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Direct Sums of Strongly Lifting Modules

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Abstract. For the recently defined notion of strongly lifting modules, it has been shown that a direct sum is not, in general, strongly lifting. In this paper we investigate the question: When are the direct sums of strongly lifting modules, also strongly lifting? We introduce the notion of a relatively strongly projective module and use it to show if $M = M_1 \oplus M_2$ is amply supplemented, then M is strongly lifting if and only if M_1 and M_2 are relatively strongly projective and strongly lifting. Also, we consider when an arbitrary direct sum of hollow (resp. local) modules is strongly lifting.

1. Introduction

Supplemented and lifting modules are worthy of study in module theory since they are the duals of complemented and extending modules. A number of results concerning lifting modules have appeared in the literature in recent years. Lifting modules were first introduced by Takeuchi [11] but under the name codirect modules. An R-module M is called *lifting* if every submodule of M lies above a direct summand. The notion of strongly extending modules was introduced in [5]. In this paper, we study modules with properties that are dual to strongly extending modules. The notion of strongly lifting modules was introduced in [8, 15]. An R-module M is called a strongly lifting module if for any submodule N of M, there exists a fully invariant direct summand K of M such that $K \subseteq N$ and N/K is a small submodule of M/K.

It is of natural interest to investigate whether or not an algebraic notion for modules is inherited by direct summands and direct sums. The purpose of this paper is to study the direct sum of strongly lifting modules. The direct sum of two

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strongly lifting modules need not be strongly lifting. We look at when direct sums of finitely many strongly lifting modules are strongly lifting. It is shown that every strongly lifting module is a direct sum of hollow modules and every strongly lifting module is π-projective and a direct sum of hollow modules. We introduce the notion of relatively strongly projective modules and use it to show if $M = M_1 \oplus M_2$ is amply supplemented, then M is strongly lifting if and only if M_1 and M_2 are relatively strongly projective and strongly lifting. Also, we consider when an arbitrary direct sum of hollow (resp. local) modules is strongly lifting.

Throughout, all rings (not necessarily commutative rings) have identity and all modules are unital right modules. For completeness, we now state some definitions and notations used in this paper. Let M be a module over a ring R. For submodules N and K of M, $N \leq K$ means that N is a submodule of K and $\mathrm{End}(M)$ denotes the ring of right R-module endomorphisms of M. We denote module direct summands by "$\leq^{\oplus}$". The symbols $\mathbb{Z}$, $\mathbb{Z}_{\mathrm{n}}$ and $\mathbb{Q}$ stand for the ring of integers, the ring of residues modulo n and ring of rational numbers, respectively. Let M be a module. Let N and L be submodules of M. N is called a *supplement* of L if it is minimal with the property $M = N + L$, equivalently, $M = N + L$ and $N \cap L \ll N$. N is called a *supplement submodule* if N is a supplement of some submodule of M. A module M is called a *supplemented* module if every submodule of M has a supplement. A module M is called *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A such that $P \subseteq B$. Let M be a module and $K \leq N \leq M$. If $N/K \ll M/K$, then N is called *coessential* submodule of M and denoted by $K \overset{ce}{\hookrightarrow} N$. Also N is called *coessential* extension of K. A submodule K of M is called *coclosed* if K has no proper coessential submodule; this is denoted by $N \overset{cc}{\hookrightarrow} M$ (every supplement submodule is coclosed). Furthermore, N is called the *s-closure* of K in M, if $K \overset{ce}{\hookrightarrow} N$ and $K \overset{cc}{\hookrightarrow} M$. An idempotent $e \in R$ is called *left (resp. right) semicentral* if $re = ere$ (resp. $er = ere$), for each $r \in R$, equivalently, eR (resp. Re) is an ideal of R. The set of all left (resp. right) semicentral idempotents of R will be denoted by $S_l(R)$ (resp. $S_r(R)$). If $e^2 = e \in \mathrm{End}(M)$, then $e \in S_l(End(M))$ if and only if eM is a fully invariant direct summand. Also $e \in S_l(R)$ if and only if $1 - e \in S_r(R)$ [2, 3]. A module M is said to have the *strong summand sum property (SSSP)*, if the sum of any family of direct summands is a direct summand of M [14]. A module M is called a C3-module, if M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = \{0\}$, then $M_1 \oplus M_2$ is also a direct summand of M [5]. In [12], Talebi and Vanaja defined

