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# On $n$-Amitsur Rings

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Abstract. The concepts of an Amitsur ring and a hereditary Amitsur ring, which were introduced and studied by S. Tumurbat in a recent paper, are generalized. For a positive integer $n$, a ring $A$ is said to be an $n$-Amitsur ring if $\gamma(A[X_n]) = (\gamma(A[X_n]) \cap A)[X_n]$ for all radicals $\gamma$, where $A[X_n]$ is the polynomial ring over $A$ in $n$ commuting indeterminates. If a ring $A$ satisfies the above equation for all hereditary radicals $\gamma$, then $A$ is said to be a hereditary $n$-Amitsur ring. Characterizations and examples of these rings are provided. Moreover, new radicals associated with $n$-Amitsur rings are introduced and studied. One of these is a special radical and its semisimple class is polynomially extensible.

## 1. Introduction

Throughout this paper, all rings considered are associative and do not necessar-

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ily have an identity element. All classes of rings contain the one element ring and are closed under isomorphisms. Let us recall that a class $\gamma$ of rings is called a radical class (in the sense of Kurosh and Amitsur) if $\gamma$ is closed under homomorphisms, is closed under extensions (if $I$ is an ideal of a ring $A$ and both $I$ and $A/I$ are in $\gamma$, then $A$ is in $\gamma$) and has the inductive property (if $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_i \subseteq \ldots$ is an ascending chain of ideals of a ring $A$ and if each $I_i$ is in $\gamma$, then $\cup I_i$ is also in $\gamma$). The unique largest ideal of a ring $A$ belonging to $\gamma$, denoted by $\gamma(A)$, is called the $\gamma$-radical of $A$. In what follows, a radical class will be shortly called a radical. For a given radical $\gamma$, the semisimple class of $\gamma$, denoted by $\boldsymbol{S}\gamma$, is the class of all rings $A$ with $\gamma(A) = 0$. If $\sigma$ is a class of rings, then the smallest radical containing $\sigma$ is called the lower radical determined by $\sigma$ and is denoted by $\boldsymbol{L}(\sigma)$. The lower hereditary radical determined by $\sigma$, denoted by $\boldsymbol{L}_h(\sigma)$, is the smallest hereditary radical containing $\sigma$. A class $\alpha$ of rings is said to be hereditary if $\alpha$ is closed under ideals. If $\alpha$ is a hereditary class of rings, then $\boldsymbol{U}(\alpha)$ denotes the upper radical determined by $\alpha$, that is, the class of all rings that have no nonzero homomorphic image in $\alpha$. To denote that $I$ is an ideal of a ring $A$, we write $I \trianglelefteq A$. An ideal $I$ of a ring $A$ is said to be essential in $A$, denoted by $I \trianglelefteq^{\bullet} A$, if $I \cap J \neq 0$ for any nonzero ideal $J$ of $A$. A class $\alpha$ of rings is said to be essentially closed if $I \trianglelefteq^{\bullet} A$ and $I \in \alpha$ implies that $A \in \alpha$. A hereditary class of prime rings that is essentially closed is called a special class. A radical $\gamma$ is called special if it is the upper radical determined by a special class of rings. For other undefined terms and radical-theoretic properties used throughout this paper, we refer the reader to [4] and [14].

Let us recall that a radical $\gamma$ is said to have the *Amitsur property* if

$$\gamma(A[x]) = (\gamma(A[x]) \cap A)[x]$$

for all rings $A$. Radicals with this property have been studied in several papers, for example, in [1, 6, 7, 11, 12, 13]. Throughout this paper, $n$ denotes an arbitrary but fixed natural number $n$. Let $X_n$ denote a set of $n$ commuting indeterminates $x_1, \ldots, x_n$. We introduce the following definition.

**Definition 1.1.** A ring $A$ is said to be an *$n$-Amitsur ring* if

$$\gamma(A[X_n]) = (\gamma(A[X_n]) \cap A)[X_n]$$

for all radicals $\gamma$.

**Definition 1.2.** A ring $A$ is called a *hereditary $n$-Amitsur ring* if

$$\gamma(A[X_n]) = (\gamma(A[X_n]) \cap A)[X_n]$$

for all hereditary radicals $\gamma$.

Clearly, every $n$-Amitsur ring is a hereditary $n$-Amitsur ring. If $n = 1$, then a 1-Amitsur ring and a hereditary 1-Amitsur ring are just the concepts of Amitsur ring and hereditary Amitsur ring, respectively, which were introduced and investigated by S. Tumurbat in [9]. The purpose of this paper is to obtain generalizations of those results.

