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Characterizations of Lie Triple Higher Derivations of Triangular Algebras by Local Actions

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Abstract. Let $\mathbb{N}$ be the set of nonnegative integers and $\mathfrak{A}$ be a 2-torsion free triangular algebra over a commutative ring $\mathcal{R}$. In the present paper, under some lenient assumptions on $\mathfrak{A}$, it is proved that if $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{R}$-linear mappings $\delta_n : \mathfrak{A} \to \mathfrak{A}$ satisfying $\delta_n([[x, y], z]) = \sum\limits_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$ for all $x, y, z \in \mathfrak{A}$ with $xy = 0$ (resp. $xy = p$, where p is a nontrivial idempotent of $\mathfrak{A}$), then for each $n \in \mathbb{N}$, $\delta_n = d_n + \tau_n$; where $d_n : \mathfrak{A} \to \mathfrak{A}$ is $\mathcal{R}$-linear mapping satisfying $d_n(xy) = \sum\limits_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in \mathfrak{A}$, i.e. $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is a higher derivation on $\mathfrak{A}$ and $\tau_n : \mathfrak{A} \to Z(\mathfrak{A})$ (where $Z(\mathfrak{A})$ is the center of $\mathfrak{A}$) is an $\mathcal{R}$-linear map vanishing at every second commutator $[[x, y], z]$ with $xy = 0$ (resp. $xy = p$).

1. Introduction and Preliminaries

Let $\mathcal{R}$ be a commutative ring with unity, $\mathcal{A}$ be an algebra over $\mathcal{R}$ and $Z(\mathcal{A})$ be the center of $\mathcal{A}$. Recall that an $\mathcal{R}$-linear map $d : \mathcal{A} \to \mathcal{A}$ is called a derivation on $\mathcal{A}$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathcal{A}$. An $\mathcal{R}$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ is called a Lie derivation on $\mathcal{A}$ if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ holds for all $x, y \in \mathcal{A}$, where $[x, y] = xy - yx$ is the usual Lie product. An $\mathcal{R}$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ is called a Lie triple derivation on $\mathcal{A}$ if $\delta([[x, y], z]) = [[\delta(x), y], z] + [[x, \delta(y)], z] + [[x, y], \delta(z)]$ holds for all $x, y, z \in \mathcal{A}$. It is easy to check that every derivation on $\mathcal{A}$ is a Lie derivation on $\mathcal{A}$ and that every Lie derivation on $\mathcal{A}$ is a Lie triple derivation on $\mathcal{A}$. However, the converse need not be true in general. For example if we consider the algebra $\mathcal{A}$ of all 3×3 strictly upper triangular matrices over $\mathbb{Z}$, the ring of integers, and define

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a map $L : \mathcal{A} \to \mathcal{A}$ such that $L\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$.
Then it can easily be seen that L is a Lie triple derivation on $\mathcal{A}$ which is neither a derivation nor a Lie derivation on $\mathcal{A}$. Now, let $\mathbb{N}$ be the set of nonnegative integers and $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{R}$-linear mappings $\delta_n : \mathcal{A} \to \mathcal{A}$ such that $\delta_0 = id_{\mathcal{A}}$, the identity map on $\mathcal{A}$. Then Δ is said to be

(i) a higher derivation on $\mathcal{A}$ if

$$\delta_n(xy) = \sum_{i+j=n} \delta_i(x)\delta_j(y), \quad \text{for all } \ x, y \in \mathcal{A} \ \ \& \ \ n \in \mathbb{N};$$

(ii) a Lie higher derivation on $\mathcal{A}$ if

$$\delta_n([x, y]) = \sum_{i+j=n} [\delta_i(x), \delta_j(y)], \quad \text{for all } \ x, y \in \mathcal{A} \ \ \& \ \ n \in \mathbb{N};$$

(iii) a Lie triple higher derivation on $\mathcal{A}$ if

$$\delta_n([[x, y], z]) = \sum_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)], \quad \text{for all } \ x, y, z \in \mathcal{A} \ \ \& \ \ n \in \mathbb{N}.$$

It is also easy to observe that there exists Lie triple higher derivation on an algebra $\mathcal{A}$ which is not a Lie higher derivation on $\mathcal{A}$. For example consider the algebra $\mathcal{A}$ of all 3×3 strictly upper triangular matrices over the field $\mathcal{Q}$ of rational numbers, and consider the sequence $\mathcal{L} = \{L_n\}_{n\in\mathbb{N}}$ of linear mappings $L_n : \mathcal{A} \to \mathcal{A}$ such that $L_n = \frac{L^n}{n!}$, where L is Lie triple derivation on $\mathcal{A}$ which is not a Lie derivation on $\mathcal{A}$. Then by using induction on n, it can be easily verified that $\mathcal{L}$ is a Lie triple higher derivation on $\mathcal{A}$ but not a Lie higher derivation on $\mathcal{A}$.

The $\mathcal{R}$-algebra $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$ under the usual matrix operations is called a triangular algebra, where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras over $\mathcal{R}$ and $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule. Recall that a left (resp. right) $\mathcal{A}$-module $\mathcal{M}$ is faithful if $a\mathcal{M} = 0$ (resp. $\mathcal{M}a = 0$) implies that $a = 0$ for every $a \in \mathcal{A}$. The notion of triangular ring was first introduced by Chase [5] in 1960. Further, in the year 2000, Cheung [7] initiated the study of linear maps on triangular algebras. He described Lie derivations, commuting maps and automorphisms of triangular algebras (see for reference [8, 9]).

In the recent years, derivation and Lie derivation have been studied by several authors (see [1, 2, 3, 4, 9, 10, 12, 14, 16, 19, 20]) in various directions. One direction of investigation is to study the conditions under which derivations and Lie derivations can be completely determined by the action on some subsets of $\mathcal{A}$. We say that an $\mathcal{R}$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ is derivable at a given point $c \in \mathcal{A}$ if $\delta(x)y + x\delta(y) = \delta(c)$ for every $x, y \in \mathcal{A}$ with $xy = c$ and such c is called a derivable point of $\mathcal{A}$. This kind of maps were discussed by several authors (see [6, 15, 22]). Similarly, an $\mathcal{R}$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ is said to be a Lie derivable at a given point $c \in \mathcal{A}$ if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$ with $xy = c$. Lu and Jing [18] discussed such maps on $B(X)$ where X is a Banach space with $\dim X \geq 3$ and $B(X)$ is the algebra of all bounded linear operators acting on X and proved that if δ is Lie derivable at $c = 0$ (resp. $c = p$, where p is a fixed nontrivial idempotent of $B(X)$), then $\delta = d + \tau$, where d is a derivation of $B(X)$ and $\tau : B(X) \to \mathbb{C}I$ is a linear map vanishing at every commutator $[x, y]$ with $xy = 0$ (resp. $xy = p$). Ji and Qi [13] investigated this problem on triangular algebras and obtained that under some mild conditions on $\mathfrak{A}$, if $L : \mathfrak{A} \to \mathfrak{A}$ is an $\mathcal{R}$-linear map satisfying $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for any $x, y \in \mathfrak{A}$ with $xy = 0$ (resp. $xy = p$, where p is a fixed nontrivial idempotent of $\mathfrak{A}$), then $\delta = d + \tau$, where d is a derivation of $\mathfrak{A}$ and $\tau : \mathfrak{A} \to Z(\mathfrak{A})$ (where $Z(\mathfrak{A})$ is the center of $\mathfrak{A}$) is an $\mathcal{R}$-linear map vanishing at commutators $[x, y]$ with $xy = 0$ (resp. $xy = p$). Furthermore, in [17] Liu analysed Lie triple derivation on factor von Neumann algebra $\mathcal{A}$ of dimension greater than one and found that if a linear map $\delta : \mathcal{A} \to \mathcal{A}$ satisfies $\delta([[x, y], z]) = [[\delta(x), y], z] + [[x, \delta(y)], z] + [[x, y], \delta(z)]$ for any $x, y, z \in \mathcal{A}$ with $xy = 0$ (resp. $xy = p$, where p is a fixed nontrivial projection of $\mathcal{A}$), then there exist an operator $r \in \mathcal{A}$ and a linear map $f : \mathcal{A} \to \mathbb{C}I$ (where $\mathbb{C}I$ is the center of $\mathcal{A}$) vanishing at every second commutator $[[x, y], z]$ with $xy = 0$ (resp. $xy = p$) such that $\delta(x) = xr - rx + f(x)$ for any $x \in \mathcal{A}$.

An $\mathcal{R}$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ is Lie triple derivable at a given point $c \in \mathcal{A}$ if $\delta([[x, y], z]) = [[\delta(x), y], z] + [[x, \delta(y)], z] + [[x, y], \delta(z)]$ for all $x, y, z \in \mathcal{A}$ with $xy = c$. It is obvious that the condition of being a Lie triple derivable map at some point is much weaker than the condition of being a Lie triple derivation. So far, there has been no result on the study of the local actions of Lie triple derivations on triangular algebras. Motivated by these observations, the purpose of the present paper is to characterize the additive mapping δ_n on triangular algebra $\mathfrak{A}$ satisfying $\delta_n([[x, y], z]) = \sum\limits_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$ for any $x, y, z \in \mathfrak{A}$ with $xy = 0$ (resp. $xy = p$, where p is a fixed nontrivial idempotent).

Throughout the present paper $\mathfrak{A}$ will denote a triangular algebra which is 2-torsion free. Define two natural projections $\pi_{\mathcal{A}} : \mathfrak{A} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathfrak{A} \to \mathcal{B}$ by $\pi_{\mathcal{A}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = a$ and $\pi_{\mathcal{B}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = b$. The center of $\mathfrak{A}$ coincides with

$$Z(\mathfrak{A}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a \in \mathcal{A}, b \in \mathcal{B}, am = mb \text{ for all } m \in \mathcal{M} \right\}.$$

Moreover, $\pi_{\mathcal{A}}(Z(\mathfrak{A})) \subseteq Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathfrak{A})) \subseteq Z(\mathcal{B})$, and there exists a unique algebra isomorphism $\eta : \pi_{\mathcal{B}}(Z(\mathfrak{A})) \to \pi_{\mathcal{A}}(Z(\mathfrak{A}))$ such that $\eta(b)m = mb$ for all $m \in \mathcal{M}$.

Let $1_{\mathcal{A}}$ (resp.$1_{\mathcal{B}}$) be the identity of the algebra $\mathcal{A}$ (resp.$\mathcal{B}$) and let $I = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$ be the identity of triangular algebra $\mathfrak{A}$. Throughout this paper, we shall use the following notations: $p_1 = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}$ and $p_2 = I - p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$. Set $\mathfrak{A}_{11} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a \in \mathcal{A} \right\}$, $\mathfrak{A}_{12} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \middle| m \in \mathcal{M} \right\}$ and $\mathfrak{A}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \middle| b \in \mathcal{B} \right\}$. Then we can write $\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{22}$, where $\mathfrak{A}_{11}$ is a subalgebra of $\mathfrak{A}$ isomorphic to $\mathcal{A}$, $\mathfrak{A}_{22}$ is a subalgebra of $\mathfrak{A}$ isomorphic to $\mathcal{B}$ and $\mathfrak{A}_{12}$ is a $(\mathfrak{A}_{11}, \mathfrak{A}_{22})$-bimodule isomorphic to the bimodule $\mathcal{M}$. To simplify the notation we will use the following convention: $a_{11} \in \mathcal{A} = \mathfrak{A}_{11}$, $a_{22} \in \mathcal{B} = \mathfrak{A}_{22}$ and $a_{12} \in \mathcal{M} = \mathfrak{A}_{12}$. Then each element $x \in \mathfrak{A}$ can be represented in the form $x = a_{11} + a_{12} + a_{22}$, where $a_{11} \in \mathfrak{A}_{11}, a_{22} \in \mathfrak{A}_{22}$ and $a_{12} \in \mathfrak{A}_{12}$.

In what follows, we write a_{ij}, it indicates $a_{ij} \in \mathfrak{A}_{ij}$ and the corresponding element in $\mathcal{A}, \mathcal{B}$ or $\mathcal{M}$. Note that $a_{ij}a_{kl} = 0$ if $j \neq k$.

The proof of the following lemma can be seen in [8, Propositon 3].