$$\overline{Z}(M) = \cap\{Ker(\varphi) : \varphi \in \mathrm{Hom}(M, N), N \ll E(N)\}$$

and $\overline{Z}^2(M) = \overline{Z}(\overline{Z}(M))$, where $E(N)$ is the injective hull of N. A module M is called a *cosingular* (resp. *noncosingular*) module, if $Z(M) = 0$ (resp. $Z(M) = M$) [12]. A module M is called a $\mathcal{T}$ *-non-cosingular* module if, for every non-zero endomorphism f of M, $\mathrm{Im}(f)$ is not small in M [14]. A submodule N of a module M is called *t-small* in M, denoted by $N \ll_t M$, if for every submodule K of M, $\overline{Z}^2(M) \subseteq N + K$ implies that $\overline{Z}^2(M) \subseteq K$ [1]. A module M is called *t-lifting* if every submodule N of M contains a direct summand K of M such that $N/K \ll_t M/K$ [1]. A module M is called *dual Rickart* (resp. *t-dual Rickart*) if for each $\varphi \in \mathrm{End}(M)$, $\varphi(M)$ (resp. $\varphi(\overline{Z}^2(M))$) is a direct summand of M [9] ([6]).

The following are used in the sequel.

Proposition 1.1.

(i) ([13, Proposition 1.5]) *Let M be an amply supplemented module. Then every submodule of M has an s-closure.*

(ii) ([4, 3.7(6)]) *Let M be an R-module and $K \leq L \leq M$. If $K \overset{cc}{\hookrightarrow} M$, then $K \overset{cc}{\hookrightarrow} L$ and the converse is true if $L \overset{cc}{\hookrightarrow} M$.*

(iii) ([13, Lemma 1.4(2)]) *Let M be an amply supplemented module and $B \leq C$ submodules of M such that C/B is co-closed in M/B and B is co-closed in M. Then C is co-closed in M.*

(iv) ([2, Lemma 1.1]) *If $M = \oplus_{i \in I} M_i$ and N is a fully invariant submodule of M, then $N = \oplus_{i \in I}(N \cap M_i)$.*

Theorem 1.2.([8, 15]) *The following are equivalent for an R-module M with $S = \mathrm{End}(M)$.*

(1) *M is strongly lifting;*

(2) *For each $N \leq M$, there exists $e \in S_l(S)$ such that $eM \subseteq N$ and $(1-e)M \cap N \ll (1-e)M$.*

(3) *M is lifting and each direct summand of M is fully invariant.*

(4) *M is lifting and S is Abelian.*

(5) *M is amply supplemented and each coclosed submodule is a fully invariant direct summand in M.*

Proposition 1.3.([8, 15]) *Let M be a strongly lifting module. Then each direct summand of M is strongly lifting.*

Theorem 1.4.([8, 15]) *If M is strongly lifting with $S = \mathrm{End}(M)$, then M has SSSP.*

Lemma 1.5.([8, 15]) *Let $M = M_1 \oplus M_2$ be an amply supplemented module. Suppose that for every co-closed submodule N of M such that either $M = N + M_1$ or $M = N + M_2$, N is a fully invariant direct summand of M. Then M is strongly lifting.*

2. Direct Sums of Strongly Lifting Modules

This section is devoted to investigate when direct sums of strongly lifting modules are strongly lifting. While it was shown in [8, 15] that every direct summand of a strongly lifting module is always strongly lifting, the following examples show that in general, the direct sum of strongly lifting modules is not a strongly lifting module.