Let $\gamma$ be a radical. If $A \in \boldsymbol{S}\gamma$ and $\gamma(A[X_n]) \neq 0$, then $A$ is not an $n$-Amitsur ring. Indeed, suppose that $A$ is an $n$-Amitsur ring. Then $0 \neq \gamma(A[X_n]) = (\gamma(A[X_n]) \cap A)[X_n]$ and hence $0 \neq \gamma(A[X_n]) \cap A \in \gamma$, which contradicts the assumption that $A \in \boldsymbol{S}\gamma$. Hence, for example, $\mathbb{Z}_p$ (the ring of integers modulo $p$, where $p$ is a prime number) is not an $n$-Amitsur ring. In fact, for $\gamma = \boldsymbol{U}(\{\mathbb{Z}_p\})$, it is clear that $\mathbb{Z}_p \in \boldsymbol{S}\gamma$ and we claim that $\mathbb{Z}_p[X_n] \notin S\gamma$. Since $\gamma = \boldsymbol{U}(\{\mathbb{Z}_p\})$ is a special radical,

$$\gamma(\mathbb{Z}_p[X_n]) = \cap\{I \trianglelefteq \mathbb{Z}_p[X_n] : \mathbb{Z}_p[X_n]/I \cong \mathbb{Z}_p\}$$

and so we can see that, for $x_i \in X_n$, $0 \neq x_i^p - x_i \in \gamma(\mathbb{Z}_p[X_n])$, since every $a \in \mathbb{Z}_p$ satisfies the polynomial equation $x_i^p - x_i = 0$ (see [2, 6]). Therefore the natural question arises as to whether $n$-Amitsur rings exist.

## 2. $n$-Amitsur Rings and Radicals

In this section, we show that $n$-Amitsur rings do indeed exist and we also provide some characterizations. Moreover, we shall introduce new radicals, associated with these rings, and study some of their properties.

We shall require the following known result:

**Proposition 2.1.**([6, 13]) *If $\gamma$ is a radical, then $(\gamma(A[X_n]) \cap A)[X_n] \subseteq \gamma(A[X_n])$ for any ring $A$.*

For any radical $\gamma$, consider the radical class

$$\gamma_n = \{A : A \text{ is a ring with } A[X_n] \in \gamma\},$$

defined by Tumurbat and Wisbauer in [13]. It is clear that, for any ring $A$ and any radical $\gamma$, $\gamma_n(A) = \gamma(A)$ if and only if $\gamma(A)[X_n] \in \gamma$ for every radical $\gamma$.

Using the method of proof of ([4] , Proposition 4.9.18), we may show the following:

**Proposition 2.2.** *If $\gamma$ is a hereditary radical, then $\gamma_n(A) = \gamma(A[X_n]) \cap A$ for any ring $A$.*

**Proposition 2.3.** *If $A$ is an $n$-Amitsur (respectively, hereditary $n$-Amitsur) ring, then*

$$\gamma_n(A) = \gamma(A[X_n]) \cap A \subseteq \gamma(A),$$

*for any radical (respectively, hereditary radical) $\gamma$.*

*Proof.* We prove the result for the case when $A$ is an $n$-Amitsur ring. Let $\gamma$ be an arbitrary radical. We have $\gamma(A[X_n]) = (\gamma(A[X_n]) \cap A)[X_n]$. Notice that since $\gamma_n(A) \in \gamma_n$, it is clear that $\gamma_n(A)[X_n] \in \gamma$. Therefore $\gamma_n(A)[X_n] \subseteq \gamma(A[X_n])$.

Now

$$\begin{aligned}\gamma\left(\left(A/\gamma_n(A)\right)[X_n]\right) &\cong \gamma\left(A[X_n]/\gamma_n(A)[X_n]\right)\\ &= \gamma\left(A[X_n]\right)/\gamma_n(A)[X_n]\\ &= \left(\gamma\left(A[X_n]\right)\cap A\right)[X_n]/\gamma_n(A)[X_n]\\ &\cong \left(\left(\gamma\left(A[X_n]\right)\cap A\right)/\gamma_n(A)\right)[X_n]\end{aligned}$$

Hence $\left(\left(\gamma\left(A[X_n]\right)\cap A\right)/\gamma_n(A)\right)[X_n] \in \gamma$ and so $\left(\gamma\left(A[X_n]\right)\cap A\right)/\gamma_n(A) \in \gamma_n$. Thus $\left(\gamma\left(A[X_n]\right)\cap A\right)/\gamma_n(A) \subseteq \gamma_n\left(A/\gamma_n(A)\right) = \overline{0}$, whence $\gamma\left(A[X_n]\right)\cap A = \gamma_n(A)$.

The assertion $\gamma\left(A[X_n]\right)\cap A \subseteq \gamma(A)$ now follows, taking into account that $\gamma_n \subseteq \gamma$. □

Let $I$ be an ideal of a ring $A$. If there exists a radical (respectively, hereditary radical) $\gamma$ such that $I = \gamma(A)$, then $I$ is called a radical ideal (respectively, $h$-radical ideal) of $A$. From Proposition 2.3, it is clear that if $A$ is an $n$-Amitsur (respectively, hereditary $n$-Amitsur) ring, then $\gamma\left(A[X_n]\right)\cap A$ is a radical ideal (respectively, $h$-radical ideal) of $A$, for any radical (respectively, hereditary radical) $\gamma$.