Lemma 1.1. *Let $\mathfrak{A}$ be a triangular algebra $Tri(\mathfrak{A}_{11}, \mathfrak{A}_{12}, \mathfrak{A}_{22})$. If $\pi_{\mathfrak{A}_{11}}(Z(\mathfrak{A})) = Z(\mathfrak{A}_{11})$ and $\pi_{\mathfrak{A}_{22}}(Z(\mathfrak{A})) = Z(\mathfrak{A}_{22})$, then there is a unique algebra isomorphism $\eta : Z(\mathfrak{A}_{22}) \to Z(\mathfrak{A}_{11})$ such that $\eta(b) \oplus b \in Z(\mathfrak{A})$ for any $b \in Z(\mathfrak{A}_{22})$.*

2. Characterizations of Lie Triple Higher Derivations by Action on Zero Product

The main result of the present paper states as follows:

Theorem 2.1. *Let $\mathfrak{A} = Tri(\mathfrak{A}_{11}, \mathfrak{A}_{12}, \mathfrak{A}_{22})$ be a 2-torsion free triangular algebra consisting of algebras $\mathfrak{A}_{11}$ and $\mathfrak{A}_{22}$ over a commutative ring $\mathcal{R}$ with unity $1_{\mathfrak{A}_{11}}$ and $1_{\mathfrak{A}_{22}}$ respectively and $\mathfrak{A}_{12}$ be a faithful $(\mathfrak{A}_{11}, \mathfrak{A}_{22})$-bimodule, which is faithful as a left $\mathfrak{A}_{11}$-module and also as a right $\mathfrak{A}_{22}$-module. Suppose that*

(i) *$\pi_{\mathfrak{A}_{11}}(Z(\mathfrak{A})) = Z(\mathfrak{A}_{11})$ and $\pi_{\mathfrak{A}_{22}}(Z(\mathfrak{A})) = Z(\mathfrak{A}_{22})$.*

(ii) *For any $a \in \mathfrak{A}_{11}$, if $[a, \mathfrak{A}_{11}] \in Z(\mathfrak{A}_{11})$, then $a \in Z(\mathfrak{A}_{11})$ or for any $b \in \mathfrak{A}_{22}$, if $[b, \mathfrak{A}_{22}] \in Z(\mathfrak{A}_{22})$, then $b \in Z(\mathfrak{A}_{22})$.*

If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{R}$-linear maps $\delta_n : \mathfrak{A} \to \mathfrak{A}$ such that $\delta_n[[x, y], z] = \sum\limits_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$ for all $x, y, z \in \mathfrak{A}$ with $xy = 0$, then for each $n \in \mathbb{N}$, $\delta_n = d_n + \tau_n$; where $d_n : \mathfrak{A} \to \mathfrak{A}$ is $\mathcal{R}$-linear mapping satisfying $d_n(xy) = \sum\limits_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in \mathfrak{A}$, i.e., $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is a higher derivation of $\mathfrak{A}$ and $\tau_n : \mathfrak{A} \to Z(\mathfrak{A})$ is an $\mathcal{R}$-linear map vanishing at every second commutator $[[x, y], z]$ with $xy = 0$.

The proof of Theorem 2.1 is based on the induction on n. We provide the proof, for $n = 1$, through several claims. Indeed, we show that under the given assumptions of our theorem every Lie triple derivation $\delta_1 = \delta$ on $\mathfrak{A}$ there exists a derivation d on $\mathfrak{A}$ and a linear mapping $\tau : \mathfrak{A} \to Z(\mathfrak{A})$ vanishing on second commutators such that $\delta(x) = d(x) + \tau(x)$ for all $x \in \mathfrak{A}$.

Proof of Theorem 2.1. **Claim 1.** *$p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 \in Z(\mathfrak{A})$; $\delta(a_{12}) = p_1\delta(a_{12})p_2 \in \mathfrak{A}_{12}$ for any $a_{12} \in \mathfrak{A}_{12}$.*

Since $a_{12}p_1 = 0$ for any $a_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}\delta(a_{12}) &= \delta([[a_{12}, p_1], p_1]) \\ &= [[\delta(a_{12}), p_1], p_1] + [[a_{12}, \delta(p_1)], p_1] + [[a_{12}, p_1], \delta(p_1)] \\ &= \delta(a_{12})p_1 - p_1\delta(a_{12})p_1 - p_1\delta(a_{12})p_1 + p_1\delta(a_{12}) + a_{12}\delta(p_1)p_1 \\ &\quad - a_{12}\delta(p_1) + p_1\delta(p_1)a_{12} - a_{12}\delta(p_1) + \delta(p_1)a_{12}.\end{aligned} \tag{2.1}$$

On multiplying the above equality from left by p_1 and right by p_2, we get

$$2\big(p_1\delta(p_1)a_{12} - a_{12}\delta(p_1)p_2\big) = 0.$$

This implies that $p_1\delta(p_1)a_{12} - a_{12}\delta(p_1)p_2 = 0$, that is, $p_1\delta(p_1)p_1a_{12} - a_{12}p_2\delta(p_1)p_2 = 0$, and hence $p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 \in Z(\mathfrak{A})$. By putting $\delta(a_{12}) = r_{11} + r_{12} + r_{22}$, $\delta(p_1) = p_{11} + p_{12} + p_{22}$ in (2.1), we get $r_{11} = 0 = r_{22}$ and so we have $p_i\delta(a_{12})p_i = 0$ for $i \in \{1, 2\}$, $\delta(a_{12}) = p_1\delta(a_{12})p_2$.

Similarly we can get the following result:

Claim 2. $p_1\delta(p_2)p_1 + p_2\delta(p_2)p_2 \in Z(\mathfrak{A})$.

Claim 3. $\delta(I) = p_1\delta(I)p_1 + p_2\delta(I)p_2 \in Z(\mathfrak{A})$.

Since $p_1(I - p_1) = 0$, we have

$$\begin{aligned}0 &= \delta([[p_1, I - p_1], p_1]) \\ &= [[\delta(p_1), I - p_1], p_1] + [[p_1, \delta(I - p_1)], p_1] + [[p_1, I - p_1], \delta(p_1)] \\ &= [[p_1, \delta(I)], p_1] \\ &= -p_1\delta(I)p_2.\end{aligned}$$

This yields that $\delta(I) = p_1\delta(I)p_1 + p_2\delta(I)p_2$. By Claims 1 & 2, we have $\delta(I) = p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 + p_1\delta(p_2)p_1 + p_2\delta(p_2)p_2 \in Z(\mathfrak{A})$.

In the sequel, we define

$$f(x) = \delta(x) + \delta_{p_1\delta(p_1)p_2}(x),$$

where $\delta_{p_1\delta(p_1)p_2}(x)$ is the inner derivation determined by $p_1\delta(p_1)p_2$, that is,

$$\delta_{p_1\delta(p_1)p_2}(x) = [p_1\delta(p_1)p_2, x] = p_1\delta(p_1)p_2x - xp_1\delta(p_1)p_2 \quad \text{for all} \quad x \in \mathfrak{A}.$$

One can verify that

$$f([[x, y], z]) = [[f(x), y], z] + [[x, f(y)], z] + [[x, y], f(z)]$$

for all $x, y, z \in \mathfrak{A}$ with $xy = 0$. Moreover by Claim 1, we have

$$f(p_1) = \delta(p_1) + p_1\delta(p_1)p_2p_1 - p_1\delta(p_1)p_2 = p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 \in Z(\mathfrak{A}).$$

By Claim 3, we get $f(I) = \delta(I) \in Z(\mathfrak{A})$. Consequently $f(p_2) = f(I) - f(p_1) \in Z(\mathfrak{A})$.

Clearly for any $a_{12} \in \mathfrak{A}_{12}$, $\delta_{p_1\delta(p_1)p_2}(a_{12}) = [p_1\delta(p_1)p_2, a_{12}] = 0$. By Claim 1, we have $f(a_{12}) = \delta(a_{12}) \in \mathfrak{A}_{12}$, and hence

Claim 4. *For any $a_{12} \in \mathfrak{A}_{12}$, $f(a_{12}) \in \mathfrak{A}_{12}$.*

Claim 5. $f(\mathfrak{A}_{ii}) \subseteq \mathfrak{A}_{ii} \oplus \mathfrak{A}_{jj}$. *There exists an $\mathcal{R}$-linear map $\tau_i : \mathfrak{A}_{ii} \to Z(\mathfrak{A})$ such that $f(a_{ii}) - \tau_i(a_{ii}) \in \mathfrak{A}_{ii}$ for all $a_{ii} \in \mathfrak{A}_{ii}$, where $i, j = 1, 2$ and $i \neq j$.*

First we show for $i = 1$. Since $a_{11}p_2 = 0$ for any $a_{11} \in \mathfrak{A}_{11}$ and $f(p_2) \in Z(\mathfrak{A})$, it follows that

$$\begin{aligned} 0 &= f([[a_{11}, p_2], p_2]) \\ &= [[f(a_{11}), p_2], p_2] + [[a_{11}, f(p_2)], p_2] + [[a_{11}, p_2], f(p_2)] \\ &= f(a_{11})p_2 - p_2 f(a_{11})p_2 - p_2 f(a_{11})p_2 + p_2 f(a_{11}). \end{aligned}$$

Multiplying by p_1 from the left we get $p_1 f(a_{11})p_2 = 0$ and hence $f(a_{11}) \in \mathfrak{A}_{11} \oplus \mathfrak{A}_{22}$. Similarly, we can prove the result for $i = 2$. Thus, $f(\mathfrak{A}_{ii}) \subseteq \mathfrak{A}_{ii} \oplus \mathfrak{A}_{jj}$.
Now we can write $f(a_{11}) = p_1 f(a_{11})p_1 + p_2 f(a_{11})p_2$. Moreover, since $a_{11}a_{22} = 0$, for any $a_{22} \in \mathfrak{A}_{22}$ and $x \in \mathfrak{A}$, it is easy to check that

$$\begin{aligned} 0 &= f([[a_{11}, a_{22}], x]) = [[f(a_{11}), a_{22}], x] + [[a_{11}, f(a_{22})], x] \\ &= [[f(a_{11}), a_{22}] + [a_{11}, f(a_{22})], x]. \end{aligned}$$

Multiplying by p_2 on both the sides of the above equation, we get

$$\begin{aligned} 0 &= p_2[[f(a_{11}), a_{22}] + [a_{11}, f(a_{22})], x]p_2 \\ &= [[p_2 f(a_{11})p_2, p_2 a_{22} p_2] + [p_2 a_{11} p_2, p_2 f(a_{22})p_2], p_2 x p_2] \\ &= [[p_2 f(a_{11})p_2, a_{22}], p_2 x p_2]. \end{aligned}$$

This implies that $[p_2 f(a_{11})p_2, a_{22}] \in Z(\mathfrak{A}_{22})$. Hence by hypothesis (ii), we find that

$$p_2 f(a_{11})p_2 \in Z(\mathfrak{A}_{22}).$$

Define $\tau_1 : \mathfrak{A}_{11} \to Z(\mathfrak{A})$ such that $\tau_1(a_{11}) = \eta(p_2 f(a_{11})p_2) \oplus p_2 f(a_{11})p_2$, where η is the map defined in Lemma 1.1. Thus, we get

$$\begin{aligned} f(a_{11}) - \tau_1(a_{11}) &= p_1 f(a_{11})p_1 + p_2 f(a_{11})p_2 - \eta(p_2 f(a_{11})p_2) - p_2 f(a_{11})p_2 \\ &= p_1 f(a_{11})p_1 - \eta(p_2 f(a_{11})p_2) \in \mathfrak{A}_{11}. \end{aligned}$$

Since f is $\mathcal{R}$-linear, one can verify that τ_1 is $\mathcal{R}$-linear. Similarly, we can define $\mathcal{R}$-linear map $\tau_2 : \mathfrak{A}_{22} \to Z(\mathfrak{A})$ by $\tau_2(a_{22}) = p_1 f(a_{22})p_1 \oplus \eta^{-1}(p_1 f(a_{22})p_1)$. Then

$$\begin{aligned} f(a_{22}) - \tau_2(a_{22}) &= p_1 f(a_{22})p_1 + p_2 f(a_{22})p_2 - p_1 f(a_{22})p_1 - \eta^{-1}(p_1 f(a_{22})p_1) \\ &= p_2 f(a_{22})p_2 - \eta^{-1}(p_1 f(a_{22})p_1) \in \mathfrak{A}_{22}. \end{aligned}$$

Now, for any $x = a_{11} + a_{12} + a_{22} \in \mathfrak{A}$, we define two $\mathcal{R}$-linear maps $\tau : \mathfrak{A} \to Z(\mathfrak{A})$ and $d : \mathfrak{A} \to \mathfrak{A}$ by

$$\tau(x) = \tau_1(a_{11}) + \tau_2(a_{22}) \quad \text{and} \quad d(x) = f(x) - \tau(x) \text{ respectively.}$$

Then, $d(\mathfrak{A}_{ij}) \subseteq \mathfrak{A}_{ij}$ for $1 \leq i \leq j \leq 2$ and $d(a_{12}) = f(a_{12})$.