Example 2.1.

(1) Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$, $M_1 = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. Then M_1 and M_2 are strongly lifting R-modules, however $R = M_1 \oplus M_2$ is not strongly lifting, by Theorem 1.2.

(2) Let R be a uniserial ring and I a nonzero proper right ideal of R. Then R and I are strongly lifting R-modules. If $R \oplus I$ is strongly lifting, then $R \oplus I$ has SSSP. So it is a C3-module. This implies that $I \leq^{\oplus} R$, by [7, Corollary 3.2], which is a contradiction.

Proposition 2.2.

(i) *Let $M = \oplus_{i \in I} M_i$. If M is a strongly lifting module, then $Hom(M_i, M_j) = 0$ for each $i \neq j$ of I.*

(ii) *A free R-module F is strongly lifting if and only if R_R is strongly lifting and* $\text{rank}(F) = 1$.

Proof. (i) As M is strongly lifting, every direct summand of M is fully invariant in M. Hence the result is clear.

(ii) Assume that F is a strongly lifting free R-module. If $\text{rank}(F) \geq 2$, then (i) gives $\text{Hom}(R, R) = 0$, a contradiction. Thus $\text{rank}(F) = 1$, and R is strongly lifting. The converse is clear. □

Theorem 2.3.

(i) *Let M be an R-module. The following are equivalent:*

(1) *M is strongly lifting;*

(2) *If $M = N + K$ $(N, K \leq M)$, then there exists a fully invariant direct summand X say $M = X \oplus Y$ such that $X \subseteq N$, $Y \subseteq K$ and $Y \cap N \ll Y$.*

(ii) *Let M be a strongly lifting module. Then M is π-projective.*

Proof. (i) (1) $\Rightarrow$ (2) Let $M = N + K$. As M is amply supplemented, there exists $Y \subseteq K$ such that $M = Y + N$ and $Y \cap N \ll Y$. Also, there exists $X \subseteq N$ such that $M = X + Y$ and $Y \cap X \ll X$. Therefore $X \cap Y \ll M$. As X is supplement of Y and Y is supplement of X, X and Y are coclosed submodules of M. Hence X and Y are fully invariant direct summands of M by Theorem 1.2. Let $X = eM$ and $Y = fM$ for some $e, f \in S_l(\text{End}(M))$. It can be seen $X \cap Y = efM \leq^{\oplus} M$. As $X \cap Y \ll M$, $X \cap Y = 0$. Therefore $M = X \oplus Y$ and $X \subseteq N$ and $Y \subseteq K$ and $Y \cap N \ll Y$.

(2) $\Rightarrow$ (1) is clear.

(ii) is clear from (i). □

Theorem 2.4. *Let M be a strongly lifting module. Then there is a decomposition $M = \oplus_{i\in I} M_i$ with hollow modules M_i, and, for every direct summand N of M, there exists a subset J of I with $M = (\oplus_{i\in J} M_i) \oplus N$.*

Proof. Let M be a strongly lifting module. Then M has SSSP by Theorem 1.4. So by [10, Theorem 2.17], $M = \oplus_{i\in I} M_i$, where M_i is indecomposable for each $i \in I$. By Proposition 1.3, M_i is strongly lifting for each $i \in I$. An inspection shows that M_i is hollow for each $i \in I$.