**Proposition 2.4.** *Let $A$ be an $n$-Amitsur (respectively, hereditary $n$-Amitsur) ring. Then $A \in \boldsymbol{S}\gamma_n$ implies that $A[X_n] \in \boldsymbol{S}\gamma$, for any radical (respectively, hereditary radical) $\gamma$.*

*Proof.* Assume that $A$ is an $n$-Amitsur ring and let $\gamma$ be an arbitrary radical Suppose that $A \in \boldsymbol{S}\gamma_n$. Since

$$\left(\gamma\left(A[X_n]\right)\cap A\right)[X_n] = \gamma\left(A[X_n]\right) \in \gamma,$$

it is clear that $\gamma\left(A[X_n]\right)\cap A \in \gamma_n$. Thus $\gamma\left(A[X_n]\right)\cap A \subseteq \gamma_n(A) = 0$. Hence $\gamma\left(A[X_n]\right) = 0$; that is $A[X_n] \in \boldsymbol{S}\gamma$. The proof is similar when $A$ is a hereditary $n$-Amitsur ring. □

In what follows, for a radical $\gamma$ and a ring $A$, we put

$$\begin{aligned}A_\gamma &= \gamma(A[X_n])\cap A,\\ \overline{A}_\gamma &= A/A_\gamma,\\ \overline{\gamma(A[X_n])} &= \gamma(A[X_n])/A_\gamma[X_n].\end{aligned}$$

**Lemma 2.5.** $\gamma(\overline{A}_\gamma[X_n])\cap\overline{A}_\gamma = \overline{0}$*, for any ring $A$ and any radical $\gamma$.*

*Proof.* Suppose that there exists a radical $\gamma$ and a ring $A$ such that $\gamma((\overline{A}_\gamma[X_n])\cap \overline{A}_\gamma \neq \overline{0}$. Then there is a nonzero element $\overline{a} \in \gamma(\overline{A}_\gamma[X_n])\cap\overline{A}_\gamma$. Hence there exists $0 \neq a \notin A\cap\gamma(A[X_n]$, which is a pre-image of $\overline{a}$. By Proposition 2.1, $((\gamma(A[X_n])\cap A)[X_n] \subseteq \gamma(A[X_n])$. Taking into account that radical classes are closed under extensions, we have $\gamma(\overline{A}_\gamma[X_n]) \cong \overline{\gamma(A[X_n])}$. Consequently, $a \in \gamma(A[X_n]$ and $a \in A$, which is a contradiction. □

**Theorem 2.6.** *For a ring $A$, the following conditions are equivalent:*

(i) *$A$ is a $n$-Amitsur ring;*

(ii) *$\gamma(\overline{A}_\gamma[X_n]) = \overline{0}$ for every radical $\gamma$;*

(iii) *$\gamma_0(\overline{A}_{\gamma_0}[X_n]) = \overline{0}$, where $\gamma_0 = \boldsymbol{L}(H)$ and $H$ is any ideal of $A[X_n]$;*

(iv) *every radical ideal $I$ of $A[X_n]$ is a polynomial ring;*

(v) *$\gamma\,(A[X_n]) = \gamma_n\,(A)\,[X_n]$ for every radical $\gamma$.*

*Proof.* (ii) implies (i). If $\gamma$ is a radical, then we know that $(\gamma(A[X_n]) \cap A)[X_n] \subseteq \gamma(A[X_n]$. Since radical classes are closed under extensions, $\gamma(\overline{A}_\gamma[X_n]) \cong \overline{\gamma(A[X_n])}$. Hence, if $\gamma(\overline{A}_\gamma[X_n]) = \overline{0}$, then $\overline{\gamma(A[X_n])} = \overline{0}$ and so $\gamma(A[X_n]) \subseteq (\gamma(A[X_n]) \cap A)[X_n]$. Therefore $A$ is a $n$-Amitsur ring.

(iii) implies (ii). Let $\gamma$ be an arbitrary radical. Since $\gamma(A[X_n]) \trianglelefteq A[X_n]$, we take $H = \gamma(A[X_n])$ and $\gamma_0 = \boldsymbol{L}(H)$. Then $\gamma_0(A[X_n]) = H = \gamma(A[X_n])$. Thus, if (iii) holds, then $\gamma(\overline{A}_\gamma[X_n]) = \overline{0}$.

(i) implies (iii). This is clear.

(i) implies (iv). Let $I$ be radical ideal of $A[X_n]$. Then $I = \gamma(A[X_n])$ for some radical $\gamma$. If $A$ is a $n$-Amitsur ring, then $\gamma(A[X_n]) = (\gamma(A[X_n]) \cap A)[X_n]$ and so $I$ is a polynomial ring.

(iv) implies (i). Let $\gamma$ be a radical such that $I = \gamma(A[X_n])$, that is, $I$ is a radical ideal of the ring $A[X_n]$. Then, by condition (iv), $I$ is a polynomial ring. Hence there exists $I' \trianglelefteq A$ such that $I'[X_n] = \gamma(A[X_n])$. Now it is easy to see that $I' = A \cap \gamma(A[X_n])$.

(i) implies (v). If $A$ is an $n$-Amitsur ring, then (v) follows from Proposition 2.3.

(v) implies (i). Suppose that $\gamma\,(A[X_n]) = \gamma_n\,(A)\,[X_n]$ for any radical $\gamma$. Then

$$\begin{aligned}\gamma_n\,(A)\,[X_n] &= (\gamma_n\,(A) \cap A)\,[X_n]\\ &\subseteq (\gamma_n\,(A)\,[X_n] \cap A)\,[X_n] = (\gamma\,(A[X_n]) \cap A)\,[X_n].\end{aligned}$$

Taking into account Proposition 2.1, we have the desired result. □

The next theorem may be proved in a similar way to the theorem above, and so we omit its proof. Recall that a subring $I$ of a ring $A$ is called an accessible subring of $A$ if there exists a finite sequence $I_0, I_1, ..., I_n$ of subrings such that $I = I_0 \trianglelefteq I_1 \trianglelefteq \ldots \trianglelefteq I_n = A$.