Claim 6. *d is a derivation.*

Since f & τ are $\mathcal{R}$-linear and $d(x) = f(x) - \tau(x)$, d is $\mathcal{R}$-linear. It remains to show that $d(xy) = d(x)y + xd(y)$, for all $x, y \in \mathfrak{A}$. Now, we divide the proof into the following three steps:

Step 1. Since $a_{12}a_{11} = 0$ for any $a_{11} \in \mathfrak{A}_{11}$, $a_{12} \in \mathfrak{A}_{12}$ and $\tau(x)$ is in $Z(\mathfrak{A})$, we have

$$\begin{aligned}
-d(a_{11}a_{12}) &= d([[a_{11}, a_{12}], p_1]) = f([[a_{11}, a_{12}], p_1]) \\
&= [[f(a_{11}), a_{12}], p_1] + [[a_{11}, f(a_{12})], p_1] + [[a_{11}, a_{12}], f(p_1)] \\
&= [[d(a_{11}) + \tau(a_{11}), a_{12}], p_1] + [[a_{11}, d(a_{12})], p_1] \\
&= -d(a_{11})a_{12} - a_{11}d(a_{12}).
\end{aligned}$$

Hence, $d(a_{11}a_{12}) = d(a_{11})a_{12} + a_{11}d(a_{12})$. Similarly, we can get

$$d(a_{12}a_{22}) = d(a_{12})a_{22} + a_{12}d(a_{22}) \quad \text{for all} \quad a_{12} \in \mathfrak{A}_{12}, a_{22} \in \mathfrak{A}_{22}.$$

Step 2. Let $a_{11}, b_{11} \in \mathfrak{A}_{11}$. For any $a_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}
d(a_{11}b_{11}a_{12}) &= d(a_{11})b_{11}a_{12} + a_{11}d(b_{11}a_{12}) \\
&= d(a_{11})b_{11}a_{12} + a_{11}d(b_{11})a_{12} + a_{11}b_{11}d(a_{12}).
\end{aligned} \tag{2.2}$$

On the other hand,

$$d(a_{11}b_{11}a_{12}) = d(a_{11}b_{11})a_{12} + a_{11}b_{11}d(a_{12}). \tag{2.3}$$

Comparing (2.2) and (2.3), we have

$$\big(d(a_{11}b_{11}) - d(a_{11})b_{11} - a_{11}d(b_{11})\big)a_{12} = 0 \quad \text{for all} \quad a_{12} \in \mathfrak{A}_{12}.$$

Since $\mathfrak{A}_{12}$ is a faithful left $\mathfrak{A}_{11}$-module, we get

$$d(a_{11}b_{11}) = d(a_{11})b_{11} + a_{11}d(b_{11}) \quad \text{for all} \quad a_{11}, b_{11} \in \mathfrak{A}_{11}.$$

Similarly, one can arrive at

$$d(a_{22}b_{22}) = d(a_{22})b_{22} + a_{22}d(b_{22}) \quad \text{for all} \quad a_{22}, b_{22} \in \mathfrak{A}_{22}.$$

Step 3. Let $x = a_{11} + a_{12} + a_{22}$, $y = b_{11} + b_{12} + b_{22}$ be in $\mathfrak{A}$.
By Steps 1 & 2, we have

$$\begin{aligned} d(xy) &= d((a_{11} + a_{12} + a_{22})(b_{11} + b_{12} + b_{22})) \\ &= d(a_{11}b_{11}) + d(a_{11}b_{12}) + d(a_{12}b_{22}) + d(a_{22}b_{22}) \\ &= d(a_{11})b_{11} + a_{11}d(b_{11}) + d(a_{11})b_{12} + a_{11}d(b_{12}) \\ &\quad + d(a_{12})b_{22} + a_{12}d(b_{22}) + d(a_{22})b_{22} + a_{22}d(b_{22}). \end{aligned}$$

On the other hand, $d(\mathfrak{A}_{ij}) \subseteq \mathfrak{A}_{ij}$ for $1 \leq i \leq j \leq 2$, we have

$$\begin{aligned} d(x)y + xd(y) &= d(a_{11} + a_{12} + a_{22})(b_{11} + b_{12} + b_{22}) \\ &\quad + (a_{11} + a_{12} + a_{22})d(b_{11} + b_{12} + b_{22}) \\ &= d(a_{11})b_{11} + a_{11}d(b_{11}) + d(a_{11})b_{12} + a_{11}d(b_{12}) \\ &\quad + d(a_{12})b_{22} + a_{12}d(b_{22}) + d(a_{22})b_{22} + a_{22}d(b_{22}). \end{aligned}$$

Hence, we find that $d(xy) = d(x)y + xd(y)$, i.e., d is a derivation.

Claim 7. *τ vanishes at second commutator $[[x, y], z]$ with $xy = 0$ for all $x, y, z \in \mathfrak{A}$.*

Suppose $xy = 0$. Since $\tau(x) \in Z(\mathfrak{A})$, we have

$$\begin{aligned} \tau([[x, y], z]) &= f([[x, y], z]) - d([[x, y], z]) \\ &= [[f(x), y], z] + [[x, f(y)], z] + [[x, y], f(z)] - d([[x, y], z]) \\ &= [[d(x) + \tau(x), y], z] + [[x, d(y) + \tau(y)], z] + [[x, y], d(z) + \tau(z)] \\ &\quad - d([[x, y], z]) \\ &= [[d(x), y], z] + [[x, d(y)], z] + [[x, y], d(z)] - d([[x, y], z]) \\ &= 0 \end{aligned}$$

for all $x, y, z \in \mathfrak{A}$. The proof of our theorem for $n = 1$ is now complete.

Now, suppose that the conclusion holds for all $m < n \in \mathbb{N}$. That is, there exist linear maps $d_m : \mathfrak{A} \to \mathfrak{A}$ and $\tau_m : \mathfrak{A} \to Z(\mathfrak{A})$ such that $\delta_m(x) = d_m(x) + \tau_m(x)$, $\tau_m([[x, y], z]) = 0$ with $xy = 0$ and $d_m(xy) = \sum\limits_{i+j=m} d_i(x)d_j(y)$ for all $x, y, z \in \mathfrak{A}$. Moreover, δ_m has the following properties:

$$p_1\delta_m(p_1)p_1 + p_2\delta_m(p_1)p_2 \in Z(\mathfrak{A}); \delta_m(a_{12}) = p_1\delta_m(a_{12})p_2 \in \mathfrak{A}_{12};$$

$$p_1\delta_m(p_2)p_1 + p_2\delta_m(p_2)p_2 \in Z(\mathfrak{A});$$
$$\delta_m(I) = p_1\delta_m(I)p_1 + p_2\delta_m(I)p_2 \in Z(\mathfrak{A}).$$

We will show that δ_n also satisfies the similar properties. We prove this through the following claims:

Claim 8. *$p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2 \in Z(\mathfrak{A})$; $\delta_n(a_{12}) = p_1\delta_n(a_{12})p_2 \in \mathfrak{A}_{12}$ for any $a_{12} \in \mathfrak{A}_{12}$.*

Since $a_{12}p_1 = 0$ for any $a_{12} \in \mathfrak{A}_{12}$, by induction hypothesis, we have

$$\begin{aligned}\delta_n(a_{12}) &= \delta_n([[a_{12}, p_1], p_1]) \\ &= [[\delta_n(a_{12}), p_1], p_1] + [[a_{12}, \delta_n(p_1)], p_1] + [[a_{12}, p_1], \delta_n(p_1)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[\delta_i(a_{12}), \delta_j(p_1)], \delta_k(p_1)] \\ &= \delta_n(a_{12})p_1 - p_1\delta_n(a_{12})p_1 - p_1\delta_n(a_{12})p_1 + p_1\delta_n(a_{12}) + a_{12}\delta_n(p_1)p_1 \\ &\quad - a_{12}\delta_n(p_1) + p_1\delta_n(p_1)a_{12} - a_{12}\delta_n(p_1) + \delta_n(p_1)a_{12}. \end{aligned} \tag{2.4}$$

On multiplying the above equality by p_1 and p_2 from left and right respectively, we get

$$2(p_1\delta_n(p_1)a_{12} - a_{12}\delta_n(p_1)p_2) = 0.$$

This implies that $p_1\delta_n(p_1)a_{12} - a_{12}\delta_n(p_1)p_2 = 0$, that is, $p_1\delta_n(p_1)p_1a_{12} - a_{12}p_2\delta_n(p_1)p_2 = 0$, and hence $p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2 \in Z(\mathfrak{A})$. By putting $\delta_n(a_{12}) = s_{11} + s_{12} + s_{22}$, $\delta_n(p_1) = t_{11} + t_{12} + t_{22}$ in (2.4), we get $s_{11} = 0 = s_{22}$ and so, we have $p_i\delta_n(a_{12})p_i = 0$ for $i \in \{1, 2\}$. Hence, $\delta_n(a_{12}) = p_1\delta_n(a_{12})p_2$.

Since $p_2a_{12} = 0$ for any $a_{12} \in \mathfrak{A}_{12}$. Similarly, we can get the following result.

Claim 9. $p_1\delta_n(p_2)p_1 + p_2\delta_n(p_2)p_2 \in Z(\mathfrak{A})$.

Claim 10. $\delta_n(I) = p_1\delta_n(I)p_1 + p_2\delta_n(I)p_2 \in Z(\mathfrak{A})$.

Since $p_1(I - p_1) = 0$, by using the induction hypothesis, we find that

$$\begin{aligned} 0 &= \delta_n([[p_1, I - p_1], p_1]) \\ &= [[\delta_n(p_1), I - p_1], p_1] + [[p_1, \delta_n(I - p_1)], p_1] + [[p_1, I - p_1], \delta_n(p_1)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[\delta_i(p_1), \delta_j(I - p_1)], \delta_k(p_1)] \\ &= p_1\delta_n(I)p_1 - \delta_n(I)p_1 - p_1\delta_n(I) + p_1\delta_n(I)p_1. \end{aligned}$$

On multiplying by p_1 from the left and by p_2 from the right, we have $p_1\delta_n(I)p_2 = 0$, and hence we get that $\delta_n(I) = p_1\delta_n(I)p_1 + p_2\delta_n(I)p_2$. By Claims 8 & 9, we have $\delta_n(I) = p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2 + p_1\delta_n(p_2)p_1 + p_2\delta_n(p_2)p_2 \in Z(\mathfrak{A})$.

In the sequel, we define

$$f_n(x) = \delta_n(x) + \delta_{p_1\delta_n(p_1)p_2}(x),$$

where $\delta_{p_1\delta_n(p_1)p_2}(x)$ is the inner derivation determined by $p_1\delta_n(p_1)p_2$, that is,

$$\delta_{p_1\delta_n(p_1)p_2}(x) = [p_1\delta_n(p_1)p_2, x] = p_1\delta_n(p_1)p_2x - xp_1\delta_n(p_1)p_2 \quad \text{for all} \quad x \in \mathfrak{A}.$$

One can verify that

$$f_n([[x,y],z]) = [[f_n(x),y],z] + [[x,f_n(y)],z] + [[x,y],f_n(z)] + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(x),f_j(y)],f_k(z)]$$

for all $x,y,z \in \mathfrak{A}$ with $xy=0$. Moreover by Claim 8, we have

$$f_n(p_1) = \delta_n(p_1) + p_1\delta_n(p_1)p_2p_1 - p_1\delta_n(p_1)p_2 = p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2 \in Z(\mathfrak{A}).$$

By Claim 10, we find that $f_n(I) = \delta_n(I) \in Z(\mathfrak{A})$. Consequently $f_n(p_2) = f_n(I) - f_n(p_1) \in Z(\mathfrak{A})$. Clearly for any $a_{12} \in \mathfrak{A}_{12}$, $\delta_{p_1\delta_n(p_1)p_2}(a_{12}) = [p_1\delta_n(p_1)p_2, a_{12}] = 0$. By Claim 8, we have $f_n(a_{12}) = \delta_n(a_{12}) \in \mathfrak{A}_{12}$, and hence

Claim 11. *For any* $a_{12} \in \mathfrak{A}_{12}$, $f_n(a_{12}) \in \mathfrak{A}_{12}$.

Claim 12. $f_n(\mathfrak{A}_{ii}) \subseteq \mathfrak{A}_{ii} \oplus \mathfrak{A}_{jj}$. *There exists an* $\mathcal{R}$*-linear map* $\tau_{ni} : \mathfrak{A}_{ii} \to Z(\mathfrak{A})$ *such that* $f_n(a_{ii}) - \tau_{ni}(a_{ii}) \in \mathfrak{A}_{ii}$ *for all* $a_{ii} \in \mathfrak{A}_{ii}$, *where* $i,j = 1,2$ *and* $i \neq j$.

First we show the result for $i=1$. Since $a_{11}p_2 = 0$ for any $a_{11} \in \mathfrak{A}_{11}$ and $f_n(p_2) \in Z(\mathfrak{A})$, using induction hypothesis, it follows that

$$\begin{aligned} 0 &= f_n([[a_{11},p_2],p_2]) \\ &= [[f_n(a_{11}),p_2],p_2] + [[a_{11},f_n(p_2)],p_2] + [[a_{11},p_2],f_n(p_2)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(a_{11}),f_j(p_2)],f_k(p_2)] \\ &= f_n(a_{11})p_2 - p_2f_n(a_{11})p_2 - p_2f_n(a_{11})p_2 + p_2f_n(a_{11}). \end{aligned}$$

Multiply by p_1 from the left to get $p_1f_n(a_{11})p_2 = 0$. So, $f_n(a_{11}) \in \mathfrak{A}_{11} \oplus \mathfrak{A}_{22}$. Similarly we can find the result for $i=2$. Thus, $f_n(\mathfrak{A}_{ii}) \subseteq \mathfrak{A}_{ii} \oplus \mathfrak{A}_{jj}$.

