

A DOUBLE INTEGRAL CHARACTERIZATION OF A BERGMAN TYPE SPACE AND ITS MÖBIUS INVARIANT SUBSPACE

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ABSTRACT. This paper shows that if $1 < p < \infty$, $\alpha \geq -n-2$, $\alpha > -1 - \frac{p}{2}$ and f is holomorphic on the unit ball \mathbb{B}_n , then

$$\int_{\mathbb{B}_n} |Rf(z)|^p (1 - |z|^2)^{p+\alpha} dv_\alpha(z) < \infty$$

if and only if

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{n+1+s+t-\alpha}} (1 - |w|^2)^s (1 - |z|^2)^t dv(z) dv(w) < \infty,$$

where $s, t > -1$ with $\min(s, t) > \alpha$.

1. Introduction

The purpose of this note is to give a double integral characterization of a Bergman space, which extends some previous results in [2–4].

Let \mathbb{B}_n be the unit ball of the n -dimensional complex Euclidean space \mathbb{C}^n . Let $H(\mathbb{B}_n)$ be the space of holomorphic functions on \mathbb{B}_n . For $f \in H(\mathbb{B}_n)$, the radial derivative of f , denoted by Rf , is given by

$$Rf(z) = \sum_{j=1}^n z_j \frac{\partial f(z)}{\partial z_j}, \quad z = (z_1, z_2, \dots, z_n) \in \mathbb{B}_n.$$

We employ the definition of Bergman spaces given in [9]. For $0 < p < \infty$ and $\alpha > -1 - p$, the Bergman space $A_\alpha^p := A_\alpha^p(\mathbb{B}_n)$ consists of those holomorphic functions f in \mathbb{B}_n with

$$(1) \quad \|f\|_{A_\alpha^p} = |f(0)| + \|f\|_{\alpha,p}$$

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$$= |f(0)| + \left(\int_{\mathbb{B}_n} |Rf(z)|^p (1 - |z|^2)^{p+\alpha} dv(z) \right)^{\frac{1}{p}} < \infty,$$

where dv is the normalized volume measure on \mathbb{B}_n so that $v(\mathbb{B}_n) = 1$. When $\alpha > -1$, it is well known that $f \in A_\alpha^p$ if and only if

$$(2) \quad \|f\|_{A_{\alpha,*}^p} = \left(\int_{\mathbb{B}_n} |f(z)|^p (1 - |z|^2)^\alpha dv_\alpha(z) \right)^{\frac{1}{p}} < \infty.$$

Here

$$dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dv(z) = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} (1 - |z|^2)^\alpha dv(z).$$

When $\alpha = -(ps + 1)$ (with $s < 1$), the spaces A_α^p are exactly the diagonal Besov spaces B_p^s . Moreover, $A_{-1}^2 = H^2$, the Hardy space on \mathbb{B}_n . See [9] for more details of A_α^p .

It is proved in [3, 4] that for $\alpha > -1$ and $f \in H(\mathbb{B}_n)$,

(1) (proved in [3]) if $p > n + 1 + \alpha$, then $f \in A_\alpha^p$ if and only if

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(w) - f(z)|^p}{|1 - \langle z, w \rangle|^p} ((1 - |z|^2)(1 - |w|^2))^{\frac{p+\alpha-n-1}{2}} dv(z) dv(w) < \infty;$$

(2) (proved in [4]) if $0 < p < n + 1 + \alpha$, then $f \in A_\alpha^p$ if and only if

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(w) - f(z)|^p}{|1 - \langle z, w \rangle|^p} dv_\alpha(z) dv_\alpha(w) < \infty.$$

The main result of this paper is the following:

Theorem 1.1. *Let $1 < p < \infty$, $\alpha \geq -n - 2$, $\alpha > -1 - \frac{p}{2}$ and let f be holomorphic on \mathbb{B}_n . Let $s, t > -1$ such that $\min(s, t) > \alpha$. Then $f \in A_\alpha^p$ if and only if*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{n+1+s+t-\alpha}} (1 - |w|^2)^s (1 - |z|^2)^t dv(z) dv(w) < \infty.$$

Remark 1.2. (1) It is proved in [2] that for $\alpha \geq -2$ and $f \in H(\mathbb{B}_n)$,

$$\int_{\mathbb{B}_n} |Rf(z)|^2 (1 - |z|^2)^{2+\alpha} dv(z)$$

is comparable to

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^2}{|1 - \langle z, w \rangle|^{n+1+s+t-\alpha}} (1 - |z|^2)^s (1 - |w|^2)^t dv(z) dv(w),$$

where $s, t > -1$ with $\min\{s, t\} > \alpha$. The proof in [2] relies on Hilbert space techniques. It is worth to note that our result can not cover the result in [2] when $p = 2$ and $\alpha = -2$, which is contained in [2] but not contained in our theorem.