**Theorem 2.7.** *For a ring $A$, the following statements are equivalent:*

(i) *$A$ is a hereditary $n$-Amitsur ring;*

(ii) *$\gamma(\overline{A}_\gamma[X_n]) = \overline{0}$ for every hereditary radical $\gamma$;*

(iii) *$\gamma_1(\overline{A}_{\gamma_1}[X_n]) = \overline{0}$, where $\gamma_1 = \boldsymbol{L}_h(H)$ and $H$ is any accessible subring of $A[X_n]$;*

(iv) *every h-radical ideal $I$ of $A[X_n]$ is a polynomial ring.*

(v) $\gamma\left(A[X_n]\right) = \gamma_n\left(A\right)\left[X_n\right]$ *for every hereditary radical $\gamma$.*

**Proposition 2.8.** *If $F$ is a finite field, then $F$ is not a hereditary $n$-Amitsur ring, for every natural number $n$.*

*Proof.* If $F$ is a finite field, then $\gamma = \boldsymbol{U}\left(\{F\}\right)$ is a special radical. Now $x_n^{p^m} - x_n \in \gamma(F[X_n])$, where $p^m$ is the number of elements in $F$. Therefore $\gamma(F[X_n]) \nsubseteq (\gamma(F[X_n]) \cap F)[X_n]$ and hence $F$ is not a hereditary $n$-Amitsur ring. □

Following the reasoning in the proof of Proposition 2.6 of [13], we may prove the next result.

**Lemma 2.9.** *Let $Y$ be any subset of $X_n$ of cardinality 1, where $n > 1$. If $A$ is a ring such that $A[X_n\backslash Y]$ is an Amitsur ring (respectively, hereditary Amitsur ring), then $\gamma(A[X_n]) \neq 0$ implies that $A \cap \gamma(A[X_n]) \neq 0$, for any radical (respectively, hereditary radical) $\gamma$.*

*Proof.* Let $\gamma$ be an arbitray radical and $A$ be a ring such that $A[X_n\backslash Y]$ is a an Amitsur ring. Assume that $\gamma(A[X_n]) \neq 0$. Since $A[X_n\backslash Y]$ is an Amitsur ring,

$$\gamma(A[X_n]) = \gamma((A[X_n\backslash Y])[Y]) = (A[X_n\backslash Y] \cap \gamma(A[X_n]))[Y].$$

For any $0 \neq a \in \gamma(A[X_n])$, there exist elements $x_{i_1}, ..., x_{i_{n(a)}} \in X_n$ and $a_{\alpha_1,...,\alpha_{n(a)}} \in A$ such that

$$a = \sum a_{\alpha_1,...,\alpha_{n(a)}} x_{i_1}^{\alpha_1} ... x_{i_{n(a)}}^{\alpha_{n(a)}}$$

where, for each $x_{i_j}$ there exists an exponent $\alpha_j \neq 0$ such that

$$a_{\alpha_1,...,\alpha_{n(a)}} x_{i_1}^{\alpha_1} ... x_{i_{n(a)}}^{\alpha_{n(a)}} \neq 0$$

or $a \in A$. The number of nonzero summands of $a$ is called the length of $a$ and is denoted by $\ell(a)$.

Suppose that $A \cap \gamma(A[X_n]) = 0$. For each $0 \neq a \in \gamma(A[X_n])$, $\ell(a) \geqslant 1$. Choose $0 \neq a \in \gamma(A[X_n])$ with $\ell(a)$ minimal. If $a$ depends on only one indeterminate, then the coefficients of $a$ belong to $A \cap \gamma(A[X_n]) = 0$ and so $a = 0$, which is a contradiction. Therefore $a$ depends on at least two indeterminates, that is, $n(a) \geqslant 2$. We can write

$$0 \neq a = \sum f_{\alpha_{n(a)}}(x_{i_1} ... x_{i_{n(a)-1}}) x_{i_{n(a)}}^{\alpha_{n(a)}} \in (A[X_n\backslash Y] \cap \gamma(A[X_n]))[Y],$$

where $Y = \left\{x_{i_{n(a)}}\right\}$ and $f_{\alpha_{n(a)}} = f_{\alpha_{n(a)}}(x_{i_1} ... x_{i_{n(a)-1}}) \in A[X_n\backslash Y] \cap \gamma(A[X_n])$. If $\ell(f_{\alpha_{n(a)}}) = \ell(a)$, take $f_{\alpha_{n(a)}}$ instead of $a$. The number of indeterminates in $f_{\alpha_{n(a)}}$ is less than the number of indeterminates in $a$. Continuing with this procedure, we can find $0 \neq f_k \in \gamma(A[X_n])$ such that either $f_k \in A[Y])$ for some subset $Y$ of $X_n$ of cardinality 1, or $\ell(f_k) < \ell(a)$. In the first case, $f_k \in A \cap \gamma(A[X_n]) = 0$, which is a contradiction. The second case contradicts the minimality of $\ell(a)$. Thus we conclude that $A \cap \gamma(A[X_n]) \neq 0$, as desired. □

Let $n$ be a natural number. We recall that a class $\alpha$ of rings said to be $n$-polynomially extensible if $A[X_n] \in \alpha$ for all $A \in \alpha$. If $n = 1$, the class $\alpha$ of rings is said to be polynomially extensible. It is clear that if a class $\alpha$ of rings is polynomially extensible, then it is $n$-polynomially extensible, for any natural number $n$.