Now we can write $f_n(a_{11}) = p_1f_n(a_{11})p_1 + p_2f_n(a_{11})p_2$. Moreover, since $a_{11}a_{22} = 0$ for any $a_{22} \in \mathfrak{A}_{22}$ and $x \in \mathfrak{A}$, it is easy to observe that

$$\begin{aligned} 0 = f_n([[a_{11},a_{22}],x]) &= [[f_n(a_{11}),a_{22}],x] + [[a_{11},f_n(a_{22})],x] + [[a_{11},a_{22}],f_n(x)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(a_{11}),f_j(a_{22})],f_k(x)] \\ &= [[f_n(a_{11}),a_{22}],x] + [[a_{11},f_n(a_{22})],x] \\ &= [[f_n(a_{11}),a_{22}] + [a_{11},f_n(a_{22})],x]. \end{aligned}$$

Multiplying by p_2 on both the sides, we get

$$\begin{aligned} 0 &= p_2[[f_n(a_{11}),a_{22}] + [a_{11},f_n(a_{22})],x]p_2 \\ &= [[p_2f_n(a_{11})p_2, p_2a_{22}p_2] + [p_2a_{11}p_2, p_2f_n(a_{22})p_2], p_2xp_2] \\ &= [[p_2f_n(a_{11})p_2, a_{22}], p_2xp_2]. \end{aligned}$$

This implies that $[p_2f_n(a_{11})p_2, a_{22}] \in Z(\mathfrak{A}_{22})$. Hence by assumption (ii), we get $p_2f_n(a_{11})p_2 \in Z(\mathfrak{A}_{22})$. Define $\tau_{n1} : \mathfrak{A}_{11} \to Z(\mathfrak{A})$ by $\tau_{n1}(a_{11}) = \eta(p_2f_n(a_{11})p_2) \oplus p_2f_n(a_{11})p_2$, where η is the map defined in Lemma 1.1. Thus, we get

$$\begin{aligned} f_n(a_{11}) - \tau_{n1}(a_{11}) &= p_1f_n(a_{11})p_1 + p_2f_n(a_{11})p_2 - \eta(p_2f_n(a_{11})p_2) - p_2f_n(a_{11})p_2 \\ &= p_1f_n(a_{11})p_1 - \eta(p_2f_n(a_{11})p_2) \in \mathfrak{A}_{11}. \end{aligned}$$

Since f_n is $\mathcal{R}$-linear, one can verify that τ_{n1} is $\mathcal{R}$-linear. Similarly, we can define $\mathcal{R}$-linear map $\tau_{n2} : \mathfrak{A}_{22} \to Z(\mathfrak{A})$ by $\tau_{n2}(a_{22}) = p_1f_n(a_{22})p_1 \oplus \eta^{-1}(p_1f_n(a_{22})p_1)$. Then

$$\begin{aligned} f_n(a_{22}) - \tau_{n2}(a_{22}) &= p_1f_n(a_{22})p_1 + p_2f_n(a_{22})p_2 - p_1f_n(a_{22})p_1 - \eta^{-1}(p_1f_n(a_{22})p_1) \\ &= p_2f_n(a_{22})p_2 - \eta^{-1}(p_1f_n(a_{22})p_1) \in \mathfrak{A}_{22}. \end{aligned}$$

Now, for any $x = a_{11} + a_{12} + a_{22} \in \mathfrak{A}$, we define two $\mathcal{R}$-linear maps $\tau_n : \mathfrak{A} \to Z(\mathfrak{A})$ and $d_n : \mathfrak{A} \to \mathfrak{A}$ by

$$\tau_n(x) = \tau_{n1}(a_{11}) + \tau_{n2}(a_{22}) \quad \text{and} \quad d_n(x) = f_n(x) - \tau_n(x) \ \text{ respectively.}$$

Then, $d_n(\mathfrak{A}_{ij}) \subseteq \mathfrak{A}_{ij}$ for $1 \le i \le j \le 2$ and $d_n(a_{12}) = f_n(a_{12})$.

Claim 13. *$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in \mathfrak{A}$.*

Since f_n & τ_n are $\mathcal{R}$-linear and $d_n(x) = f_n(x) - \tau_n(x)$, d_n is an $\mathcal{R}$-linear. It remains to show that $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in \mathfrak{A}$.

Now, we divide the proof into the following three steps:

Step 1. Since $a_{12}a_{11} = 0$ for any $a_{11} \in \mathfrak{A}_{11}$, $a_{12} \in \mathfrak{A}_{12}$ and $\tau_n(x)$ is in $Z(\mathfrak{A})$, by induction hypothesis, we have

$$\begin{aligned} -d_n(a_{11}a_{12}) &= f_n([[a_{11}, a_{12}], p_1]) \\ &= [[f_n(a_{11}), a_{12}], p_1] + [[a_{11}, f_n(a_{12})], p_1] + [[a_{11}, a_{12}], f_n(p_1)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(a_{11}), f_j(a_{12})], f_k(p_1)] \\ &= [[f_n(a_{11}), a_{12}], p_1] + [[a_{11}, f_n(a_{12})], p_1] \\ &\quad + \sum_{\substack{i+j=n \\ 0<i,j<n}} [[f_i(a_{11}), f_j(a_{12})], p_1] \\ &= [[d_n(a_{11}) + \tau_n(a_{11}), a_{12}], p_1] + [[a_{11}, d_n(a_{12}) + \tau_n(a_{12})], p_1] \\ &\quad + \sum_{\substack{i+j=n \\ 0<i,j<n}} [[d_n(a_{11}) + \tau_n(a_{11}), d_n(a_{12}) + \tau_n(a_{12})], p_1] \\ &= [[d_n(a_{11}), a_{12}], p_1] + [[a_{11}, d_n(a_{12})], p_1] \end{aligned}$$

$$+ \sum_{\substack{i+j=n\\0<i,j<n}} [[d_n(a_{11}), d_n(a_{12})], p_1]$$

$$= -d_n(a_{11})a_{12} - a_{11}d_n(a_{12}) - \sum_{\substack{i+j=n\\0<i,j<n}} d_i(a_{11})d_j(a_{12}).$$

Hence, $d_n(a_{11}a_{12}) = \sum_{i+j=n} d_i(a_{11})d_j(a_{12})$. Similarly, we can get

$$d_n(a_{12}a_{22}) = \sum_{i+j=n} d_i(a_{12})d_j(a_{22}) \quad \text{for any} \quad a_{12} \in \mathfrak{A}_{12}, a_{22} \in \mathfrak{A}_{22}.$$

Step 2. Let $a_{11}, b_{11} \in \mathfrak{A}_{11}$. For any $a_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned} d_n(a_{11}b_{11}a_{12}) &= \sum_{i+k=n} d_i(a_{11}b_{11})d_k(a_{12}) \\ &= d_n(a_{11}b_{11})a_{12} + \sum_{\substack{i+k=n\\k\neq 0}} d_i(a_{11}b_{11})d_k(a_{12}) \\ &= d_n(a_{11}b_{11})a_{12} + \sum_{\substack{l+t+k=n\\k\neq 0}} d_l(a_{11})d_t(b_{11})d_k(a_{12}). \end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned} d_n(a_{11}b_{11}a_{12}) &= \sum_{i+j=n} d_i(a_{11})d_j(b_{11}a_{12}) \\ &= \sum_{i+j+k=n} d_i(a_{11})d_j(b_{11})d_k(a_{12}) \\ &= \sum_{i+j=n} d_i(a_{11})d_j(b_{11})a_{12} + \sum_{\substack{i+j+k=n\\k\neq 0}} d_i(a_{11})d_j(b_{11})d_k(a_{12}). \end{aligned} \tag{2.6}$$

Comparing (2.5) and (2.6), we have

$$\big(d_n(a_{11}b_{11}) - \sum_{i+j=n} d_i(a_{11})d_j(b_{11})\big)a_{12} = 0.$$

Since $\mathfrak{A}_{12}$ is a faithful left $\mathfrak{A}_{11}$-module, we get

$$d_n(a_{11}b_{11}) = \sum_{i+j=n} d_i(a_{11})d_j(b_{11}).$$

Similarly, we can calculate

$$d_n(a_{22}b_{22}) = \sum_{i+j=n} d_i(a_{22})d_j(b_{22}).$$

Step 3. Let $x = a_{11} + a_{12} + a_{22}$, $y = b_{11} + b_{12} + b_{22}$ be in $\mathfrak{A}$.
By Steps 1 & 2, we have

$$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y).$$

Claim 14. *τ_n vanishes at second commutator $[[x,y],z]$ with $xy = 0$ for all $x, y, z \in \mathfrak{A}$.*

Since $xy = 0$, we find that

$$\begin{aligned}
\tau_n([[x,y],z]) &= f_n([[x,y],z]) - d_n([[x,y],z]) \\
&= \sum_{i+j+k=n} ([[f_i(x), f_j(y)], f_k(z)]) - d_n([[x,y],z]) \\
&= \sum_{i+j+k=n} ([[d_i(x) + \tau_i(x), d_j(y) + \tau_j(y)], d_k(z) + \tau_k(z)]) \\
&\quad - d_n([[x,y],z]) \\
&= \sum_{i+j+k=n} [[d_i(x), d_j(y)], d_k(z)] - d_n([[x,y],z]) \\
&= 0
\end{aligned}$$

for all $x, y, z \in \mathfrak{A}$. The proof is now complete. □

3. Characterizations of Lie Triple Higher Derivations by Action on Idempotent Product

The proof of the following theorem shares the same outline as that of Theorem 2.1 but requires different technique.

Theorem 3.1. *Let $\mathfrak{A} = Tri(\mathfrak{A}_{11}, \mathfrak{A}_{12}, \mathfrak{A}_{22})$ be a 2-torsion free triangular algebra consisting of algebras $\mathfrak{A}_{11}$ and $\mathfrak{A}_{22}$ over a commutative ring $\mathcal{R}$ with unity $1_{\mathfrak{A}_{11}}$ and $1_{\mathfrak{A}_{22}}$ respectively and $\mathfrak{A}_{12}$ be a faithful $(\mathfrak{A}_{11}, \mathfrak{A}_{22})$-bimodule. Suppose that*

(i) $\pi_{\mathfrak{A}_{11}}(Z(\mathfrak{A})) = Z(\mathfrak{A}_{11})$ *and* $\pi_{\mathfrak{A}_{22}}(Z(\mathfrak{A})) = Z(\mathfrak{A}_{22})$.

(ii) *For any $a \in \mathfrak{A}_{11}$, if $[a, \mathfrak{A}_{11}] \in Z(\mathfrak{A}_{11})$, then $a \in Z(\mathfrak{A}_{11})$ or for any $b \in \mathfrak{A}_{22}$, if $[b, \mathfrak{A}_{22}] \in Z(\mathfrak{A}_{22})$, then $b \in Z(\mathfrak{A}_{22})$.*

(iii) *For every $a \in \mathfrak{A}_{11}$, there exists an integer t such that $t1_{\mathfrak{A}_{11}} - a$ is invertible.*

If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{R}$-linear mappings $\delta_n : \mathfrak{A} \to \mathfrak{A}$ such that $\delta_n[[x,y],z] = \sum\limits_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$ for all $x, y, z \in \mathfrak{A}$ with $xy = p$, then for each $n \in \mathbb{N}$, $\delta_n = d_n + \tau_n$; where $d_n : \mathfrak{A} \to \mathfrak{A}$ is $\mathcal{R}$-linear mapping satisfying $d_n(xy) = \sum\limits_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in \mathfrak{A}$, i.e., $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is a higher derivation *of* $\mathfrak{A}$ *and* $\tau_n : \mathfrak{A} \to Z(\mathfrak{A})$ *is an* $\mathcal{R}$*-linear map vanishing at every second commutator* $[[x, y], z]$ *with* $xy = p$.

For the proof of Theorem 3.1, we proceed by induction on n. We provide the proof, for $n = 1$, through several claims. Indeed, we show that for every Lie triple derivation $\delta_1 = \delta$ on $\mathfrak{A}$ there exist a derivation d on $\mathfrak{A}$ and a linear mapping $\tau : \mathfrak{A} \to Z(\mathfrak{A})$ vanishing on second commutators such that $\delta(x) = d(x) + \tau(x)$ for all $x \in \mathfrak{A}$.

Proof of Theorem 3.1. **Claim 1.** $p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 \in Z(\mathfrak{A})$*;* $\delta(a_{12}) = p_1\delta(a_{12})p_2 \in \mathfrak{A}_{12}$ *for any* $a_{12} \in \mathfrak{A}_{12}$.