Now, let N be a direct summand of M. Then by Theorem 1.2, N is fully invariant in M. Hence by Proposition 1.1(iv), $N = \oplus_{i\in I}(N \cap M_i)$. As $N \leq^{\oplus} M$, $N \cap M_i \leq^{\oplus} M_i$ for each $i \in I$. Since M_i is indecomposable, either $N \cap M_i = M_i$ or $N \cap M_i = 0$. Thus $N = \oplus_{i\in K} M_i$ for some $K \subseteq I$ and so $M = (\oplus_{i\in J} M_i) \oplus N$ for some $J \subseteq I$. □

Corollary 2.5. *Let R be a ring. Then R is strongly lifting as an R-module if and only if R is a direct product of finite local rings.*

Proof. Let R be strongly lifting as an R-module. By Theorem 2.4, $R = \oplus_{i=1}^{n} H_i$ where $n \in \mathbb{N}$ and H_i is a hollow module, for each $1 \leq i \leq n$. By Theorem 1.2, every H_i is fully invariant. It is clear that H_i is finitely generated, for each $1 \leq i \leq n$. Therefore $R = R_1 \times R_2 \times ... \times R_n$, where each R_i is a local ring. Conversely, assume that R is a direct product of finite local rings. This implies that R is a semiperfect ring by [16, 42.6]. Hence R is lifting as an R-module. Moreover, R is Abelian. Hence by Theorem 1.2, R is strongly lifting. □

By Corollary 2.5, we can see that every strongly lifting ring (considered as a module over itself) is semiperfect, the following example shows that the converse is not true, in general.

Example 2.6. Let $R = \begin{pmatrix} F & F \\ F & F \end{pmatrix}$, where F is a field. Then R is a semiperfect ring, which is not strongly lifting as an R-module.

We now define a relative version of particular projective condition which is useful in our main theorems.

Definition 2.7. Let M and N be two R-modules. Then M is called *N-strongly projective* (or *strongly projective relative to* N) if $\mathrm{Hom}(M, T) = 0$ for each factor module T of N.

This is obviously true if and only if M is N-projective and $\mathrm{Hom}(M, N) = 0$. R-modules $\{M_i : i \in I\}$ are called relatively strongly projective if M_i is M_j-strongly projective for all distinct $i, j \in I$.

Lemma 2.8.

(i) *Let M and N be two modules. If M is N-strongly projective and $N' \leq N$, then M is N'-strongly projective and N/N'-strongly projective.*

(ii) $\oplus_{i\in I}M_i$ *is* N*-strongly projective if and only if* M_i *is* N*-strongly projective for each* $i \in I$.

(iii) M *is* $\oplus_{i=1}^{n}N_i$*-strongly projective if and only if* M *is* N_i*-strongly projective for each* $1 \leq i \leq n$.

(iv) *Let* M *be a finitely generated module. Then* M *is* $\oplus_{i\in I}N_i$*-strongly projective if and only if* M *is* N_i*-strongly projective for each* $i \in I$.

Proof. (i) Suppose that $\mathrm{Hom}(M, T) = 0$ for each factor module T of N and $N' \leq N$. Then $\mathrm{Hom}(M, T') = 0$ for each factor module T' of N', because every factor module of N' is a submodule of a factor module of N. Now, since every factor module of N/N' is a factor module of N, $\mathrm{Hom}(M, T'') = 0$ for each factor module T'' of N/N'.

(ii) Let T be a factor module of N. Then $\mathrm{Hom}(\oplus_{i\in I}M_i, T) = 0$ if and only if $\mathrm{Hom}(M_i, T) = 0$ for each $i \in I$. Hence $\oplus_{i\in I}M_i$ is N-strongly projective if and only if M_i is N-strongly projective for each $i \in I$.

(iii) The necessity is clear by (i). For the sufficiency, it is sufficient to show that if M is relative strongly projective to N_1 and N_2, then M is $N_1 \oplus N_2$-strongly projective. Let $f \in \mathrm{Hom}(M, (N_1 \oplus N_2)/L)$ where $L \leq N_1 \oplus N_2$. Let $\pi : (N_1 \oplus N_2)/L \to (N_1 \oplus N_2)/(N_1 + L)$ be natural homomorphism. As $(N_1 \oplus N_2)/(N_1 + L)$ is a factor module of N_2 and M is N_2-strongly projective, $\pi f = 0$. Hence $f(M) \subseteq \mathrm{Ker}(\pi) = (N_1 + L)/L$. Since M is N_1-strongly projective, $f = 0$. Hence M is $N_1 \oplus N_2$-strongly projective.

