\mathcal{A}$.

PROPOSITION 3.2. *If a state* ϕ *of* $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ *is a ground state for* σ, *then*

$$\phi(s^xt^y\upsilon_p\upsilon_q^*t^{*y'}s^{*x'}) = 0$$

for $x, x', y, y' \in \mathbb{N}$ *and* $p, q \in \mathbb{P}$.

Proof. Let ϕ be a ground state for σ. By Proposition 3.1, $s^xt^y\upsilon_p$ and $\upsilon_q^*t^{*y'}s^{*x'}$ are all analytic for $x, y, x', y' \in \mathbb{N}$ and $p,\ q \in \mathbb{P}$. The expression $\phi(s^xt^y\upsilon_p\sigma_{\alpha+i\beta}(\upsilon_q^*t^{*y'}s^{*x'})) = (\frac{1}{q})^{i\alpha-\beta}\phi(s^xt^y\upsilon_p\upsilon_q^*t^{*y'}s^{*x'})$ is bounded on the upper half plane $(\beta > 0)$ if and only if $\phi(s^xt^y\upsilon_p\upsilon_q^*t^{*y'}s^{*x'}) = 0$. □

Moreover, if a state ϕ of $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ is a ground state for σ, then $\phi(s^xt^y\upsilon_b\upsilon_b^*t^{*y'}s^{*x'}) = 0$ whenever $b \neq 1$ for $x, y, x', y' \in \mathbb{N}$ and $b \in \mathbb{N}^\times$.

PROPOSITION 3.3. *If a state* ϕ *of* $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ *satisfies the condition* $\phi(s^x t^y \upsilon_b \upsilon_b^* t^{*y'} s^{*x'}) = 0$ *whenever* $x, x', y, y', a, b \in \mathbb{N}$ *and* $b \neq 1$, *then* ϕ *is a ground state for* σ.

Proof. Suppose that $\phi(s^x t^y \upsilon_b \upsilon_b^* t^{*y'} s^{*x'}) = 0$ whenever b is not 1. We choose two analytic elements $\mathrm{X} = s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'}$ and Y for σ. The Cauchy-Schwarz inequality yields

$$\begin{aligned} |\phi(\mathrm{Y}^* \sigma_{\alpha+i\beta}(\mathrm{X}))|^2 &= |(\frac{a}{b})^{i\alpha-\beta} \phi(\mathrm{Y}^*\mathrm{X})|^2 \\ &\leq (\frac{a}{b})^{-\beta} \phi(\mathrm{Y}^*\mathrm{Y})\phi(\mathrm{X}^*\mathrm{X}) \\ &= (\frac{b}{a})^{\beta} \phi(\mathrm{Y}^*\mathrm{Y})\phi(\mathrm{X}^*\mathrm{X}), \end{aligned}$$

and

$$\begin{aligned} \mathrm{X}^*\mathrm{X} &= (s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'})^* (s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'}) \\ &= s^{x'} t^{y'} \upsilon_b \upsilon_a^* t^{*y} s^{*x} s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'} \\ &= s^{x'} t^{y'} \upsilon_b \upsilon_b^* t^{*y'} s^{*x'}. \end{aligned}$$

Since the last factor $\phi(\mathrm{X}^*\mathrm{X}) = \phi(s^{x'} t^{y'} \upsilon_b \upsilon_b^* t^{*y'} s^{*x'})$ vanishes for $b \neq 1$, the function $\alpha + i\beta \mapsto \phi(\mathrm{Y}^* \sigma_{\alpha+i\beta}(\mathrm{X}))$ is bounded for $\beta > 0$. This implies that ϕ is a ground state. □

THEOREM 3.4. *If a state* ϕ *of* $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$ *is a ground state for* σ, *then* $\phi(s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'}) = 0$ *whenever* $a \neq 1$ *or* $b \neq 1$.