(2) If $\alpha = -n - 1 > -1 - \frac{p}{2}$, then $p > 2n$ and A_α^p is the Besov space B_p . It is shown in [10, Theorem 6.28] that if f is holomorphic in \mathbb{B}_n , then $f \in B_p$ if and only if

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p dv_t(z) dv_t(w)}{|1 - \langle z, w \rangle|^{2(n+1+t)}} < \infty$$

for $t > -1$. Thus Theorem 1.1 extends [10, Theorem 6.28].

(3) Let $s = t = \frac{p+\alpha-n-1}{2}$ in Theorem 1.1, we obtain the characterization given in [3]. Thus our characterization can be viewed as a generalization of [3]. However, the strategy of the verification is quite different from [3].

The main idea of the proof of Theorem 1.1 is inspired by [1]. Similar characterization of the Dirichlet type space in the unit disk is given in [7].

Let X be a Banach space of holomorphic functions in \mathbb{B}_n . The Möbius invariant subspace of X is defined as

$$M_X = \{f \in X : \sup_{a \in \mathbb{B}_n} \|f \circ \varphi_a\| < \infty\},$$

where

$$\varphi_a(z) = \begin{cases} \frac{a - \frac{\langle z, a \rangle a}{|a|^2} - \sqrt{1 - |a|^2} (z - \frac{\langle z, a \rangle a}{|a|^2})}{1 - \langle z, a \rangle}, & \text{if } a \neq 0, \\ -z, & \text{if } a = 0, \end{cases}$$

for $z \in \mathbb{B}_n$. For $a \in \mathbb{B}_n$, the transform φ_a is usually called the involution on \mathbb{B}_n . Let $\text{Aut}(\mathbb{B}_n)$ denote the set of all automorphisms on \mathbb{B}_n . It is well known that every automorphism φ of \mathbb{B}_n is of the form

$$\varphi = U \varphi_a = \varphi_b V,$$

where U and V are unitary transformations of \mathbb{C}^n , and φ_a and φ_b are involutions.

It is known that $M_{A_\alpha^p}$ is the Bloch space when $\alpha > -1$ and $p \geq 1$. $M_{A_{-1}^2} = BMOA$, the space of holomorphic functions of bounded mean oscillation.

We have the following theorem, which is motivated by [8, Theorem 2.5.2].

Theorem 1.3. *Let $1 < p < \infty$, $\alpha \geq -n - 2$, $\alpha > -1 - \frac{p}{2}$ and let f be holomorphic on \mathbb{B}_n . Let $s, t > -1$ such that $\min(s, t) > \alpha$. Then $f \in M_{A_\alpha^p}$ if and only if*

$$\sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p (1 - |a|^2)^{n+1+\alpha} (1 - |w|^2)^s (1 - |z|^2)^t dv(z) dv(w)}{|1 - \langle z, w \rangle|^{n+1+s+t-\alpha} |1 - \langle a, w \rangle|^{n+1+s-t+\alpha} |1 - \langle a, z \rangle|^{n+1+t-s+\alpha}} < \infty.$$

Notation. Throughout this paper, we only write $U \lesssim V$ (or $V \gtrsim U$) for $U \leq cV$ for a positive constant c , and moreover $U \approx V$ for both $U \lesssim V$ and $V \lesssim U$. \square

2. Preliminaries

For $f \in H(\mathbb{B}_n)$ with the homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z),$$

it is easy to check that

$$Rf(z) = \sum_{k=1}^{\infty} k f_k(z), \quad Rf(0) = 0$$

and

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt$$

for all $z \in \mathbb{B}_n$. According to [10, page 51], for $f \in A_{\alpha}^p$ with $\alpha > -1$, there exists a sufficiently large β satisfying

$$Rf(z) = \int_{\mathbb{B}_n} \frac{Rf(w) dv_{\beta}(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}}, \quad z \in \mathbb{B}_n,$$

where

$$dv_{\beta}(z) = \frac{\Gamma(n+1+\beta)}{n! \Gamma(\beta+1)} (1 - |z|^2)^{\beta} dv(z).$$