Since $(p_1 + a_{12})p_1 = p_1$ for any $a_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}\delta(a_{12}) &= \delta([[p_1 + a_{12}, p_1], p_1]) \\ &= [[\delta(p_1) + \delta(a_{12}), p_1], p_1]) + [[p_1 + a_{12}, \delta(p_1)], p_1] + [[p_1 + a_{12}, p_1], \delta(p_1)] \\ &= [[\delta(a_{12}), p_1], p_1] + [[a_{12}, \delta(p_1)], p_1] + [[a_{12}, p_1], \delta(p_1)] \\ &= \delta(a_{12})p_1 - p_1\delta(a_{12})p_1 - p_1\delta(a_{12})p_1 + p_1\delta(a_{12}) + a_{12}\delta(p_1)p_1 \\ &\quad - a_{12}\delta(p_1) + p_1\delta(p_1)a_{12} - a_{12}\delta(p_1) + \delta(p_1)a_{12}. \end{aligned} \tag{3.1}$$

On multiplying the above equality by p_1 and p_2 from the left and the right respectively, we get $2(p_1\delta(p_1)a_{12} - a_{12}\delta(p_1)p_2) = 0$. This gives that $p_1\delta(p_1)a_{12} - a_{12}\delta(p_1)p_2 = 0$, that is, $p_1\delta(p_1)p_1a_{12} - a_{12}p_2\delta(p_1)p_2 = 0$. It follows that $p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 \in Z(\mathfrak{A})$. By putting $\delta(a_{12}) = r_{11} + r_{12} + r_{22}$, $\delta(p_1) = p_{11} + p_{12} + p_{22}$ in (3.1), we get $r_{11} = 0 = r_{22}$ and so we have $p_i\delta(a_{12})p_i = 0$ for $i \in \{1, 2\}$. Hence, $\delta(a_{12}) = p_1\delta(a_{12})p_2$. Now, define

$$f(x) = \delta(x) + \delta_{p_1\delta(p_1)p_2}(x),$$

where $\delta_{p_1\delta(p_1)p_2}$ is the inner derivation determined by $p_1\delta(p_1)p_2$. Then, we have

$$f(p_1) = \delta(p_1) + p_1\delta(p_1)p_2p_1 - p_1\delta(p_1)p_2 = p_1\delta(p_1)p_1 + p_2\delta(p_1)p_2 \in Z(\mathfrak{A})$$

and $f([[x, y], z]) = [[f(x), y], z] + [[x, f(y)], z] + [[x, y], f(z)]$ for all $x, y, z \in \mathfrak{A}$ with $xy = p$. Moreover, for any $a_{12} \in \mathfrak{A}_{12}$, by Claim 1, we have

$$f(a_{12}) = \delta(a_{12}) + \delta_{p_1\delta(p_1)p_2}(a_{12}) = \delta(a_{12}) \in \mathfrak{A}_{12}.$$

Claim 2. $f(I) = p_1f(I)p_1 + p_2f(I)p_2 \in Z(\mathfrak{A})$ *and* $f(p_2) \in Z(\mathfrak{A})$.

Since $Ip_1 = p_1$, we have

$$\begin{aligned}0 &= f([[I, p_1], p_1]) \\ &= [[f(I), p_1], p_1] + [[I, f(p_1)], p_1] + [[I, p_1], f(p_1)] \\ &= f(I)p_1 - p_1f(I)p_1 - p_1f(I)p_1 + p_1f(I).\end{aligned}$$

This yields that $p_1(f(I)p_1 - p_1f(I)p_1 - p_1f(I)p_1 + p_1f(I))p_2 = 0$ and hence we find that $p_1f(I)p_2 = 0$. So, we get $f(I) = p_1f(I)p_1 + p_2f(I)p_2$. For any $a_{12} \in \mathfrak{A}_{12}$, since $(p_1 - a_{12})(I + a_{12}) = p_1$, we have

$$\begin{aligned} -f(a_{12}) &= f([[p_1 - a_{12}, I + a_{12}], p_1]) \\ &= [[f(p_1) - f(a_{12}), I + a_{12}], p_1] + [[p_1 - a_{12}, f(I) + f(a_{12})], p_1] \\ &\quad + [[p_1 - a_{12}, I + a_{12}], f(p_1)] \\ &= -f(a_{12})p_1 - a_{12}f(p_1)p_1 - p_1f(p_1)a_{12} + p_1f(a_{12})a_{12} + a_{12}f(p_1) \\ &\quad + p_1f(I)p_1 + p_1f(a_{12})p_1 - a_{12}f(I)p_1 - f(I)p_1 - p_1f(I) \\ &\quad - p_1f(a_{12}) + a_{12}f(I) + p_1f(I)p_1 - p_1f(I)a_{12} + p_1f(a_{12})p_1) \\ &\quad - p_1f(a_{12})a_{12} + a_{12}f(p_1) - f(p_1)a_{12}. \end{aligned}$$

On multiplying above equality by p_1 and p_2 from the left and the right respectively, we get $a_{12}f(I)p_2 - p_1f(I)a_{12} = 0$ and $a_{12}p_2f(I)p_2 - p_1f(I)p_1a_{12} = 0$.

This implies that $f(I) = p_1f(I)p_1 + p_2f(I)p_2 \in Z(\mathfrak{A})$. Consequently, $f(p_2) = f(I) - f(p_1) \in Z(\mathfrak{A})$.

Claim 3. $f(\mathfrak{A}_{ii}) \subseteq \mathfrak{A}_{ii} \oplus \mathfrak{A}_{jj}$. *There exists an* $\mathcal{R}$*-linear map* $\tau_i : \mathfrak{A}_{ii} \to Z(\mathfrak{A})$ *such that* $f(a_{ii}) - \tau_i(a_{ii}) \in \mathfrak{A}_{ii}$ *for all* $a_{ii} \in \mathfrak{A}_{ii},$ *where* $i, j = 1, 2$ *and* $i \neq j$.

First we show for $i = 1$. Suppose that a_{11} is invertible in $\mathfrak{A}_{11}$, that is, there exists an element $a_{11}^{-1} \in \mathfrak{A}_{11}$ such that $a_{11}a_{11}^{-1} = a_{11}^{-1}a_{11} = p_1$. From $a_{11}a_{11}^{-1} = p_1$ and $(a_{11}^{-1} + p_2)a_{11} = p_1$, we have

$$\begin{aligned} 0 &= f([[a_{11}^{-1}, a_{11}], p_1]) \\ &= [[f(a_{11}^{-1}), a_{11}], p_1] + [[a_{11}^{-1}, f(a_{11})], p_1] + [[a_{11}^{-1}, a_{11}], f(p_1)] \end{aligned}$$

and hence by Claim 2,

$$\begin{aligned} 0 &= f([[a_{11}^{-1} + p_2, a_{11}], p_1]) \\ &= [[f(a_{11}^{-1}) + f(p_2), a_{11}], p_1] + [[a_{11}^{-1} + p_2, f(a_{11})], p_1] + [[a_{11}^{-1} + p_2, a_{11}], f(p_1)] \\ &= [[f(a_{11}^{-1}), a_{11}], p_1] + [[a_{11}^{-1}, f(a_{11})], p_1] + [[a_{11}^{-1}, a_{11}], f(p_1)] + [[p_2, f(a_{11})], p_1] \\ &= p_2f(a_{11})p_1 + p_1f(a_{11})p_2. \end{aligned}$$

On multiplying by p_1 from the left and by p_2 from the right, we get $p_1f(a_{11})p_2 = 0$, and hence we find that $f(a_{11}) \subseteq \mathfrak{A}_{11} \oplus \mathfrak{A}_{22}$.

If a_{11} is not invertible in $\mathfrak{A}_{11}$, by the hypothesis (iii), there exists an integer t such that $tp_1 - a_{11}$ is invertible in $\mathfrak{A}_{11}$. It follows from the preceding case that $f(tp_1 - a_{11}) \in \mathfrak{A}_{11} \oplus \mathfrak{A}_{22}$. Therefore, we have

$$f(a_{11}) = tf(p_1) - f(tp_1 - a_{11}) \in \mathfrak{A}_{11} \oplus \mathfrak{A}_{22}.$$

Similarly, we can prove the result for $i = 2$. Now we can write $f(a_{11}) = p_1f(a_{11})p_1 + p_2f(a_{11})p_2$. First suppose that a_{11} is invertible in $\mathfrak{A}_{11}$ with inverse element a_{11}^{-1}.

Note that $a_{11}a_{11}^{-1}=p_1$ and $(a_{11}^{-1}+a_{22})a_{11}=p_1$, we get

$$\begin{aligned}0&=f([[a_{11}^{-1},a_{11}],x])\\&=[[f(a_{11}^{-1}),a_{11}],x]+[[a_{11}^{-1},f(a_{11})],x]+[[a_{11}^{-1},a_{11}],f(x)],\end{aligned}$$

and hence

$$\begin{aligned}0&=f([[a_{11}^{-1}+a_{22},a_{11}],x])\\&=[[f(a_{11}^{-1})+f(a_{22}),a_{11}],x]+[[a_{11}^{-1}+a_{22},f(a_{11})],x]+[[a_{11}^{-1}+a_{22},a_{11}],f(x)]\\&=[[f(a_{22}),a_{11}],x]+[[a_{22},f(a_{11})],x]\\&=[[f(a_{11}),a_{22}]+[a_{11},f(a_{22})],x].\end{aligned}$$

Multiplying by p_2 on both the sides, we get

$$\begin{aligned}0&=p_2[[f(a_{11}),a_{22}]+[a_{11},f(a_{22})],x]p_2\\&=[[p_2f(a_{11})p_2,p_2a_{22}p_2]+[p_2a_{11}p_2,p_2f(a_{22})p_2],p_2xp_2]\\&=[[p_2f(a_{11})p_2,a_{22}],p_2xp_2].\end{aligned}$$

This implies that $[p_2f(a_{11})p_2,a_{22}]\in Z(\mathfrak{A}_{22})$. Hence by hypothesis (ii), we get $p_2f(a_{11})p_2\in Z(\mathfrak{A}_{22})$. If a_{11} is not invertible in $\mathfrak{A}_{11}$, by the hypothesis (iii), there exists an integer t such that (tp_1-a_{11}) is invertible in $\mathfrak{A}_{11}$. It follows from the preceding case that

$$\begin{aligned}0&=f([[a_{22},tp_1-a_{11}],x])\\&=[[f(a_{22}),tp_1-a_{11}],x]+[[a_{22},tf(p_1)-f(a_{11})],x]+[[a_{22},tp_1-a_{11}],f(x)]\\&=-[[f(a_{22}),a_{11}],x]-[[a_{22},f(a_{11})],x].\end{aligned}$$

Multiplying by p_2 on both the sides, we get

$$\begin{aligned}0&=p_2[[f(a_{11}),a_{22}]+[a_{11},f(a_{22})],x]p_2\\&=[[p_2f(a_{11})p_2,p_2a_{22}p_2]+[p_2a_{11}p_2,p_2f(a_{22})p_2],p_2xp_2]\\&=[[p_2f(a_{11})p_2,a_{22}],p_2xp_2].\end{aligned}$$

This implies that $[p_2f(a_{11})p_2,a_{22}]\in Z(\mathfrak{A}_{22})$. Hence by hypothesis (ii), we get $p_2f(a_{11})p_2\in Z(\mathfrak{A}_{22})$.
Define $\tau_1:\mathfrak{A}_{11}\to Z(\mathfrak{A})$ by $\tau_1(a_{11})=\eta(p_2f(a_{11})p_2)\oplus p_2f(a_{11})p_2$, where η is the map defined in Lemma 1.1. Thus, we get

$$\begin{aligned}f(a_{11})-\tau_1(a_{11})&=p_1f(a_{11})p_1+p_2f(a_{11})p_2-\eta(p_2f(a_{11})p_2)-p_2f(a_{11})p_2\\&=p_1f(a_{11})p_1-\eta(p_2f(a_{11})p_2)\in\mathfrak{A}_{11}.\end{aligned}$$

Since f is an $\mathcal{R}$-linear, one can verify that τ_1 is $\mathcal{R}$-linear.

Similarly, we can define $\mathcal{R}$-linear map $\tau_2 : \mathfrak{A}_{22} \to Z(\mathfrak{A})$ by $\tau_2(a_{22}) = p_1 f(a_{22}) p_1 \oplus \eta^{-1}(p_1 f(a_{22}) p_1)$. Then

$$\begin{aligned} f(a_{22}) - \tau_2(a_{22}) &= p_1 f(a_{22}) p_1 + p_2 f(a_{22}) p_2 - p_1 f(a_{22}) p_1 - \eta^{-1}(p_1 f(a_{22}) p_1) \\ &= p_2 f(a_{22}) p_2 - \eta^{-1}(p_1 f(a_{22}) p_1) \in \mathfrak{A}_{22}. \end{aligned}$$

Now, for any $x = a_{11} + a_{12} + a_{22} \in \mathfrak{A}$, we define two $\mathcal{R}$-linear mappings $\tau : \mathfrak{A} \to Z(\mathfrak{A})$ and $d : \mathfrak{A} \to \mathfrak{A}$ by $\tau(x) = \tau_1(a_{11}) + \tau_2(a_{22})$ and $d(x) = f(x) - \tau(x)$ respectively. Then, $d(\mathfrak{A}_{ij}) \subseteq \mathfrak{A}_{ij}$ for $1 \le i \le j \le 2$ and $d(a_{12}) = f(a_{12})$.