(iv) Assume that M is $\oplus_{i\in I}N_i$-strongly projective. Then M is N_i-strongly projective by (i). Conversely, let M be N_i-strongly projective for each $i \in I$ and $f \in \mathrm{Hom}(M, (\oplus_{i\in I}N_i)/L)$ where $L \leq \oplus_{i\in I}N_i$. Since M is finitely generated, $f(M) \subseteq (\oplus_{i\in F}N_i + L)/L$ for some finite subset F of I. By (iii), M is $\oplus_{i\in F}N_i$-strongly projective; hence $f = 0$. Thus M is $\oplus_{i\in I}N_i$-strongly projective. □

The result of Lemma 2.8(iii) does not extend to infinite direct sums, as the following example shows.

Example 2.9. It is clear that $\mathbb{Q}$ is $\mathbb{Z}$-strongly projective because $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_{\mathrm{n}}) = 0$ for each integer n. As Q is a homomorphic image of a free $\mathbb{Z}$-module $\mathbb{Z}^{(\mathrm{I})}$ for some infinite set I, $\mathrm{Hom}(\mathbb{Q}, \mathrm{T}) \neq 0$ for some factor module T of $\mathbb{Z}^{(\mathrm{I})}$. Hence $\mathbb{Q}$ is not $\mathbb{Z}^{(\mathrm{I})}$-strongly projective.

Lemma 2.10. *Let* $M = M_1 \oplus M_2$. *Then the following are equivalent:*

(1) M_2 *is* M_1*-strongly projective;*

(2) *If* $M = K + M_1$ *for some* $K \leq M$, *then* $M_2 \subseteq K$.

Proof. (1) ⇒ (2) Let K be a submodule of M such that $M = K+M_1$. Hence we have $M_2 \cong M/M_1 = (K+M_1)/M_1 \cong K/(M_1 \cap K)$. We show $K = (K \cap M_1) \oplus (K \cap M_2)$. Define $f : K/(M_1 \cap K) \to M_1/(M_1 \cap K)$ so that $f(k + M_1 \cap K) = k_1 + M_1 \cap K$ where $k = k_1 + k_2$, $k_1 \in M_1$ and $k_2 \in M_2$. Clearly, f is a homomorphism. Since $\mathrm{Hom}(M_2, M_1/(M_1 \cap K)) = 0$ and $M_2 \cong K/(K \cap M_1)$, we have $f = 0$. This implies that for each $k = k_1 + k_2$ where $k_1 \in M_1$ and $k_2 \in M_2$, $k_1 \in M_1 \cap K$. Therefore $k_2 \in M_2 \cap K$. Hence $K = (K \cap M_1) \oplus (K \cap M_2)$. As $M = K + M_1$, $M = M_1 \oplus (K \cap M_2)$. By using modular law we have $M_2 = K \cap M_2$. Thus $M_2 \subseteq K$.

(2) $\Rightarrow$ (1) Let $f : M_2 \to L$ be a homomorphism and L a factor module of M_1. Let $\pi : M_1 \to L$ be natural homomorphism. Set $K = \{m_2 - m_1 : m_2 \in M_2,\ m_1 \in M_1 \text{ and } f(m_2) = \pi(m_1)\}$. It can be seen that K is a submodule of M and $M = K + M_1$. By (2), $M_2 \subseteq K$. This implies that $f(m_2) = 0$ for each $m_2 \in M_2$. Thus $f = 0$, as desired. □

In the following theorem, we present necessary and sufficient conditions under which direct sum of finite strongly lifting modules is strongly lifting.

