JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **33**, No. 3, August 2020
http://dx.doi.org/10.14403/jcms.2020.33.3.319

ON STABILITY OF A GENERALIZED QUADRATIC FUNCTIONAL EQUATION WITH n-VARIABLES AND m-COMBINATIONS IN QUASI-β-NORMED SPACES

Heejeong Koh* and Yonghoon Lee**

Abstract. In this paper, we establish a general solution of the following functional equation

$$\begin{aligned} &mf\left(\sum_{k=1}^{n} x_k\right) + \sum_{t=1}^{m} f\left(\sum_{k=1}^{n-i_t} x_k - \sum_{k=n-i_t+1}^{n} x_k\right) \\ =&2\sum_{t=1}^{m}\left(f\left(\sum_{k=1}^{n-i_t} x_k\right) + f\left(\sum_{k=n-i_t+1}^{n} x_k\right)\right) \end{aligned}$$

where $m, n, t, i_t \in \mathbb{N}$ such that $1 \leq t \leq m < n$. Also, we study Hyers-Ulam-Rassias stability for the generalized quadratic functional equation with n-variables and m-combinations form in quasi-β-normed spaces and then we investigate its application.

1. Introduction

The stability problem of functional equations concerning the stability of group homomorphism was proposed by Ulam in 1940. In 1941, Hyers [4] partially solved the stability of the linear functional equation for the case when the groups are Banach spaces. Hyers's theorem was generalized by Aoki [2] for additive mapping and Rassias [10] for linear mapping by considering unbounded Cauchy differences. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings in various spaces[1, 2, 3, 8, 10, 12, 13].

Received September 30, 2019; Accepted June 09, 2020.
2010 Mathematics Subject Classification: 39B72, 39B82, 39B52, 47H09.
Key words and phrases: generalized quadratic functional equation with n-variable, Hyers-Ulam-Rassias stability, quasi-β-normed space.
**Corresponding author : yonghoon@dankook.ac.kr.

Let X and Y be vector spaces and let $f : X \to Y$ be a mapping. The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called the quadratic functional equation. Every solution of the equation (1.1) is said to be a quadratic mapping. The Hyers-Ulam stability theorem for the quadratic functional equation was proved by Skof [14] and Czerwik [3].

Before we present our results, we introduce some basic facts concerning quasi-β-normed space. We fix a real number β with $0<\beta\le 1$ and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over a field $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following:
(1) $\|x\|\ge 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$;
(2) $\|\lambda x\|=|\lambda|^{\beta}\cdot\|x\|$ for all $\lambda\in\mathbb{K}$ and $x\in X$;
(3) there exists a constant $K\ge 1$ such that $\|x+y\|\le K(\|x\|+\|y\|)$ for all $x,y\in X$.

The pair $(X,\|\cdot\|)$ is called a quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. In fact, a quasi-β-Banach space is a complete quasi-β-normed space. A quasi-β-norm is called a (β,p)-norm $(0<p\le 1)$ if $\|x+y\|^p\le\|x\|^p+\|y\|^p$ for all $x,y\in X$.

In this case, a quasi-β-Banach space is called a (β,p)-Banach space. In this paper, we will introduce a generalized quadratic functional equation with n-variable. The purpose of this paper, we establish a solution of

$$\begin{aligned} &mf\left(\sum_{k=1}^{n}x_k\right)+\sum_{t=1}^{m}f\left(\sum_{k=1}^{n-i_t}x_k-\sum_{k=n-i_t+1}^{n}x_k\right)\\ =&2\sum_{t=1}^{m}\left(f\left(\sum_{k=1}^{n-i_t}x_k\right)+f\left(\sum_{k=n-i_t+1}^{n}x_k\right)\right)\end{aligned} \tag{1.2}$$

where $m,n,t,i_t\in\mathbb{N}$ such that $1\le t\le m<n$ and $1\le i_t<n$. We note that the order of $i_1,i_2,\cdots,i_m$ does not have to be the order of the positive integers and $i_1,i_2,\cdots,i_m$ do not have to equal. Also, we study Hyers-Ulam-Rassias stability for the generalized quadratic functional equation with n-variables and m-combinations form in quasi-β-normed spaces and its application.

2. Main theorem

Throughout this section, X is a normed space and Y is a quasi-β-Banach space. In this section, we will establish a general solution of the equation (1.2) and then we will point out the Hyers-Ulam-Rassias stability results controlled by approximately mappings for a quadratic functional equation with n-variables and m-combinations form in quasi-β-normed space.