Claim 4. *d is a derivation.*

Since f & τ are $\mathcal{R}$-linear and $d(x) = f(x) - \tau(x)$, d is an $\mathcal{R}$-linear. It remains to show that $d(xy) = d(x)y + xd(y)$, for all $x, y \in \mathfrak{A}$. We divide the proof into the following three Steps:

Step 1. If a_{11} is invertible in $\mathfrak{A}_{11}$ with inverse element a_{11}^{-1}, then $(a_{11}^{-1} + a_{11}^{-1}a_{12})a_{11} = p_1$ for any $a_{11} \in \mathfrak{A}_{11}$, $a_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned} -d(a_{12}) &= d([[a_{11}, a_{11}^{-1} + a_{11}^{-1}a_{12}], p_1]) = f([[a_{11}, a_{11}^{-1} + a_{11}^{-1}a_{12}], p_1]) \\ &= [[f(a_{11}), a_{11}^{-1} + a_{11}^{-1}a_{12}], p_1] + [[a_{11}, f(a_{11}^{-1}) + f(a_{11}^{-1}a_{12})], p_1] \\ &\quad + [[a_{11}, a_{11}^{-1} + a_{11}^{-1}a_{12}], f(p_1)]. \end{aligned}$$

Since $[[f(a_{11}), a_{11}^{-1}], p_1] + [[a_{11}, f(a_{11}^{-1})], p_1] + [[a_{11}, a_{11}^{-1}], f(p_1)] = 0$, we have

$$\begin{aligned} -d(a_{12}) &= [[f(a_{11}), a_{11}^{-1}a_{12}], p_1] + [[a_{11}, f(a_{11}^{-1}a_{12})], p_1] \\ &= [[d(a_{11}), a_{11}^{-1}a_{12}], p_1] + [[a_{11}, d(a_{11}^{-1}a_{12})], p_1] \\ &= -d(a_{11})a_{11}^{-1}a_{12} - a_{11}d(a_{11}^{-1}a_{12}). \end{aligned}$$

Hence, $d(a_{12}) = d(a_{11})a_{11}^{-1}a_{12} + a_{11}d(a_{11}^{-1}a_{12})$. Replacing a_{12} by $a_{11}a_{12}$, we arrive at

$$d(a_{11}a_{12}) = d(a_{11})a_{12} + a_{11}d(a_{12}).$$

For any $a_{11} \in \mathfrak{A}_{11}$, let $tp_1 - a_{11}$ be invertible in $\mathfrak{A}_{11}$. Then

$$d((tp_1 - a_{11})a_{12}) = d(tp_1 - a_{11})a_{12} + (tp_1 - a_{11})d(a_{12}).$$

Since $d(p_1a_{12}) = d(p_1)a_{12} + p_1d(a_{12})$, we have

$$d(a_{11}a_{12}) = d(a_{11})a_{12} + a_{11}d(a_{12}).$$

Step 2. Let $a_{12} \in \mathfrak{A}_{12}$ and $a_{22} \in \mathfrak{A}_{22}$. Observe that $(p_1 + a_{12})(p_1 + a_{22} - a_{12}a_{22}) = p_1$

and $(p_1 + a_{22} - a_{12}a_{22})(p_1 + a_{12}) = p_1 + a_{12}$. Since $f(p_1) \in Z(\mathfrak{A})$, we have

$$\begin{aligned}
-d(a_{12}) &= d([[p_1 + a_{22} - a_{12}a_{22}, p_1 + a_{12}], p_1]) \\
&= f([[p_1 + a_{22} - a_{12}a_{22}, p_1 + a_{12}], p_1]) \\
&= [[f(p_1) + f(a_{22}) - f(a_{12}a_{22}), p_1 + a_{12}], p_1] \\
&\quad + [[p_1 + a_{22} - a_{12}a_{22}, f(p_1) + f(a_{12})], p_1] \\
&\quad + [[p_1 + a_{22} - a_{12}a_{22}, p_1 + a_{12}], f(p_1)] \\
&= [[d(a_{22}) - d(a_{12}a_{22}), p_1 + a_{12}], p_1] + [[p_1 + a_{22} - a_{12}a_{22}, d(a_{12})], p_1] \\
&= a_{12}d(a_{22}) - d(a_{12}a_{22}) - d(a_{12}) + d(a_{12})a_{22}.
\end{aligned}$$

Thus, $d(a_{12}a_{22}) = d(a_{12})a_{22} + a_{12}d(a_{22})$ for any $a_{12} \in \mathfrak{A}_{12}, a_{22} \in \mathfrak{A}_{22}$.

With the same approach as used in the proof of Claim 6 of Theorem 2.1, we can get:

Step 3. For any $a_{11}, b_{11} \in \mathfrak{A}_{11}$ and $a_{22}, b_{22} \in \mathfrak{A}_{22}$,

(i) $d(a_{11}b_{11}) = d(a_{11})b_{11} + a_{11}d(b_{11})$,

(ii) $d(a_{22}b_{22}) = d(a_{22})b_{22} + a_{22}d(b_{22})$.

Step 4. $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathfrak{A}$.

Claim 5. *τ vanishes at second commutator $[[x, y], z]$ with $xy = p$ for all $x, y, z \in \mathfrak{A}$.*

Since $xy = p$, we find that

$$\begin{aligned}
\tau([[x, y], z]) &= f([[x, y], z]) - d([[x, y], z]) \\
&= [[f(x), y], z] + [[x, f(y)], z] + [[x, y], f(z)] - d([[x, y], z]) \\
&= [[d(x) + \tau(x), y], z] + [[x, d(y) + \tau(y)], z] + [[x, y], d(z) + \tau(z)] \\
&\quad - d([[x, y], z]) \\
&= [[d(x), y], z] + [[x, d(y)], z] + [[x, y], d(z)] - d([[x, y], z]) \\
&= 0
\end{aligned}$$

for all $x, y, z \in \mathfrak{A}$. The proof for $n = 1$ is now complete.

Now, suppose that the conclusion holds for all $m < n \in \mathbb{N}$. That is, there exist linear maps $d_m : \mathfrak{A} \to \mathfrak{A}$ and $\tau_m : \mathfrak{A} \to Z(\mathfrak{A})$ such that $\delta_m(x) = d_m(x) + \tau_m(x)$, $\tau_m([[x, y], z]) = 0$ with $xy = p$ and $d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y)$ for all $x, y, z \in \mathfrak{A}$. Moreover, δ_m has the following properties:

$$p_1\delta_m(p_1)p_1 + p_2\delta_m(p_1)p_2 \in Z(\mathfrak{A});$$

$$p_1\delta_m(p_2)p_1 + p_2\delta_m(p_2)p_2 \in Z(\mathfrak{A});$$

$$\delta_m(a_{12}) = p_1\delta_m(a_{12})p_2 \in \mathfrak{A}_{12}.$$

We shall show that δ_n also satisfies the similar properties. We prove this through the following claims:

Claim 6. *$p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2 \in Z(\mathfrak{A})$; $\delta_n(a_{12}) = p_1\delta_n(a_{12})p_2 \in \mathfrak{A}_{12}$ for any $a_{12} \in \mathfrak{A}_{12}$.*

Since $(p_1 + a_{12})p_1 = p_1$ for any $a_{12} \in \mathfrak{A}_{12}$, by induction hypothesis, we have

$$\begin{aligned}
\delta_n(a_{12}) &= \delta_n([[p_1 + a_{12}, p_1], p_1]) \\
&= [[\delta_n(p_1) + \delta_n(a_{12}), p_1], p_1]) + [[p_1 + a_{12}, \delta_n(p_1)], p_1] \\
&\quad + [[p_1 + a_{12}, p_1], \delta_n(p_1)] + \sum_{\substack{i+j+k=n \\ 0\leq i,j,k<n}} [[\delta_i(p_1 + a_{12}), \delta_j(p_1)], \delta_k(p_1)] \\
&= [[\delta_n(a_{12}), p_1], p_1] + [[a_{12}, \delta_n(p_1)], p_1] + [[a_{12}, p_1], \delta_n(p_1)] \\
&= \delta_n(a_{12})p_1 - p_1\delta_n(a_{12})p_1 - p_1\delta_n(a_{12})p_1 + p_1\delta_n(a_{12}) + a_{12}\delta_n(p_1)p_1 \\
&\quad - a_{12}\delta_n(p_1) + p_1\delta_n(p_1)a_{12} - a_{12}\delta_n(p_1) + \delta_n(p_1)a_{12}.
\end{aligned} \tag{3.2}$$

On multiplying by p_1 from the left and by p_2 from the right in the above equation, we get $2(p_1\delta_n(p_1)a_{12} - a_{12}\delta_n(p_1)p_2) = 0$. This gives that $p_1\delta_n(p_1)a_{12} - a_{12}\delta_n(p_1)p_2 = 0$, that is, $p_1\delta_n(p_1)p_1a_{12} - a_{12}p_2\delta_n(p_1)p_2 = 0$. It follows that $p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2 \in Z(\mathfrak{A})$. By putting $\delta_n(a_{12}) = s_{11} + s_{12} + s_{22}$, $\delta_n(p_1) = t_{11} + t_{12} + t_{22}$ in (3.2), we get $s_{11} = 0 = s_{22}$ and so, we have $p_i\delta_n(a_{12})p_i = 0$ for $i \in \{1, 2\}$. Hence, $\delta_n(a_{12}) = p_1\delta_n(a_{12})p_2$.

Now, define

$$f_n(x) = \delta_n(x) + \delta_{p_1\delta_n(p_1)p_2}(x),$$

where $\delta_{p_1\delta_n(p_1)p_2}$ is the inner derivation determined by $p_1\delta_n(p_1)p_2$. Then, we have

$$f_n(p_1) = \delta_n(p_1) + p_1\delta_n(p_1)p_2p_1 - p_1\delta_n(p_1)p_2 = p_1\delta_n(p_1)p_1 + p_2\delta_n(p_1)p_2 \in Z(\mathfrak{A}).$$

and

$$\begin{aligned}
f_n([[x, y], z]) = [[f_n(x), y], z] + [[x, f_n(y)], z] + [[x, y], f_n(z)] \\
+ \sum_{\substack{i+j+k=n \\ 0\leq i,j,k<n}} [[f_i(x), f_j(y)], f_k(z)]
\end{aligned}$$

for all $x, y, z \in \mathfrak{A}$ with $xy = p$. Moreover, for any $a_{12} \in \mathfrak{A}_{12}$, by Claim 6, we have

$$f_n(a_{12}) = \delta_n(a_{12}) + \delta_{p_1\delta_n(p_1)p_2}(a_{12}) = \delta_n(a_{12}) \in \mathfrak{A}_{12}.$$

Claim 7. *$f_n(I) = p_1f_n(I)p_1 + p_2f_n(I)p_2 \in Z(\mathfrak{A})$ and $f_n(p_2) \in Z(\mathfrak{A})$.*

Since $Ip_1 = p_1$, using induction hypothesis, we have

$$\begin{aligned}0 &= f_n([[I, p_1], p_1]) \\ &= [[f_n(I), p_1], p_1] + [[I, f_n(p_1)], p_1] + [[I, p_1], f_n(p_1)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(I), f_j(p_1)], f_k(p_1)] \\ &= f_n(I)p_1 - p_1 f_n(I)p_1 - p_1 f_n(I)p_1 + p_1 f_n(I).\end{aligned}$$

On multiplying by p_1 and by p_2 from the left and the right respectively, we get $p_1 f_n(I)p_2 = 0$. Hence, we find that $f_n(I) = p_1 f_n(I)p_1 + p_2 f_n(I)p_2$. For any $a_{12} \in \mathfrak{A}_{12}$, since $(p_1 - a_{12})(I + a_{12}) = p_1$, using induction hypothesis, we have

$$\begin{aligned}-f_n(a_{12}) &= f_n([[p_1 - a_{12}, I + a_{12}], p_1]) \\ &= [[f_n(p_1) - f_n(a_{12}), I + a_{12}], p_1] + [[p_1 - a_{12}, f_n(I) + f_n(a_{12})], p_1] \\ &\quad + [[p_1 - a_{12}, I + a_{12}], f_n(p_1)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(p_1 - a_{12}), f_j(I + a_{12})], f_k(p_1)] \\ &= -f_n(a_{12})p_1 - a_{12} f_n(p_1)p_1 - p_1 f_n(p_1)a_{12} + p_1 f_n(a_{12})a_{12} + a_{12} f_n(p_1) \\ &\quad + p_1 f_n(I)p_1 + p_1 f_n(a_{12})p_1 - a_{12} f_n(I)p_1 - f(I)p_1 - p_1 f_n(I) \\ &\quad - p_1 f_n(a_{12}) + a_{12} f_n(I) + p_1 f_n(I)p_1 - p_1 f_n(I)a_{12} + p_1 f_n(a_{12})p_1 \\ &\quad - p_1 f_n(a_{12})a_{12} + a_{12} f_n(p_1) - f_n(p_1)a_{12}.\end{aligned}$$

Further, multiply by p_1 from the left and by p_2 from the right, we find that

$$\begin{aligned}a_{12} f_n(I)p_2 - p_1 f_n(I)a_{12} = 0 \\ a_{12} p_2 f_n(I)p_2 - p_1 f_n(I)p_1 a_{12} = 0.\end{aligned}$$

This implies that $f_n(I) = p_1 f_n(I)p_1 + p_2 f_n(I)p_2 \in Z(\mathfrak{A})$. Consequently, $f_n(p_2) = f_n(I) - f_n(p_1) \in Z(\mathfrak{A})$.