For any ring $A$, let $A^0$ be the zero-ring on the additive group of $A$, that is, $A^0$ has the additive group of $A$ and multiplication defined by $ab = 0$ for all $a, b \in A$. Clearly, the class of all zero-rings is $n$-polynomially extensible, for any natural number $n$. Moreover, in [9], it was proved that every zero-ring is an Amitsur ring. We are now in a position to prove the following proposition.

**Proposition 2.10.** *For any ring $A$, the ring $A^0$ is an $n$-Amitsur ring.*

*Proof.* Let $\gamma$ be an arbitrary radical. From Proposition 2.1,

$$(A^0 \cap \gamma(A^0[X_n]))[X_n] \subseteq \gamma(A^0[X_n]).$$

Suppose that

$$\gamma(A^0[X_n]) \nsubseteqq (A^0 \cap \gamma(A^0[X_n]))[X_n].$$

Then $\gamma(\overline{A^0}_\gamma[X_n]) \neq \overline{0}$. Hence, by the previous lemma, $A^0 \cap \gamma(A^0[X_n] \neq \overline{0}$, which is a contradiction with Lemma 2.5. □

The Baer radical class is the upper radical determined by the class of all prime rings and also coincides with the upper radical determined by the class of all semiprime rings. Rings belonging to the Baer radical class are called Baer radical rings. It is well known that the Baer radical class is $n$-polynomially extensible, for any natural number $n$. In addition, it was proved in [9] that every Baer radical ring is a hereditary Amitsur ring. Hence we have:

**Proposition 2.11.** *If $A$ is a Baer radical ring, then $A$ is a hereditary $n$-Amitsur ring.*

*Proof.* $\overline{A}_\gamma$ is a Baer radical ring and therefore $\overline{A}_\gamma\,[X_n]$ is also a Baer radical ring. If $\gamma(\overline{A}_\gamma\,[X_n]) \neq \overline{0}$, then, by the previous lemma, $\overline{A}_\gamma \cap \gamma(\overline{A}_\gamma[X_n]) \neq \overline{0}$. This is a contradiction with Lemma 2.5. Thus $\gamma(\overline{A}_\gamma\,[X_n]) = \overline{0}$. Hence, by Theorem 2.7, $A$ is a hereditary $n$-Amitsur ring. □

The following lemma is required in the proof of the next proposition.

**Lemma 2.12.**([8]) *Let $A$ be an infinite integral domain, and let $S = A[x_i : i \in \Lambda]$ be the polynomial ring over $A$ with $|\Lambda|$ commuting indeterminates. For every radical class $\gamma$, $A \cap \gamma(S) = 0$ if and only if $\gamma(S) = 0$.*

**Proposition 2.13.** *If $A$ is an infinite integral domain whose every proper homomorphic image is a Baer radical ring, then $A$ is a hereditary $n$-Amitsur ring.*

*Proof.* Let $A$ be an infinite integral domain without proper prime homomorphic images. Let $\gamma$ be a hereditary radical and suppose that $\gamma(A[X_n]) \neq 0$. By the lemma above, $A \cap \gamma(A[X_n]) \neq 0$. We also know that $(A \cap \gamma(A[X_n]))[X_n] \subseteq \gamma(A[X_n])$. If $\gamma(A[X_n]) \nsubseteq (A \cap \gamma(A[X_n]))[X_n]$, then we have $\overline{0} \neq \overline{\gamma(A[X_n])} = \gamma(A[X_n])/(A \cap \gamma(A[X_n]))[X_n] \trianglelefteq A[X_n]/(A \cap \gamma(A[X_n]))[X_n]$. By assumption, $\overline{A}_\gamma$ is a Baer radical ring and hence $\overline{A}_\gamma[X_n]$ and $\overline{\gamma(A[X_n])} \cong \gamma\left(\overline{A}_\gamma[X_n]\right)$ are Baer radical rings. Then $\overline{\gamma(A[X_n])}$ is not a semiprime ring and so it has a nonzero ideal $\overline{I}$ such that $\overline{I}^2 = \overline{0}$. Let $\overline{I}_\gamma$ be the ideal of $\overline{A}_\gamma$ generated by the coefficients of $\overline{g(x_1, ..., x_n)} \in \overline{I}$. Therefore

$$\overline{0} \neq \overline{I} \subseteq \overline{I}_\gamma[X_n] \cap \gamma\left(\overline{A}_\gamma[X_n]\right) = \gamma\left(\overline{I}_\gamma[X_n]\right),$$

since $\gamma$ is hereditary Then, from the lemma above, $\overline{I}_\gamma \cap \gamma(\overline{I}_\gamma[X_n]) \neq \overline{0}$, where $\overline{I}_\gamma \cap \gamma(\overline{I}_\gamma[X_n]) \subseteq \overline{A}_\gamma \cap \gamma(\overline{A}_\gamma[X_n])$. However, from Lemma 2.5, we have $\overline{A}_\gamma \cap \gamma(\overline{A}_\gamma[X_n]) = \overline{0}$, which is a contradiction. □

**Example 2.14.**([4])

(i) The ring

$$W = \left\{\frac{2x}{2y+1} : \ x \text{ and } y \text{ are integers and } (2x, 2y+1) = 1\right\}$$

is an infinite integral domain and every proper homomorphic image of $W$ is a Baer radical ring. Therefore $W$ is a hereditary $n$-Amitsur ring.