THEOREM 2.1. *A mapping $f : X \to Y$ satisfies the functional equation (1.2) if and only if the mapping f satisfies the functional equation (1.1).*

Proof. Let f be a solution of the functional equation (1.2). Setting $x_2 = x_3 = \cdots = x_{n-1} = 0$ in (1.2), then we get

$$m\left[f(x_1 + x_n) + f(x_1 - x_n)\right] = 2m\left[f(x_1) + f(x_n)\right]$$

for all $x_1, x_n \in X$. Thus f satisfies (1.1).

Conversely, assume that the mapping f satisfies the functional equation (1.1). Then we have the following $n-1$ equations ;

$$\begin{aligned}
&f(x_1 + x_2 + \cdots + x_n) + f(x_1 + \cdots + x_{n-1} - x_n) \\
&\qquad = 2\left[f(x_1 + \cdots + x_{n-1}) + f(x_n)\right], \\
&f(x_1 + x_2 + \cdots + x_n) + f(x_1 + \cdots + x_{n-2} - x_{n-1} - x_n) \\
&\qquad = 2\left[f(x_1 + \cdots + x_{n-2}) + f(x_{n-1} + x_n)\right], \\
&\qquad\qquad\qquad \vdots \\
&f(x_1 + x_2 + \cdots + x_n) + f(x_1 - x_2 - \cdots - x_n) \\
&\qquad = 2\left[f(x_1) + f(x_2 + \cdots + x_n)\right].
\end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Summing up m of the above $n-1$ equations, we obtain the equation (1.2). This completes the proof of the theorem. $\square$

Now, we investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.2). Define

$$\begin{aligned}
Df(x_1, \cdots, x_n) :=& mf\left(\sum_{k=1}^{n} x_k\right) + \sum_{t=1}^{m} f\left(\sum_{k=1}^{n-i_t} x_k - \sum_{k=n-i_t+1}^{n} x_k\right) \\
&-2\sum_{t=1}^{m}\left(f\left(\sum_{k=1}^{n-i_t} x_k\right) + f\left(\sum_{k=n-i_t+1}^{n} x_k\right)\right),
\end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

THEOREM 2.2. *Let $\psi : X^n \to [0, \infty]$ be a function such that*

$$\widetilde{\psi}(x_1, \cdots, x_n) := \sum_{j=1}^{\infty} \left(\frac{K}{4^\beta}\right)^j \psi\left(2^{j-1}x_1, \cdots, 2^{j-1}x_n\right) < \infty \tag{2.1}$$

for all $x_1, \cdots, x_n \in X$ and $K \geq 1$. If $f : X \to Y$ is a mapping satisfying $f(0) = 0$ such that

$$\|Df(x_1, \cdots, x_n)\| \leq \psi(x_1, \cdots, x_n) \tag{2.2}$$

for all $x_1, \cdots, x_n \in X$, then there exists a unique generalized quadratic mapping $Q : X \to Y$ satisfying the equation (1.2) such that

$$\|f(x) - Q(x)\| \leq \frac{1}{m^\beta}\widetilde{\psi}(x, 0, \cdots, 0, x) \tag{2.3}$$

for all $x \in X$.

Proof. Letting $x_1 = x_n = x$ and $x_2 = x_3 = \cdots = x_{n-1} = 0$ in (2.2) and dividing by $(4m)^\beta$, we have

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{(4m)^\beta}\psi(x, 0, \cdots, 0, x) \tag{2.4}$$

for all $x \in X$. Replacing x by $2x$ in (2.4) and then dividing by 4^β, we get

$$\left\|\frac{1}{4}f(2x) - \frac{1}{4^2}f(2^2x)\right\| \leq \frac{1}{(4^2m)^\beta}\psi(2x, 0, \cdots, 0, 2x)$$

for all $x \in X$. Adding (2.4) and the above inequality, we have

$$\begin{aligned}&\left\|f(x) - \frac{1}{4^2}f(2^2x)\right\| \\ &\leq K\left(\frac{1}{(4m)^\beta}\psi(x, 0, \cdots, 0, x) + \frac{1}{(4^2m)^\beta}\psi(2x, 0, \cdots, 0, 2x)\right)\end{aligned}$$

for all $x \in X$. Continuing in this way, one can obtain that

$$\left\|f(x) - \frac{1}{4^l}f(2^lx)\right\| \leq \frac{1}{m^\beta}\sum_{j=1}^{l}\left(\frac{K}{4^\beta}\right)^j \psi(2^{j-1}x, 0, \cdots, 0, 2^{j-1}x) \tag{2.5}$$