Claim 8. $f_n(\mathfrak{A}_{ii}) \subseteq \mathfrak{A}_{ii} \oplus \mathfrak{A}_{jj}$. *There exists an* $\mathcal{R}$*-linear map* $\tau_{ni} : \mathfrak{A}_{ii} \to Z(\mathfrak{A})$ *such that* $f_n(a_{ii}) - \tau_{ni}(a_{ii}) \in \mathfrak{A}_{ii}$ *for all* $a_{ii} \in \mathfrak{A}_{ii}$*, where* $i, j = 1, 2$ *and* $i \neq j$*.*

First we show the result for $i = 1$. Suppose that a_{11} is invertible in $\mathfrak{A}_{11}$, that is, there exists an element $a_{11}^{-1} \in \mathfrak{A}_{11}$ such that $a_{11}a_{11}^{-1} = a_{11}^{-1}a_{11} = p_1$. From $a_{11}a_{11}^{-1} = p_1$ and $(a_{11}^{-1} + p_2)a_{11} = p_1$, by induction hypothesis, we have

$$\begin{aligned}0 &= f_n([[a_{11}^{-1}, a_{11}], p_1]) \\ &= [[f_n(a_{11}^{-1}), a_{11}], p_1] + [[a_{11}^{-1}, f_n(a_{11})], p_1] + [[a_{11}^{-1}, a_{11}], f_n(p_1)] \\ &\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(a_{11}^{-1}), f_j(a_{11})], f_k(p_1)],\end{aligned}$$

and hence by Claim 7,

$$\begin{aligned}
0 &= f_n([[a_{11}^{-1}+p_2, a_{11}], p_1]) \\
&= [[f_n(a_{11}^{-1})+f_n(p_2), a_{11}], p_1] + [[a_{11}^{-1}+p_2, f_n(a_{11})], p_1] \\
&\quad + [[a_{11}^{-1}+p_2, a_{11}], f_n(p_1)] + \sum_{\substack{i+j+k=n\\ 0\le i,j,k<n}} [[f_i(a_{11}^{-1}+p_2), f_j(a_{11})], f_k(p_1)] \\
&= [[p_2, f_n(a_{11})], p_1] + \sum_{\substack{i+j+k=n\\ 0\le i,j,k<n}} [[f_i(p_2), f_j(a_{11})], f_k(p_1)] \\
&= p_2 f_n(a_{11})p_1 + p_1 f_n(a_{11})p_2.
\end{aligned}$$

This yields that $p_1(p_2f_n(a_{11})p_1 + p_1f_n(a_{11})p_2)p_2 = 0$ and hence we find that $p_1f_n(a_{11})p_2 = 0$. From this we get $f_n(a_{11}) \subseteq \mathfrak{A}_{11} \oplus \mathfrak{A}_{22}$.

On the other hand if a_{11} is not invertible in $\mathfrak{A}_{11}$, by the hypothesis (iii), there exists an integer t such that $tp_1 - a_{11}$ is invertible in $\mathfrak{A}_{11}$. It follows from the preceding case that $f_n(tp_1 - a_{11}) \in \mathfrak{A}_{11} \oplus \mathfrak{A}_{22}$. Therefore, we have $f_n(a_{11}) = tf_n(p_1) - f_n(tp_1 - a_{11}) \in \mathfrak{A}_{11} \oplus \mathfrak{A}_{22}$. Similarly, we can prove that for $i = 2$.

Now we can write $f_n(a_{11}) = p_1f_n(a_{11})p_1 + p_2f_n(a_{11})p_2$. First, suppose that a_{11} is invertible in $\mathfrak{A}_{11}$ with inverse element a_{11}^{-1}. Note that $a_{11}a_{11}^{-1} = p_1$ and $(a_{11}^{-1} + a_{22})a_{11} = p_1$, using induction hypothesis, we get

$$\begin{aligned}
0 &= f_n([[a_{11}^{-1}, a_{11}], x]) \\
&= [[f_n(a_{11}^{-1}), a_{11}], x] + [[a_{11}^{-1}, f_n(a_{11})], x] + [[a_{11}^{-1}, a_{11}], f_n(x)] \\
&\quad + \sum_{\substack{i+j+k=n\\ 0\le i,j,k<n}} [[f_i(a_{11}^{-1}), f_j(a_{11})], f_k(x)],
\end{aligned}$$

and hence

$$\begin{aligned}
0 &= f_n([[a_{11}^{-1}+a_{22}, a_{11}], x]) \\
&= [[f_n(a_{11}^{-1})+f_n(a_{22}), a_{11}], x] + [[a_{11}^{-1}+a_{22}, f_n(a_{11})], x] \\
&\quad + [[a_{11}^{-1}+a_{22}, a_{11}], f_n(x)] + \sum_{\substack{i+j+k=n\\ 0\le i,j,k<n}} [[f_i(a_{11}^{-1}+a_{22}), f_j(a_{11})], f_k(x)] \\
&= [[f_n(a_{22}), a_{11}], x] + [[a_{22}, f_n(a_{11})], x] \\
&= [[f_n(a_{11}), a_{22}] + [a_{11}, f_n(a_{22})], x].
\end{aligned}$$

Multiplying by p_2 on both the sides, we get

$$\begin{aligned}
0 &= p_2[[f_n(a_{11}), a_{22}] + [a_{11}, f_n(a_{22})], x]p_2 \\
&= [[p_2f_n(a_{11})p_2, p_2a_{22}p_2] + [p_2a_{11}p_2, p_2f_n(a_{22})p_2], p_2xp_2] \\
&= [[p_2f_n(a_{11})p_2, a_{22}], p_2xp_2].
\end{aligned}$$

This implies that $[p_2 f_n(a_{11})p_2, a_{22}] \in Z(\mathfrak{A}_{22})$. Hence by hypothesis (ii), we get $p_2 f_n(a_{11})p_2 \in Z(\mathfrak{A}_{22})$.

If a_{11} is not invertible in $\mathfrak{A}_{11}$, by the hypothesis *(iii)*, there exists an integer t such that $(tp_1 - a_{11})$ is invertible in $\mathfrak{A}_{11}$. It follows from the preceding case that

$$\begin{aligned}
0 &= f_n([[a_{22}, tp_1 - a_{11}], x]) \\
&= [[f_n(a_{22}), tp_1 - a_{11}], x] + [[a_{22}, tf_n(p_1) - f_n(a_{11})], x] + [[a_{22}, tp_1 - a_{11}], f_n(x)] \\
&\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(a_{22}), f_j(tp_1 - a_{11})], f_k(x)] \\
&= -[[f_n(a_{22}), a_{11}], x] - [[a_{22}, f_n(a_{11})], x] + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(a_{22}), f_j(tp_1 - a_{11})], f_k(x)] \\
&= [[f_n(a_{22}), a_{11}], x] + [[a_{22}, f_n(a_{11})], x] \\
&= [[f_n(a_{11}), a_{22}] + [a_{11}, f_n(a_{22})], x].
\end{aligned}$$

Multiplying by p_2 on both the sides, we get

$$\begin{aligned}
0 &= p_2[[f_n(a_{11}), a_{22}] + [a_{11}, f_n(a_{22})], x]p_2 \\
&= [[p_2 f_n(a_{11})p_2, p_2 a_{22} p_2] + [p_2 a_{11} p_2, p_2 f_n(a_{22})p_2], p_2 x p_2] \\
&= [[p_2 f_n(a_{11})p_2, a_{22}], p_2 x p_2].
\end{aligned}$$

This implies that $[p_2 f_n(a_{11})p_2, a_{22}] \in Z(\mathfrak{A}_{22})$. Hence by hypothesis (ii), we get $p_2 f_n(a_{11})p_2 \in Z(\mathfrak{A}_{22})$.
Define $\tau_{n1} : \mathfrak{A}_{11} \to Z(\mathfrak{A})$ by $\tau_{n1}(a_{11}) = \eta(p_2 f_n(a_{11})p_2) \oplus p_2 f_n(a_{11})p_2$, where η is the map defined in Lemma 1.1. Thus, we get

$$\begin{aligned}
f_n(a_{11}) - \tau_{n1}(a_{11}) &= p_1 f_n(a_{11})p_1 + p_2 f_n(a_{11})p_2 - \eta(p_2 f_n(a_{11})p_2) - p_2 f_n(a_{11})p_2 \\
&= p_1 f_n(a_{11})p_1 - \eta(p_2 f_n(a_{11})p_2) \in \mathfrak{A}_{11}.
\end{aligned}$$

Since f_n is $\mathcal{R}$-linear, one can verify that τ_{n1} is $\mathcal{R}$-linear. Similarly, we can define $\mathcal{R}$-linear map $\tau_{n2} : \mathfrak{A}_{22} \to Z(\mathfrak{A})$ by $\tau_{n2}(a_{22}) = p_1 f_n(a_{22})p_1 \oplus \eta^{-1}(p_1 f_n(a_{22})p_1)$. Then

$$\begin{aligned}
f_n(a_{22}) - \tau_{n2}(a_{22}) &= p_1 f_n(a_{22})p_1 + p_2 f_n(a_{22})p_2 - p_1 f_n(a_{22})p_1 - \eta^{-1}(p_1 f_n(a_{22})p_1) \\
&= p_2 f_n(a_{22})p_2 - \eta^{-1}(p_1 f_n(a_{22})p_1) \in \mathfrak{A}_{22}.
\end{aligned}$$

Now, for any $x = a_{11} + a_{12} + a_{22} \in \mathfrak{A}$, we define two $\mathcal{R}$-linear maps $\tau_n : \mathfrak{A} \to Z(\mathfrak{A})$ and $d_n : \mathfrak{A} \to \mathfrak{A}$ by $\tau_n(x) = \tau_{n1}(a_{11}) + \tau_{n2}(a_{22})$ and $d_n(x) = f_n(x) - \tau_n(x)$. Then, $d_n(\mathfrak{A}_{ij}) \subseteq \mathfrak{A}_{ij}$ for $1 \le i \le j \le 2$ and $d_n(a_{12}) = f_n(a_{12})$.

Claim 9. $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ *for all* $x, y \in \mathfrak{A}$.

Since f_n & τ_n are $\mathcal{R}$-linear and $d_n(x) = f_n(x) - \tau_n(x)$, d_n is an $\mathcal{R}$-linear. It remains to show that $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$, for all $x, y \in \mathfrak{A}$.

We divide the proof into the following three Steps:

Step 1. If a_{11} is invertible in $\mathfrak{A}_{11}$ with inverse element a_{11}^{-1}, then $(a_{11}^{-1} + a_{11}^{-1}a_{12})a_{11} = p_1$ for any $a_{11} \in \mathfrak{A}_{11}$, $a_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}
-d_n(a_{12}) &= d_n([[a_{11}, a_{11}^{-1} + a_{11}^{-1}a_{12}], p_1]) = f_n([[a_{11}, a_{11}^{-1} + a_{11}^{-1}a_{12}], p_1]) \\
&= [[f_n(a_{11}), a_{11}^{-1} + a_{11}^{-1}a_{12}], p_1] + [[a_{11}, f_n(a_{11}^{-1}) + f_n(a_{11}^{-1}a_{12})], p_1] \\
&\quad + [[a_{11}, a_{11}^{-1} + a_{11}^{-1}a_{12}], f_n(p_1)] \\
&\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(a_{11}), f_j(a_{11}^{-1} + a_{11}^{-1}a_{12})], f_k(p_1)].
\end{aligned}$$

Since,

$$\begin{aligned}
0 &= [[f_n(a_{11}), a_{11}^{-1}], p_1] + [[a_{11}, f_n(a_{11}^{-1})], p_1] + [[a_{11}, a_{11}^{-1}], f_n(p_1)] \\
&\quad + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(a_{11}), f_j(a_{11}^{-1})], f_k(p_1)],
\end{aligned}$$

we find that

$$\begin{aligned}
-d_n(a_{12}) &= [[f_n(a_{11}), a_{11}^{-1}a_{12}], p_1] + [[a_{11}, f_n(a_{11}^{-1}a_{12})], p_1] \\
&\quad + [[a_{11}, a_{11}^{-1}a_{12}], f_n(p_1)] + \sum_{\substack{i+j+k=n \\ 0\le i,j,k<n}} [[f_i(a_{11}), f_j(a_{11}^{-1}a_{12})], f_k(p_1)] \\
&= [[f_n(a_{11}), a_{11}^{-1}a_{12}], p_1] + [[a_{11}, f_n(a_{11}^{-1}a_{12})], p_1] \\
&\quad + \sum_{\substack{i+j=n \\ 0<i,j<n}} [[f_i(a_{11}), f_j(a_{11}^{-1}a_{12})], p_1] \\
&= [[d_n(a_{11}) + \tau_n(a_{11}), a_{11}^{-1}a_{12}], p_1] \\
&\quad + [[a_{11}, d_n(a_{11}^{-1}a_{12}) + \tau_n(a_{11}^{-1}a_{12})], p_1] \\
&\quad + \sum_{\substack{i+j=n \\ 0<i,j<n}} [[d_i(a_{11}) + \tau_i(a_{11}), d_j(a_{11}^{-1}a_{12}) + \tau_j(a_{11}^{-1}a_{12})], p_1] \\
&= [[d_n(a_{11}), a_{11}^{-1}a_{12}], p_1] + [[a_{11}, d_n(a_{11}^{-1}a_{12})], p_1] \\
&\quad + \sum_{\substack{i+j=n \\ 0<i,j<n}} [[d_i(a_{11}), d_j(a_{11}^{-1}a_{12})], p_1] \\
&= [[d_n(a_{11}), a_{11}^{-1}a_{12}], p_1] + [[a_{11}, d_n(a_{11}^{-1}a_{12})], p_1] \\
&\quad - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(a_{11})d_j(a_{11}^{-1}a_{12})
\end{aligned}$$

$$= -d_n(a_{11})a_{11}^{-1}a_{12} - a_{11}d_n(a_{11}^{-1}a_{12}) - \sum_{\substack{i+j=n\\0<i,j<n}} d_i(a_{11})d_j(a_{11}^{-1}a_{12})$$
$$= -\sum_{i+j=n} d_i(a_{11})d_j(a_{11}^{-1}a_{12}).$$