Theorem 2.11. *Let $M = M_1 \oplus M_2$ and M be an amply supplemented module. Then M is strongly lifting if and only if*

(i) *M_1 and M_2 are relatively strongly projective.*

(ii) *M_1 and M_2 are strongly lifting.*

Proof. We show M_2 is M_1-strongly projective. Let $M = K + M_1$ for some $K \leq M$. By Theorem 2.3, there exists a fully invariant direct summand X of M say $M = X \oplus Y$ such that $X \subseteq M_1$ and $Y \subseteq K$ and $Y \cap M_1 \ll M$. As M is strongly lifting, every direct summand of M is fully invariant by Theorem 1.2, this implies that $M_1 \cap Y \leq^{\oplus} M$. Hence $M_1 \cap Y = 0$ and $M = Y \oplus M_1$. Since M_2 is fully invariant, $M_2 = (M_2 \cap Y) \oplus (M_2 \cap M_1) = M_2 \cap Y$, by Proposition 1.1(iv). Thus $M_2 \subseteq Y \subseteq K$. Therefore, by Lemma 2.10, M_2 is M_1-strongly projective. Similarly M_1 is M_2-strongly projective.

Conversely, let K be a coclosed submodule of M such that $K + M_1 = M$. By Lemma 2.10, $M_2 \subseteq K$ and so $K = M_2 \oplus (K \cap M_1)$. Since $K \cap M_1 \leq^{\oplus} K$, $K \cap M_1 \overset{cc}{\hookrightarrow} K$. As $K \cap M_1 \leq K \leq M$ and $K \cap M_1 \overset{cc}{\hookrightarrow} K$ and $K \overset{cc}{\hookrightarrow} M$, We have $K \cap M_1 \overset{cc}{\hookrightarrow} M$ by Proposition 1.1(ii). Since $K \cap M_1 \leq M_1 \leq M$, and $K \cap M_1 \overset{cc}{\hookrightarrow} M$, we have $K \cap M_1 \overset{cc}{\hookrightarrow} M_1$ by Proposition 1.1(ii). Since M_1 is strongly lifting, $K \cap M_1$ is a fully invariant direct summand of M_1. Therefore $K = K \cap M_1 \oplus M_2 \leq^{\oplus} M$. We show K is fully invariant in M. Let $f \in \mathrm{End}(M)$. Since $\mathrm{Hom}(M_1, M_2) = 0$ and $\mathrm{Hom}(M_2, M_1) = 0$, $f = f_1 \oplus f_2$ where $f_1 \in \mathrm{End}(M_1)$ and $f_2 \in \mathrm{End}(M_2)$. Thus $f(K) = f_1(K \cap M_1) \oplus f_2(M_2)$. Since $K \cap M_1$ is fully invariant in M_1, $f_1(K \cap M_1) \subseteq K \cap M_1$. This implies that $f(K) \subseteq K$ and so K is fully invariant. Similarly, if K is a coclosed submodule of M with $K + M_2 = M$, then K is a fully invariant submodule of M. Thus by Lemma 1.5, M is strongly lifting. □

Corollary 2.12. *Let $M = M_1 \oplus \ldots \oplus M_n$ be a finite direct sum of modules M_i. Then M is strongly lifting if and only if M is amply supplemented, for each $1 \leq i \leq n$, M_i is strongly lifting and modules $\{M_i\}_{i=1}^{n}$ are relatively strongly projective.*

Proof. Let M be strongly lifting. Then by Theorem 2.11, M_i is $\oplus_{j \neq i} M_i$-strongly projective. Hence M_i is M_j-strongly projective by Lemma 2.8. Hence $\{M_i\}_{1 \leq i \leq n}$

are relatively strongly projective. The other implication is clear from Theorem 1.2 and Proposition 1.3.