for all $l \in \mathbb{N}$ and all $x \in X$. Now, for $k \in \mathbb{N}$, dividing the inequality (2.5) by $4^{k\beta}$ and then substituting x by $2^k x$, we see that

$$\begin{aligned}
&\left\| \frac{1}{4^k} f(2^k x) - \frac{1}{4^{l+k}} f(2^{l+k} x) \right\| \\
&\leq \frac{1}{m^\beta} \sum_{j=1}^{l} \left(\frac{K}{4^\beta} \right)^{j+k} \psi(2^{j+k-1}x, 0, \cdots, 0, 2^{j+k-1}x)
\end{aligned}$$

for all $x \in X$. Taking $l \to \infty$ and $k \to \infty$ in the previous inequality, by (2.1) we conclude that $\left\{ \frac{1}{4^l} f(2^l x) \right\}$ is a Cauchy sequence in Y for all $x \in X$. Because of the completeness of Y, we can define a mapping $Q : X \to Y$ by

$$Q(x) := \lim_{l \to \infty} \frac{1}{4^l} f(2^l x)$$

for all $x \in X$. By (2.2) and (2.3), we obtain that

$$\begin{aligned}
\|DQ(x_1, \cdots, x_n)\| &= \lim_{l \to \infty} \frac{1}{4^l} \left\| Df(2^l x_1, \cdots, 2^l x_n) \right\| \\
&\leq \lim_{l \to \infty} \frac{1}{4^l} \psi(2^l x_1, \cdots, 2^l x_n) = 0
\end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Hence the mapping $Q : X \to Y$ satisfies (1.2). Taking $l \to \infty$ in (2.5), we get the inequality (2.3). To prove the uniqueness of the generalized quadratic mapping Q, we assume that there exists another quadratic mapping $Q' : X \to Y$ satisfying (2.3). We have

$$\begin{aligned}
\left\| Q(x) - Q'(x) \right\| &\leq K \left(\left\| \frac{Q(2^l x)}{4^l} - \frac{f(2^l x)}{4^l} \right\| + \left\| \frac{f(2^l x)}{4^l} - \frac{Q'(2^l x)}{4^l} \right\| \right) \\
&\leq \frac{2K}{(4^l m)^\beta} \widetilde{\psi}(2^l x, 0, \cdots, 0, 2^l x) \to 0 \text{ as } l \to \infty
\end{aligned}$$

for all $x \in X$. Therefore Q is unique.

□

Corollary 2.3. *Let θ, p be real numbers such that $\theta \geq 0$ and $0 < p < 2\beta - \log_2 K$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_n)\| \leq \theta(\|x_1\|^p + \cdots + \|x_n\|^p)$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique generalized quadratic mapping $Q : X \to Y$ satisfying (2.1) such that

$$\|f(x) - Q(x)\| \le \frac{2\theta K}{m^\beta(4^\beta - 2^p K)}\|x\|^p$$

for all $x \in X$.

Proof. Taking $\psi(x_1, \cdots, x_n) := \theta(\|x_1\|^p + \cdots + \|x_n\|^p)$ and applying Theorem 2.2, one can obtain the result. $\square$

COROLLARY 2.4. *Let* θ *be real number such that* $\theta > 0$. *Suppose that a mapping* $f : X \to Y$ *satisfies*

$$\|Df(x_1, \cdots, x_n)\| \le \theta$$

for all $x_1, \cdots, x_n \in X$. *Then there exists a unique generalized quadratic mapping* $Q : X \to Y$ *satisfying (2.1) such that*

$$\|f(x) - Q(x)\| \le \frac{\theta}{3m^\beta}$$

for all $x \in X$.

Proof. Taking $\psi(x_1, \cdots, x_n) := \theta$ and applying Theorem 2.2, one can obtain the result. $\square$

3. Application

Let X be a normed linear space and $\mathbb{R}_0$ be a non-negative real number. We define $H : {\mathbb{R}_0}^n \to \mathbb{R}_+$ and $\varphi_0 : \mathbb{R}_0 \to \mathbb{R}_+$ such that

$$\begin{aligned}
&\varphi_0(\lambda) > 0, \text{ for all } \lambda > 0,\\
&\varphi_0(2) < \frac{4^\beta}{K}\\
&\varphi_0(2\lambda) \le \varphi_0(2)\varphi_0(\lambda), \text{ for all } \lambda > 0,\\
&H(\lambda t_1, \cdots, \lambda t_n) \le \varphi_0(\lambda) H(t_1, \cdots, t_n), \text{ for all } t_1, \cdots, t_n \in \mathbb{R}_0, \lambda > 0.
\end{aligned}$$