Then

$$Rf(z) = \int_{\mathbb{B}_n} Rf(w) \left(\frac{1}{(1 - \langle z, w \rangle)^{n+1+\beta}} - 1 \right) dv_{\beta}(w)$$

since $Rf(0) = 0$. We have

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt = \int_{\mathbb{B}_n} Rf(w) L(w, z) dv_{\beta}(w),$$

where the kernel

$$L(z, w) = \int_0^1 \left(\frac{1}{(1 - t\langle z, w \rangle)^{n+1+\beta}} - 1 \right) \frac{dt}{t}$$

satisfies

$$(3) \quad |L(z, w)| \lesssim \frac{1}{|1 - \langle z, w \rangle|^{n+\beta}}$$

for all z and w in \mathbb{B}_n . So

$$(4) \quad |f(z) - f(0)| \lesssim \int_{\mathbb{B}_n} \frac{|Rf(w)|}{|1 - \langle z, w \rangle|^{n+\beta}} dv_{\beta}(w).$$

For a holomorphic function f in \mathbb{B}_n we write

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right)$$

and call $|\nabla f(z)|$ the holomorphic gradient of f at z . We define

$$\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0),$$

and call $|\tilde{\nabla}f(z)|$ the invariant gradient of f at z . It is shown in [10, page 49] that

$$|\tilde{\nabla}(f \circ \varphi)(z)| = |(\tilde{\nabla}f) \circ \varphi(z)|$$

for all f and $\varphi \in \text{Aut}(\mathbb{B}_n)$. Moreover,

$$(5) \quad (1 - |z|^2)|Rf(z)| \leq (1 - |z|^2)|\nabla f(z)| \leq |\tilde{\nabla}f(z)|$$

for all $z \in \mathbb{B}_n$ and $f \in H(\mathbb{B}_n)$.

Let

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}$$

be the Möbius invariant measure on \mathbb{B}_n . It is easy to check that

$$(6) \quad \int_{\mathbb{B}_n} f \circ \varphi(z) d\lambda(z) = \int_{\mathbb{B}_n} f(z) d\lambda(z),$$

where $\varphi \in \text{Aut}(\mathbb{B}_n)$.

The following lemma is quoted from [10, Exercise 2.4]

Lemma 2.1. *Suppose $0 < p < \infty$, $\alpha > -\frac{p}{2} - 1$, and f is holomorphic in \mathbb{B}_n . Then $f \in A_\alpha^p$ if and only if*

$$\int_{\mathbb{B}_n} |\tilde{\nabla}f(z)|^p (1 - |z|^2)^\alpha dv(z) < \infty.$$

The following lemma is quoted from [5], which is Lemma 2.5 there.

Lemma 2.2. *Suppose $s > -1$ and $r, t > 0$. If $t < s + n + 1 < r$, then*

$$\int_{\mathbb{B}_n} \frac{(1 - |w|^2)^s dv(w)}{|1 - \langle z, w \rangle|^r |1 - \langle \eta, w \rangle|^t} \lesssim \frac{1}{(1 - |z|^2)^{r-s-n-1} |1 - \langle \eta, z \rangle|^t}.$$

3. The derivative-free characterization for A_α^p

Inspired by Lemma 2.1 of [1], we have the following lemma.

Lemma 3.1. *Let $1 \leq p < \infty$, and let $\alpha > -1$ and $\beta \geq 0$ with $\beta < n + 1 + \alpha$. Let f be holomorphic on \mathbb{B}_n . Then*

$$(7) \quad \int_{\mathbb{B}_n} |f(z) - f(0)|^p \frac{(1 - |z|^2)^\alpha}{|1 - \langle z, w \rangle|^\beta} dv(z) \lesssim \int_{\mathbb{B}_n} |Rf(z)|^p \frac{(1 - |z|^2)^{\alpha+p}}{|1 - \langle z, w \rangle|^\beta} dv(z).$$

Proof. The case $\beta = 0$ is contained in [10, Theorem 2.16]. So we assume that $\beta > 0$.

If $p > 1$, choose $\varepsilon > 0$ with

$$\alpha - \varepsilon \max(1, p - 1) > -1 \quad \text{and} \quad \beta + \varepsilon(p - 1) < n + 1 + \alpha.$$

Without loss of generality we may assume that the right-hand side of (7) is finite. Then it follows from Hölder's inequality that $Rf \in A_{1+\alpha}^1$. Theorem

2.16 in [10] implies that $f \in A_\alpha^1$. Then it follows from (4) that there exists a sufficiently large t such that

$$\begin{aligned} & |f(z) - f(0)|^p \\ & \lesssim \left(\int_{\mathbb{B}_n} |Rf(u)| \frac{(1 - |u|^2)^t}{|1 - \langle z, u \rangle|^{n+t}} dv(u) \right)^p \\ & \leq \left(\int_{\mathbb{B}_n} |Rf(u)|^p \frac{(1 - |u|^2)^{t+(p-1)(1+\varepsilon)}}{|1 - \langle z, u \rangle|^{n+t}} dv(u) \right) \left(\int_{\mathbb{B}_n} \frac{(1 - |u|^2)^{t-1-\varepsilon}}{|1 - \langle z, u \rangle|^{n+t}} dv(u) \right)^{p-1} \\ & \lesssim \left(\int_{\mathbb{B}_n} |Rf(u)|^p \frac{(1 - |u|^2)^{t+(p-1)(1+\varepsilon)}}{|1 - \langle z, u \rangle|^{n+t}} dv(u) \right) (1 - |z|^2)^{-\varepsilon(p-1)}. \end{aligned}$$