(ii) Taking into account Lemma 2.12, it can be easily seen that any infinite field is an $n$-Amitsur ring.

Next, we introduce the following classes of rings, which rely on the concept of $n$-Amitsur ring:

$\mathcal{T}_n = \{A : A$ is a ring whose nonzero prime homomorphic images are not hereditary $n$-Amitsur rings$\}$,

$\mathcal{T}_{ns} = \{A : A$ is a ring whose prime homomorphic images have no nonzero ideals that are hereditary $n$-Amitsur rings$\}$,

It is easy to see that each of the classes $\mathcal{T}_n$ and $\mathcal{T}_{ns}$ is homomorphically closed and that $\mathcal{T}_{ns} \subseteq \mathcal{T}_n$. Moreover, we will show that they are radical classes. For this purpose, we require the following class of rings:

$\tau_n = \{A : A$ is a prime and hereditary $n$-Amitsur ring$\}$,

**Lemma 2.15.** *$\tau_n$ is a hereditary class of rings.*

*Proof.* Suppose that $0 \neq I \trianglelefteq A \in \tau_n$. Let $\gamma$ be a hereditary radical. Since $A$ is a prime ring, $A[X_n]$ is also a prime ring. So, if $\gamma(A[X_n]) \neq 0$, then $\gamma(I[X_n]) = I[X_n] \cap \gamma(A[X_n]) \neq 0$. Moreover, all the coefficients of any $g(x_1, \ldots, x_n) \in \gamma(I[X_n])$ are in $I$ and also in $\gamma(I[X_n])$ and therefore $\gamma(I[X_n]) \subseteq (I \cap \gamma(I[X_n]))[X_n]$. Taking into account Proposition 2.1, $\gamma(I[X_n]) = (I \cap \gamma(I[X_n]))[X_n]$, as desired. □

Let us recall that the essential cover of a class $\alpha$ of rings, denoted by $\epsilon\alpha$, is the class of all rings that contain an essential ideal in $\alpha$.

**Lemma 2.16.** *The essential cover $\epsilon\tau_n$ is a special class of rings.*

*Proof.* Since $\tau_n$ is a hereditary class of prime rings, $\epsilon\tau_n$ is also a hereditary class of prime rings. It remains to show that the class $\epsilon\tau_n$ is essentially closed. Let $A \trianglelefteq^{\bullet} B$ and $A \in \epsilon\tau_n$. Then there exists a nonzero ideal $I$ of $A$ such that $I \triangleleft^{\bullet} A$ and $I \in \tau_n$. Since $I$ is a prime ring, $A$ is a prime ring and hence $B$ is also a prime ring, because the class of prime rings is essentially closed. Let $I_B$ be the ideal of $B$ generated by $I$. By Andrunakievich's Lemma, $I_B^3 \subseteq I$, where $0 \neq I_B^3 \trianglelefteq^{\bullet} B$. By Lemma 2.15, $I_B^3$ is a hereditary $n$-Amitsur ring. Thus $B \in \epsilon\tau_n$. □

**Corollary 2.17.** $\epsilon\tau_n = \{A : A$ *is a prime ring having a nonzero ideal $I$ that is a hereditary $n$-Amitsur ring*$\}$.

**Theorem 2.18.** *$\mathcal{T}_n$ is a radical class, and $\mathcal{T}_{ns}$ is a special radical class.*

*Proof.* We claim that $\boldsymbol{U}(\tau_n) = \mathcal{T}_n$. If $A \in \mathcal{T}_n$, then every nonzero prime homomorphic image of $A$ is not a hereditary $n$-Amitsur ring, that is, $A \in \boldsymbol{U}(\tau_n)$. Therefore $\mathcal{T}_n \subseteq \boldsymbol{U}(\tau_n)$. Let $A \in \boldsymbol{U}(\tau_n)\backslash\mathcal{T}_n$. Since $A \notin \mathcal{T}_n$, there is a nonzero prime homomorphic image $\overline{A}$ of $A$ which is a hereditary $n$-Amitsur ring. Hence $\overline{A} \in \tau_n \cap \boldsymbol{U}(\tau_n)$ and so $\overline{A} = \overline{0}$, which is a contradiction. Therefore $\boldsymbol{U}(\tau_n) \subseteq \mathcal{T}_n$. Lemma 2.15, now yields that $\boldsymbol{U}(\tau_n) = \mathcal{T}_n$ is a radical class.