Hence, $d_n(a_{12}) = \sum_{i+j=n} d_i(a_{11})d_j(a_{11}^{-1}a_{12})$. Replacing a_{12} by $a_{11}a_{12}$, we arrive at

$$d_n(a_{11}a_{12}) = \sum_{i+j=n} d_i(a_{11})d_j(a_{12}).$$

For any $a_{11} \in \mathfrak{A}_{11}$, let $tp_1 - a_{11}$ be invertible in $\mathfrak{A}_{11}$. Then

$$d_n((tp_1 - a_{11})a_{12}) = \sum_{i+j=n} d_i(tp_1 - a_{11})d_j(a_{12}).$$

Since $d_n(p_1a_{12}) = \sum_{i+j=n} d_i(p_1)d_j(a_{12})$, we have $d_n(a_{11}a_{12}) = \sum_{i+j=n} d_i(a_{11})d_j(a_{12})$.

Step 2. Let $a_{12} \in \mathfrak{A}_{12}$ and $a_{22} \in \mathfrak{A}_{22}$. Observe that $(p_1+a_{12})(p_1+a_{22}-a_{12}a_{22}) = p_1$ and $(p_1 + a_{22} - a_{12}a_{22})(p_1 + a_{12}) = p_1 + a_{12}$. Since $f_n(p_1) \in Z(\mathfrak{A})$, we have

$$\begin{aligned}
-d_n(a_{12}) &= d_n([[p_1 + a_{22} - a_{12}a_{22}, p_1 + a_{12}], p_1])\\
&= f_n([[p_1 + a_{22} - a_{12}a_{22}, p_1 + a_{12}], p_1])\\
&= [[f_n(p_1) + f_n(a_{22}) - f_n(a_{12}a_{22}), p_1 + a_{12}], p_1]\\
&\quad + [[p_1 + a_{22} - a_{12}a_{22}, f_n(p_1) + f_n(a_{12})], p_1]\\
&\quad + [[p_1 + a_{22} - a_{12}a_{22}, p_1 + a_{12}], f_n(p_1)]\\
&\quad + \sum_{\substack{i+j+k=n\\0\le i,j,k<n}} [[f_i(p_1 + a_{22} - a_{12}a_{22}), f_j(p_1 + a_{12})], f_k(p_1)]\\
&= [[f_n(a_{22}) - f_n(a_{12}a_{22}), p_1 + a_{12}], p_1]\\
&\quad + [[p_1 + a_{22} - a_{12}a_{22}, f_n(a_{12})], p_1]\\
&\quad + \sum_{\substack{i+j+k=n\\0\le i,j,k<n}} [[f_i(p_1 + a_{22} - a_{12}a_{22}), f_j(p_1 + a_{12})], f_k(p_1)]\\
&= [[d_n(a_{22}) - d_n(a_{12}a_{22}), p_1 + a_{12}], p_1]\\
&\quad + [[p_1 + a_{22} - a_{12}a_{22}, d_n(a_{12})], p_1]\\
&\quad + \sum_{\substack{i+j+k=n\\0\le i,j,k<n}} [[d_i(p_1 + a_{22} - a_{12}a_{22}), d_j(p_1 + a_{12})], d_k(p_1)]
\end{aligned}$$

$$
\begin{aligned}
&= [[d_n(a_{22}) - d_n(a_{12}a_{22}), p_1 + a_{12}], p_1] \\
&\quad + [[p_1 + a_{22} - a_{12}a_{22}, d_n(a_{12})], p_1] + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(a_{12})d_j(a_{22}) \\
&= -d_n(a_{12}a_{22}) + a_{12}d_n(a_{22}) - d_n(a_{12}) \\
&\quad + d_n(a_{12})a_{22} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(a_{12})d_j(a_{22}).
\end{aligned}
$$

Thus, $d_n(a_{12}a_{22}) = \sum_{i+j=n} d_i(a_{12})d_j(a_{22})$ for any $a_{12} \in \mathfrak{A}_{12}, a_{22} \in \mathfrak{A}_{22}$.

Using the same approach as used in the proof of Claim 13 of Theorem 2.1, we find that

Step 3. For any $a_{11}, b_{11} \in \mathfrak{A}_{11}$ and $a_{22}, b_{22} \in \mathfrak{A}_{22}$,

(i) $d_n(a_{11}b_{11}) = \sum_{i+j=n} d_i(a_{11})d_j(b_{11})$,

(ii) $d_n(a_{22}b_{22}) = \sum_{i+j=n} d_i(a_{22})d_j(b_{22})$.

Step 4. $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in \mathfrak{A}$.

Claim 10. *τ_n vanishes at second commutator $[[x, y], z]$ with $xy = p$ for all $x, y, z \in \mathfrak{A}$.*

Since $xy = p$, we find that

$$
\begin{aligned}
\tau_n([[x, y], z]) &= f_n([[x, y], z]) - d_n([[x, y], z]) \\
&= \sum_{i+j+k=n} ([[f_i(x), f_j(y)], f_k(z)]) - d_n([[x, y], z]) \\
&= \sum_{i+j+k=n} ([[d_i(x) + \tau_i(x), d_j(y) + \tau_j(y)], d_k(z) + \tau_k(z)]) \\
&\quad - d_n([[x, y], z]) \\
&= \sum_{i+j+k=n} [[d_i(x), d_j(y)], d_k(z)] - d_n([[x, y], z]) \\
&= 0
\end{aligned}
$$

for all $x, y, z \in \mathfrak{A}$. The proof is now complete.

4. Applications

As an application of Theorems 2.1 & 3.1, we consider the nest algebra case. We know that every nontrivial nest algebra is a triangular algebra (see [11]), which satisfies the conditions of Theorems 2.1 & 3.1 and hence we have the following results.

Theorem 4.1. *Let $\mathcal{N}$ be an arbitrary nontrivial nest on a Hilbert space T of dimension greater than 2, $Alg\mathcal{N}$ be the associated nest algebra. If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{R}$-linear maps $\delta_n : Alg\mathcal{N} \to Alg\mathcal{N}$ satisfying $\delta_n[[x,y],z] = \sum_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$ for all $x, y, z \in Alg\mathcal{N}$ with $xy = 0$. Then for each $n \in \mathbb{N}$, $\delta_n(x) = h_n(x) + \tau_n(x)$ for all $x \in Alg\mathcal{N}$; where $H = \{h_n\}_{n\in\mathbb{N}}$ is an inner higher derivation on $Alg\mathcal{N}$ and $\tau_n : Alg\mathcal{N} \to \mathcal{F}I$ (where $\mathcal{F}I$ is the center of $Alg\mathcal{N}$) is an $\mathcal{R}$-linear map vanishing at the second commutator $[[x,y],z]$ with $xy = 0$.*

Proof. Since $\mathcal{N}$ is nontrivial nest, the associated nest algebra is a triangular algebra which satisfies the conditions of Theorem 2.1. Then there exists a higher derivation $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ of $Alg\mathcal{N}$ and a linear map $\tau_n : Alg\mathcal{N} \to \mathcal{F}I$ vanishing at the second commutator $[[x,y],z]$ with $xy = 0$ such that for each $n \in \mathbb{N}$, $\delta_n(x) = d_n(x) + \tau_n(x)$ for all $x \in Alg\mathcal{N}$. Since every higher derivation on $Alg\mathcal{N}$ is inner (see [10, 21]), there is an inner higher derivation $H = \{h_n\}_{n\in\mathbb{N}}$ on $Alg\mathcal{N}$. This implies that for each $n \in \mathbb{N}$ $\delta_n(x) = h_n(x) + \tau_n(x)$ for all $x \in Alg\mathcal{N}$. □

Theorem 4.2. *Let $\mathcal{N}$ be a nontrivial nest on a Hilbert space T of dimension greater than 2, $Alg\mathcal{N}$ be the associated nest algebra and p be a nontrivial projection in $\mathcal{N}$. If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{R}$-linear maps $\delta_n : Alg\mathcal{N} \to Alg\mathcal{N}$ satisfying $\delta_n[[x,y],z] = \sum_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$ for all $x, y, z \in Alg\mathcal{N}$ with $xy = p$. Then for each $n \in \mathbb{N}$, $\delta_n(x) = h_n(x) + \tau_n(x)$ for all $x \in Alg\mathcal{N}$; where $H = \{h_n\}_{n\in\mathbb{N}}$ is an inner higher derivation on $Alg\mathcal{N}$ and $\tau_n : Alg\mathcal{N} \to \mathcal{F}I$ (where $\mathcal{F}I$ is the center of $Alg\mathcal{N}$) is an $\mathcal{R}$-linear map vanishing at the second commutator $[[x,y],z]$ with $xy = p$.*

Proof. Let $\mathfrak{A}_{11} = pAlg\mathcal{N}p$, $\mathfrak{A}_{22} = (I-p)Alg\mathcal{N}(I-p)$ and $\mathfrak{A}_{12} = pAlg\mathcal{N}(I-p)$. Then $\mathfrak{A}_{11}$ and $\mathfrak{A}_{22}$ are unital algebras with unit element p and $I - p$ respectively and $Alg\mathcal{N} = Tri(\mathfrak{A}_{11}, \mathfrak{A}_{12}, \mathfrak{A}_{22})$ is a triangular algebra. Also $Alg\mathcal{N}$ satisfies the conditions of Theorem 3.1, then there exists a higher derivation $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ of $Alg\mathcal{N}$ and a linear map $\tau_n : Alg\mathcal{N} \to \mathcal{F}I$ vanishing at the second commutator $[[x,y],z]$ with $xy = p$ such that for each $n \in \mathbb{N}$, $\delta_n(x) = d_n(x) + \tau_n(x)$ for all $x \in Alg\mathcal{N}$. Since every higher derivation on $Alg\mathcal{N}$ is inner (see [10, 21]) there exists an inner higher derivation $H = \{h_n\}_{n\in\mathbb{N}}$ on $Alg\mathcal{N}$ such that for each $n \in \mathbb{N}$, $\delta_n(x) = h_n(x) + \tau_n(x)$ for all $x \in Alg\mathcal{N}$. □

If Hilbert space T is finite dimensional, then nest algebras are upper block triangular matrices algebras [7].

Theorem 4.3. *Let $\mathcal{B}_n(\mathcal{R})$ be a proper block upper triangular matrix algebra over a commutative ring $\mathcal{R}$. If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{R}$-linear maps $\delta_n : \mathcal{B}_n(\mathcal{R}) \to \mathcal{B}_n(\mathcal{R})$ satisfying $\delta_n([[x,y],z]) = \sum_{i+j+k=n} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$ for all $x, y, z \in \mathcal{B}_n(\mathcal{R})$ with $xy = 0$ (resp. $xy = p$, p be a nontrivial projection in $\mathcal{B}_n(\mathcal{R})$). Then for each $n \in \mathbb{N}$, $\delta_n(x) = h_n(x) + \tau_n(x)$ for all $x \in \mathcal{B}_n(\mathcal{R})$; where $H = \{h_n\}_{n\in\mathbb{N}}$ is an inner higher derivation on $\mathcal{B}_n(\mathcal{R})$ and $\tau_n : \mathcal{B}_n(\mathcal{R}) \to \mathcal{F}I$ (where $\mathcal{F}I$ is the center of $\mathcal{B}_n(\mathcal{R})$) is an $\mathcal{R}$-linear map vanishing at the second commutator $[[x,y],z]$ with $xy = 0$ (resp.*

$xy = p$).

Proof. It can be easily seen that conditions of Theorems 2.1 & 3.1 hold for block upper triangular matrix algebra and from [21, Proposition 2.6] all higher derivations of $\mathcal{B}_n(\mathcal{R})$ are inner. Hence δ_n is the sum of an inner higher derivation $h_n : \mathcal{B}_n(\mathcal{R}) \to \mathcal{B}_n(\mathcal{R})$ and a functional $\tau_n : \mathcal{B}_n(\mathcal{R}) \to \mathcal{F}I$ that vanishes on all second commutators of $\mathcal{B}_n(\mathcal{R})$. □

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