Proof. Let ϕ be the state of $\mathcal{T}(\mathrm{P} \rtimes \mathbb{N}^\times)$. The expression

$$\begin{aligned} &\phi(s^{x_3} t^{y_3} \upsilon_c \upsilon_d^* t^{*y_4} s^{*x_4} \sigma_{\alpha+i\beta}(s^{x_1} t^{y_1} \upsilon_a \upsilon_b^* t^{*y_2} s^{*x_2})) \\ &= (\frac{a}{b})^{i\alpha-\beta} \phi(s^{x_3} t^{y_3} \upsilon_c \upsilon_d^* t^{*y_4} s^{*x_4} s^{x_1} t^{y_1} \upsilon_a \upsilon_b^* t^{*y_2} s^{*x_2}) \end{aligned}$$

is bounded on the upper half plane ($\beta > 0$) if and only if

$$\phi(s^{x_3} t^{y_3} \upsilon_c \upsilon_d^* t^{*y_4} s^{*x_4} s^{x_1} t^{y_1} \upsilon_a \upsilon_b^* t^{*y_2} s^{*x_2}) = 0 \tag{3.1}$$

where $a < b$. Suppose that ϕ is a ground state. We consider $x_1 = x_4$, $y_1 = y_4$, and $d = a = 1$. Then $\phi(s^{x_3} t^{y_3} \upsilon_c \upsilon_b^* t^{*y_2} s^{*x_2}) = 0$ for $1 < b$. Taking adjoints (3.1), we have

$$\phi(s^{x_2} t^{y_2} \upsilon_b \upsilon_a^* t^{*y_1} s^{*x_1} s^{x_4} t^{y_4} \upsilon_d \upsilon_c^* t^{*y_3} s^{*x_3}) = 0$$

whenever $a < b$. Put again $x_1 = x_4$, $y_1 = y_4$, and $d = a = 1$. Then we have $\phi(s^{x_2} t^{y_2} \upsilon_b \upsilon_c^* t^{*y_3} s^{*x_3}) = 0$ for $1 < b$. □

THEOREM 3.5. *If a state ϕ of $\mathcal{T}(\mathrm{P}\rtimes\mathbb{N}^{\times})$ satisfies $\phi(s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'}) = 0$ whenever $a \neq 1$ or $b \neq 1$, then ϕ is a ground state for σ.*

Proof. Suppose that $\phi(s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'}) = 0$ whenever a or b is not 1. We choose two analytic elements $\mathrm{X} = s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'}$ and Y for σ. By the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\phi(\mathrm{Y}^*\sigma_{\alpha+i\beta}(\mathrm{X}))|^2 &= |(\frac{a}{b})^{i\alpha-\beta}\phi(\mathrm{Y}^*\mathrm{X})|^2 \\ &\leq (\frac{a}{b})^{-\beta}\phi(\mathrm{Y}^*\mathrm{Y})\phi(\mathrm{X}^*\mathrm{X}) \\ &= (\frac{b}{a})^{\beta}\phi(\mathrm{Y}^*\mathrm{Y})\phi(\mathrm{X}^*\mathrm{X}), \end{aligned}$$

and

$$\begin{aligned} \mathrm{X}^*\mathrm{X} &= (s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'})^*(s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'}) \\ &= s^{x'} t^{y'} \upsilon_b \upsilon_a^* t^{*y} s^{*x} s^x t^y \upsilon_a \upsilon_b^* t^{*y'} s^{*x'} \\ &= s^{x'} t^{y'} \upsilon_b \upsilon_b^* t^{*y'} s^{*x'}. \end{aligned}$$

Since the last factor $\phi(\mathrm{X}^*\mathrm{X}) = \phi(s^{x'} t^{y'} \upsilon_b \upsilon_b^* t^{*y'} s^{*x'})$ vanishes for $b \neq 1$, the function $\alpha+i\beta \mapsto \phi(\mathrm{Y}^*\sigma_{\alpha+i\beta}(\mathrm{X}))$ is bounded for $\beta > 0$. This implies that ϕ is a ground state. □

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