Next, we show that $\boldsymbol{U}(\epsilon\tau_n) = \mathcal{T}_{ns}$. Suppose that $\boldsymbol{U}(\epsilon\tau_n) \nsubseteq \mathcal{T}_{ns}$. Then there exists $0 \neq A \in \boldsymbol{U}(\epsilon\tau_n)$ such that $A \notin \mathcal{T}_{ns}$. Since $A \notin \mathcal{T}_{ns}$, there is a homomorphic image $\overline{A}$ of $A$ such that $\overline{0} \neq \overline{A} \in \epsilon\tau_n$. Therefore $\overline{0} \notin \overline{A} \in \epsilon\tau_n \cap \boldsymbol{U}(\epsilon\tau_n) = 0$, which is a contradiction. Hence $\boldsymbol{U}(\epsilon\tau_n) \subseteq \mathcal{T}_{ns}$. Now suppose that $\mathcal{T}_{ns} \nsubseteq \boldsymbol{U}(\epsilon\tau_n)$. Then there exists $0 \neq A \in \mathcal{T}_{ns}$ such that $A \notin \boldsymbol{U}(\epsilon\tau_n)$. Since $A \notin \boldsymbol{U}(\epsilon\tau_n)$, there is a nonzero prime homomorphic image $\overline{A}$ of $A$ such that $\overline{A} \in \epsilon\tau_n$. Since $\mathcal{T}_{ns}$ is homomorphically closed, $\overline{A} \in \mathcal{T}_{ns}$. On the other hand, every homomorphic image of $A$ is not in $\epsilon\tau_n$, which is a contradiction. Thus $\mathcal{T}_{ns} \subseteq \boldsymbol{U}(\epsilon\tau_n)$. By Lemma 2.16, $\epsilon\tau_n$ is a special class of rings and so $\mathcal{T}_{ns} = \boldsymbol{U}(\epsilon\tau_n)$ is a special radical. □

**Theorem 2.19.** *The semisimple class $\boldsymbol{S}\mathcal{T}_{ns}$ is $n$-polynomially extensible.*

*Proof.* Let $A \in \boldsymbol{S}\mathcal{T}_{ns}$. By Theorem 2.18, $\mathcal{T}_{ns}$ is a special radical. Therefore there exist ideals $H_i$ $(i \in \Lambda)$ of $A$ such that, for each $i \in \Lambda$, $B_i = A/H_i \in \epsilon\tau_n$ and $\cap_{i\in\Lambda} H_i = 0$. We claim that $\mathcal{T}_{ns}(A[X_n]) \subseteq H_i[X_n]$, for any $i \in \Lambda$. If $\mathcal{T}_{ns}(A[X_n]) \nsubseteq H_i[X_n]$ for some $i \in \Lambda$, then

$$\mathcal{T}_{ns}((A/H_i)[X_n]) \cong \mathcal{T}_{ns}\left(\frac{A[X_n]}{H_i[X_n]}\right) \neq \overline{0}.$$

Thus $0 \neq \mathcal{T}_{ns}(B_i[X_n])$. Since $\epsilon\tau_n$ is a special class of rings, $A/H_i$ is a prime ring. On the other hand, there exists an ideal $J_i$ of $B_i$ such that $0 \neq J_i \trianglelefteq^{\bullet} B_i$ and $J_i$ is a prime hereditary $n$-Amitsur ring. We have, for any $i \in \Lambda$,

$$0 \neq \mathcal{T}_{ns}(J_i[X_n]) = (J_i \cap \mathcal{T}_{ns}(J_i[X_n])[X_n] \in \mathcal{T}_{ns}.$$

Thus $0 \neq J_i \cap \mathcal{T}_{ns}(J_i[X_n]) \in \mathcal{T}_{ns}$, where $J_i \cap \mathcal{T}_{ns}(J_i[X_n]) \trianglelefteq B_i$. But $J_i \cap \mathcal{T}_{ns}(J_i[X_n]) \subseteq \mathcal{T}_{ns}(B_i) = 0$, which is contradiction. Thus $B_i[X_n] \in \boldsymbol{S}\mathcal{T}_{ns}$ and consequently

$\mathcal{T}_{ns}(A[X_n] \subseteq H_i[X_n]$, for any $i \in \Lambda$. If $0 \neq \mathcal{T}_{ns}(A[X_n])$, then there exists $0 \neq g(X_n) \in \mathcal{T}_{ns}(A[X_n])$. But all the coefficients $a_i$ of $g(X_n)$ are in $\cap_{i\in\Lambda} H_i = 0$. Hence $a_i = 0$ and so $\mathcal{T}_n(A[X_n]) = 0$, that is, $A[X_n] \in \boldsymbol{S}\mathcal{T}_{ns}$. □

We recall from [5] that a radical $\gamma$ is called *small* if $\gamma \vee \gamma' \neq A_{ss}$ for each proper radical $\gamma'$, where $A_{ss}$ denotes the class of all associative rings. Dually, a nonzero radical $\gamma$ is large if $\gamma \cap \gamma' \neq 0$ for each proper radical $\gamma'$.

Let $\mathbb{L}_s$ denote the collection of all strongly hereditary (that is, closed under subrings) and large radicals, and let $\mathbb{L}$ denote the collection of all radicals $\gamma$ such that $\gamma \cap \gamma_\alpha \neq 0$ for every $\gamma_\alpha \in \mathbb{L}_s$. In [10], S. Tumurbat et al. proved that $\mathbb{L}$ is a complete sublattice in the lattice of all radicals. Now we have the following:

**Proposition 2.20.** *Each of the radicals $\mathcal{T}_n$ and $\mathcal{T}_{ns}$ belongs to $\mathbb{L}$.*

*Proof.* The radical $\mathcal{T}_{ns}$ contains all fields $\mathbb{Z}_p$ and all rings $\mathbb{Z}_p^0$ of prime order $p$ (where $\mathbb{Z}_p^0$ denotes the ring with the additive group of $\mathbb{Z}_p$ and with multiplication defined by $ab = 0$ for all $a, b \in \mathbb{Z}_p$). In [10], it was shown that every nonzero strongly hereditary radical contains a prime field or a simple zero ring of prime order. Hence $\mathcal{T}_{ns} \cap \gamma_\alpha \neq 0$ for every $\gamma_\alpha \in \mathbb{L}_s$. Thus $\mathcal{T}_{ns} \in \mathbb{L}$ and also $\mathcal{T}_n \in \mathbb{L}$. □

**Definition 2.21.** We call a radical $\gamma$ an *n-bad radical* if, for every prime ring $A \in \gamma$, there exists a hereditary radical $\gamma_A$ such that $\gamma_A(A[X_n]) \neq (\gamma_A(A[X_n]) \cap A)[X_n]$.