Conversely, assume that M is amply supplemented and M_i is strongly lifting and relatively strongly projective. Then by induction on n, it is enough to prove that M is strongly lifting when $n = 2$. This follows from Theorem 2.11 and Lemma 2.8. □

In [5], it is shown an R-module M is strongly extending if and only if $M = Z_2(M) \oplus N$ for some $N \leq M$ where $Z_2(M)$ (second singular submodule of M) and N are both strongly extending and $\mathrm{Hom}(K, Z_2(M)) = 0$ for each submodule K of N. The next theorem is dual of this result and similar to [12, Theorem 4.1].

Theorem 2.13. *Let M be an R-module. Then M is strongly lifting if and only if $M = \overline{Z}^2(M) \oplus N$ for some submodule N of M, where $\overline{Z}^2(M)$ and N are both strongly lifting and N is $\overline{Z}^2(M)$-strongly projective and M is amply supplemented.*

Proof. The necessity is clear by Theorem 2.11 and Theorem 1.2 (since $\overline{Z}^2(M)$ is coclosed by [12, Corollary 3.4], $\overline{Z}^2(M) \leq^{\oplus} M$). For the sufficiency, it suffices to show that $\overline{Z}^2(M)$ is N-strongly projective. Let $L \leq N$ and N/L be a factor module of N and $f \in \mathrm{Hom}(\overline{Z}^2(M), N/L)$. By [12, Theorem 3.5], $\overline{Z}^2(N/L) = (\overline{Z}^2(N) + L)/L$. As $\overline{Z}^2(N) = 0$, $\overline{Z}^2(N/L) = 0$. Since $f(\overline{Z}^2(M)) \subseteq \overline{Z}^2(N/L) = 0$, $f = 0$. Thus $\mathrm{Hom}(\overline{Z}^2(M), N/L) = 0$. Hence by Theorem 2.11, M is strongly lifting. □

Corollary 2.14. *Let M be a strongly lifting module. Then M is t-dual Rickart.*

Proof. By Theorem 2.13, $M = \overline{Z}^2(M) \oplus N$ for some submodule N of M, where $\overline{Z}^2(M)$ and N are both strongly lifting and M is amply supplemented. Therefore by [1, Theorem 1], M is t-lifting. As $\overline{Z}^2(M)$ is noncosingular (and so $\mathcal{T}$-noncosingular), $\overline{Z}^2(M)$ is dual Rickart. Therefore by [6, Theorem 3.2], M is t-dual Rickart. □

In the next theorem, it is considered when direct sum of arbitrary hollow modules is strongly lifting.

Theorem 2.15. *Let $M = \oplus_{i \in I} M_i$ with M_i hollow. Then the following are equivalent:*

(1) *M is strongly lifting;*

(2) *M_i is $\oplus_{j \in I, j \neq i} M_j$-strongly projective.*

Proof. (1) ⇒ (2) is from Theorem 2.11.

(2) ⇒ (1) For each $i \in I$, M_i is hollow, hence $\mathrm{End}(M_i)$ is Abelian. Also for each $i, j \in I$ with $i \neq j$, $\mathrm{Hom}(M_i, M_j) = 0$ by Lemma 2.8(i). This follows that $\mathrm{End}(M)$ is Abelian. It suffices to show that M is lifting. Let $N \leq M$. Let $\{N_\alpha\}_{\alpha \in \Gamma}$ be all direct summands of M that are contained in N. Let $K = \sum_{\alpha \in \Gamma} N_\alpha$. We will show