We take in our theorem

$$\psi(x_1, \cdots, x_n) = H(\|x_1\|, \cdots, \|x_n\|)$$

Then

$$\begin{aligned}
\psi(2^{j-1}x_1, \cdots, 2^{j-1}x_n) &= H(2^{j-1}\|x_1\|, \cdots, 2^{j-1}\|x_n\|)\\
&\le \varphi_0(2^{j-1}) H(\|x_1\|, \cdots, \|x_n\|)\\
&\le (\varphi_0(2))^{j-1} H(\|x_1\|, \cdots, \|x_n\|),
\end{aligned}$$

and because $\frac{K}{4^\beta}\varphi_0(2) < 1$ we have

$$\begin{aligned}\widetilde{\psi}(x_1, \cdots, x_n) &\le \sum_{j=1}^{\infty} \left(\frac{K}{4^\beta}\right)^j (\varphi_0(2))^{j-1} H(\|x_1\|, \cdots, \|x_n\|) \\ &= \frac{K}{4^\beta - \varphi_0(2)K} H(\|x_1\|, \cdots, \|x_n\|)\end{aligned}$$

and the inequality (2.4) becomes

$$\begin{aligned}\|f(x) - Q(x)\| &\le \frac{1}{m^\beta}\widetilde{\psi}(x, 0, \cdots, 0, x) \\ &\le \frac{K}{m^\beta(4^\beta - \varphi_0(2)K)} H(\|x\|, 0, \cdots, 0, \|x\|)\end{aligned}$$

or

$$\|f(x) - Q(x)\| \le \frac{K}{m^\beta(4^\beta - \varphi_0(2)K)} \varphi_0(\|x\|) H(1, 0, \cdots, 0, 1).$$

References

[1] Z. Alizadeh and A. G. Ghazanfari, *On the stability of a radical cubic functional equation in quasi-β-spaces,* Journal of Fixed Point Theory and Applications, **18** (2016), no. 4, 843-853.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Japan, **2** (1950), 64-66.

[3] St. Czerwik, *On the stability of the quadratic mapping in normed spaces,* Abhandlungen aus dem Mathematischen Seminar der Universitat Hamburg, **62** (1992), 59-64.

[4] D. H. Hyers, *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci, **27** (1941), 222-224.

[5] S.M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property,* Journal of Mathematical Analysis and Applications, **222** (1998), no. 1, 126-137.

[6] K.W. Jun and Y.H. Lee, *On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality,* Mathematical Inequalities and Applications, **4** (2001), no. 1, pp. 93-118.

[7] K.W. Jun and H. M. Kim, *On the stability of an n-dimensional quadratic and additive functional equation,* Mathematical Inequalities and Applications, **9** (2006), no. 1, 153-165.

[8] D. S. Kang, *On the Stability of Generalized Quartic Mappings in Quasi-β-Normed Spaces,* Journal of Inequalities and Applications, **2010** (2010), no. 2.

[9] R. Krishnan, E. Thandapani, and B.V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* Journal of Nonlinear Science and Applications, **7** (2014), no. 1.

[10] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc, **72** (1978), 297-300.

[11] Th. M. Rassias, *On the stability of the quadratic functional equation and its applications,* Studia Universitatis Babe-Bolyai, **43** (1998), no. 3, 89-124.

[12] B. V. Senthil Kumar and K. Ravi, *Ulam stability of a reciprocal functional equation in quasi-beta-normed spaces,* Global Journal of Pure and Applied Mathematics, **12** (2016), no. 1, 125-128.

[13] D. Y. Shin, C.K. Park, and R. A. Aghjeubeh, S. Farhadabadi, *Fuzzy stability of functional equations in n-variable fuzzy Banach spaces,* Journal of Computational Analysis and Applications, **19** (2015), no. 1, 186-196.

[14] F. Skof, *Proprietá locali e approssimazione di operatori,* Rend. Sem. Mat. Fis. Milano, **53** (1983), 113-129.

[15] S. M. Ulam, *A Collection of Mathematical Problems,* Interscience Publ. New York, 1960.

Heejeong Koh*
College of Liberal Arts
Dankook University
152, Jukjeon, Suji, Yongin, Gyeonggi, 16890, Korea
E-mail: khjmath@dankook.ac.kr

Yonghoon Lee**
Department of Mathematics
Dankook University
119, Dandae, Dongnam, Cheonan, Chungnam, 31116, Korea
E-mail: yonghoon@dankook.ac.kr