In what follows, $\mathbb{L}_b$ and $\mathbb{L}_{sb}$ denote, respectively, the class of all $n$-bad radicals and the class of all special radicals that are $n$-bad radicals.

**Proposition 2.22.** *The classes $\mathbb{L}_b$ and $\mathbb{L}_{sb}$ satisfy the following conditions:*

(i) *If $\gamma_i \in \mathbb{L}_b$ for each $i \in \Lambda$, then $\bigcap_{i\in\Lambda} \gamma_i \in \mathbb{L}_b$;*

(ii) *If $\gamma_i \in \mathbb{L}_{sb}$ for each $i \in \Lambda$, then $\bigcap_{i\in\Lambda} \gamma_i \in \mathbb{L}_{sb}$.*

*Proof.* (i) Let $A$ be a prime ring in $\bigcap_{i\in\Lambda} \gamma_i$. For each $i \in \Lambda$, there exists a hereditary radical $(\gamma_i)_A$ such that

$$(\gamma_i)_A(A[X_n]) \neq ((\gamma_i)_A(A[X_n]) \cap A)[X_n].$$

Then we may choose any one of $(\gamma_i)_A$, $i \in \Lambda$.

Statement (ii) may be proved in a similar way. □

**Theorem 2.23.** *Both $\mathbb{L}_b$ and $\mathbb{L}_{sb}$ are complete sublattices in the lattice of all radicals.*

*Proof.* We have $\gamma_i \subseteq \mathcal{T}_n$ for each $n$-bad radical $\gamma_i$. Therefore $\mathcal{L}(\cup\gamma_i) \subseteq \mathcal{T}_n$ and $\mathcal{L}(\cup\gamma_i) \in \mathbb{L}_b$. By Proposition 2.22, $\cap\gamma_i \in \mathbb{L}_b$. Thus $\mathbb{L}_b$ is a complete sublattice in the lattice of all radicals. The proof that $\mathbb{L}_{sb}$ is a complete sublattice in the lattice of all radicals is similar. □

In [3], B.J. Gardner, J. Krempa and R. Wiegandt posed the question as to whether there exists a (hereditary) radical $\gamma$ with polynomially extensible semisimple class $\boldsymbol{S}\gamma$ such that $\gamma$ does not have the Amitsur property. If $\mathcal{T}_{1s}$ does not have the Amitsur property, then we have a positive answer to this question.

## References


[1] N. Divinsky and A. Suliński, *Kurosh radicals of rings with operators*, Canad. J. Math., **17**(1965), 278–280.

[2] H. France-Jackson, T. Khulan and S. Tumurbat, *On α-like radicals of rings*, Bull. Aust. Math. Soc., **88(2)**(2013), 331–339.

[3] B. J. Gardner, J. Krempa and R. Wiegandt, *Open problems in radical theory*, Algebra Discrete Math., **(3)**(2007), 15–17.

[4] B. J. Gardner and R. Wiegandt, *Radical theory of rings*, Marcel Dekker, New York, 2004.

[5] B. J. Gardner and L. Zhian, *Small and large radical classes*, Comm. Algebra, **20**(1992), 2533–2551.

[6] J. Krempa, *On radical properties of polynomial rings*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., **20**(1972), 545–548.

[7] N. V. Loi and R. Wiegandt, *On the Amitsur property of radicals*, Algebra Discrete Math., **(3)**(2006), 92–100.

[8] N. R. McConnell and T. Stokes, *Radical ideals of radically simple rings and their extensions*, Theory of radicals, 185–196, Colloquia Mathematica Societatis Janos Bolyai **61**, North-Holland, Amsterdam, 1993.

[9] S. Tumurbat, *On Amitsur rings*, Quaest. Math., **42(5)**(2019), 665–672.

[10] S. Tumurbat, D. Dayantsolmon and T. Khulan, *Notes on chain rings and radicals*, Kyungpook Math. J., **58**(2018), 473–479.

[11] S. Tumurbat and R. Wiegandt, *Radicals of polynomial rings*, Soochow J. Math., **29(4)**(2003), 425–434.

[12] S. Tumurbat and R. Wiegandt. *On radicals with Amitsur property*, Comm. Algebra, **32(3)**(2004), 1219–1227.

[13] S. Tumurbat and R. Wisbauer, *Radicals with the α-Amitsur property*, J. Algebra Appl., **7(3)**(2008), 347–361.

[14] R. Wiegandt, *Radical and semisimple classes of rings,* Queen's Papers in Pure and Applied Mathematics **37**, Queen's University, Kingston Ontario, 1974.