$K \leq^{\oplus} M$ and $K = \oplus_{i \in T} M_i$ for some $T \subseteq I$. Since $\mathrm{End}(M)$ is Abelian, N_α is fully invariant for each $\alpha \in \Gamma$. Hence $N_\alpha = \oplus_{i \in I}(N_\alpha \cap M_i)$ by Proposition 1.1(iv). Since $N_\alpha \leq^{\oplus} M$, $N_\alpha \cap M_i \leq^{\oplus} M_i$ for each $i \in I$. Therefore M_i is hollow gives either $N_\alpha \cap M_i = M_i$ or $N_\alpha \cap M_i = 0$. This implies that $N_\alpha = \oplus_{i \in T_\alpha} M_i$ for some $T_\alpha \subseteq I$. Set $T = \cup_{\alpha \in \Gamma} T_\alpha$. Then it can be easily seen $K = \oplus_{i \in T} M_i$ and so $K \leq^{\oplus} M$ say $M = K \oplus K'$. We show $K' \cap N \ll M$. Assume that $K' \cap N + S = M$ for some submodule S of M. Let $Q = \sum_{S' \leq^{\oplus} M,\ S' \subseteq S} S'$. Then $Q \leq^{\oplus} M$ and $Q = \oplus_{i \in J} M_j$ for some $J \subseteq I$ by argument mentioned before. Let $j \notin J$ and $\pi : M \to M_j$ be natural projection. As $K' \cap N + S = M$, $M_j = \pi(K' \cap N) + \pi(S)$. Since M_j is hollow, $\pi(K' \cap N) = M_j$ or $\pi(S) = M_j$. Hence $M = \mathrm{Ker}(\pi) + K' \cap N$ or $M = \mathrm{Ker}(\pi) + S$. By (2) and Lemma 2.10, $M_j \subseteq K' \cap N$ or $M_j \subseteq S$. If $M_j \subseteq K' \cap N$, then $M_j \subseteq N$ and $M_j \subseteq K'$. Hence $M_j \subseteq K$ and so $M_j = 0$. If $M_j \subseteq S$, then $M_j \subseteq Q$ and so $M_j = 0$. Hence $M = Q$. This gives $S = M$, as desired. □

In the following, it is considered when direct sum of an arbitrary local modules is strongly lifting.

Corollary 2.16. *Let $M = \oplus_{i \in I} M_i$ where each M_i is local. Then M is strongly lifting if and only if M_i is M_j-strongly projective where $j \neq i$.*

Proof. Let M be strongly lifting. Then by Theorem 2.11, M_i is $\oplus_{i \neq j \in I} M_j$-strongly projective. Thus by Lemma 2.8(i), M_i is M_j-strongly projective for each $i \neq j \in I$.

Conversely, let M_i be M_j-strongly projective for each $i \neq j$ of I. By Lemma 2.8(iv), M_i is $\oplus_{i \neq \in I} M_j$-strongly projective. Hence M is strongly lifting by Theorem 2.15. □

In the next theorem, we use Theorem 2.15 to show that each factor module by a fully invariant submodule of a strongly lifting module is strongly lifting.

Theorem 2.17. *Let M be a strongly lifting module and N a fully invariant submodule of M. Then M/N is strongly lifting.*

Proof. Let M be a strongly lifting module and N a fully invariant submodule of M. Then by [4, 22.2(4)], M/N is lifting. We show $\mathrm{End}(M/N)$ is Abelian. By Theorem 2.4, $M = \oplus_{i \in I} H_i$ where H_i is hollow for each $i \in I$. Since N is fully invariant, $N = \oplus_{i \in I}(N \cap H_i)$ by Proposition 1.1(iv). Hence $M/N = \oplus_{i \in I} H_i/(N \cap H_i)$. Since $H_i/(N \cap H_i)$ is indecomposable for each $i \in I$, it suffices to show that $\mathrm{Hom}(H_i/(N \cap H_i), H_j/(N \cap H_j)) = 0$ for each distinct i, j of I. Let $f : H_i/(N \cap H_i) \longrightarrow H_j/(N \cap H_j)$ be a homomorphism and $i \neq j \in I$. Let $\pi : H_i \longrightarrow H_i/(N \cap H_i)$ be natural homomorphism. By Theorem 2.15, H_i is $\oplus_{j \in I, i \neq j} H_j$-strongly projective, so H_i is H_j-strongly projective for each distinct i, j of I by Lemma 2.8(i). Hence $f\pi = 0$. Thus $f = 0$, as desired. □

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