Now the Fubini's theorem and Lemma 2.2 imply that

$$\begin{aligned} & \int_{\mathbb{B}_n} |f(z) - f(0)|^p \frac{(1 - |z|^2)^\alpha}{|1 - \langle z, w \rangle|^\beta} dv(z) \\ & \lesssim \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} |Rf(u)|^p \frac{(1 - |u|^2)^{t+(p-1)(1+\varepsilon)}}{|1 - \langle z, u \rangle|^{n+t}} dv(u) \right) \frac{(1 - |z|^2)^{-\varepsilon(p-1)+\alpha}}{|1 - \langle z, w \rangle|^\beta} dv(z) \\ & = \int_{\mathbb{B}_n} |Rf(u)|^p (1 - |u|^2)^{t+(p-1)(1+\varepsilon)} \left(\int_{\mathbb{B}_n} \frac{(1 - |z|^2)^{-\varepsilon(p-1)+\alpha} dv(z)}{|1 - \langle z, w \rangle|^\beta |1 - \langle z, u \rangle|^{n+t}} \right) dv(u) \\ & \lesssim \int_{\mathbb{B}_n} |Rf(u)|^p (1 - |u|^2)^{t+(p-1)(1+\varepsilon)} \frac{dv(u)}{(1 - |u|^2)^{t-1-\alpha+\varepsilon(p-1)} |1 - \langle u, w \rangle|^\beta} \\ & = \int_{\mathbb{B}_n} |Rf(u)|^p \frac{(1 - |u|^2)^{p+\alpha}}{|1 - \langle u, w \rangle|^\beta} dv(u). \end{aligned}$$

If $p = 1$, by choosing γ big enough and applying (4), we have

$$\begin{aligned} & \int_{\mathbb{B}_n} |f(z) - f(0)| \frac{(1 - |z|^2)^\alpha}{|1 - \langle z, w \rangle|^\beta} dv(z) \\ & \lesssim \int_{\mathbb{B}_n} |Rf(u)| (1 - |u|^2)^\gamma \left(\int_{\mathbb{B}_n} \frac{(1 - |z|^2)^\alpha dv(z)}{|1 - \langle z, w \rangle|^\beta |1 - \langle z, u \rangle|^{n+\gamma}} \right) dv(u) \\ & \lesssim \int_{\mathbb{B}_n} |Rf(u)| (1 - |u|^2)^\gamma \frac{dv(u)}{(1 - |u|^2)^{\gamma-1-\alpha} |1 - \langle u, w \rangle|^\beta} \\ & = \int_{\mathbb{B}_n} |Rf(u)|^p \frac{(1 - |u|^2)^{1+\alpha}}{|1 - \langle u, w \rangle|^\beta} dv(u). \end{aligned}$$

The proof is completed. \square

We can get the following corollary from Lemma 3.1 and (5).

Corollary 3.2. *Let $1 \leq p < \infty$, and let $\alpha > -1$ and $\beta \geq 0$ with $\beta < n + 1 + \alpha$. Let f be holomorphic on \mathbb{B}_n . Then we have*

$$(8) \quad \int_{\mathbb{B}_n} |f(z) - f(0)|^p \frac{(1 - |z|^2)^\alpha}{|1 - \langle z, w \rangle|^\beta} dv(z) \lesssim \int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p \frac{(1 - |z|^2)^\alpha}{|1 - \langle z, w \rangle|^\beta} dv(z).$$

It is also shown in [10, page 50] that if $f \in A_\alpha^p$ for $\alpha > -1$, then

$$(9) \quad |\tilde{\nabla} f(z)|^p \lesssim (1 - |z|^2)^{n+1+\beta} \int_{\mathbb{B}_n} \frac{|f(w)|^p dv_\beta(w)}{|1 - \langle z, w \rangle|^{2(n+1+\beta)}}$$

for $\beta > \alpha$. We have the following corollary.

Corollary 3.3. *If $p > 0$, $\alpha > -1$ and $f \in A_\alpha^p$, then for $\beta > \alpha$, we have*

$$(10) \quad \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^p \frac{(1 - |z|^2)^\alpha}{|1 - \langle z, w \rangle|^\beta} dv(z) \lesssim \int_{\mathbb{B}_n} |f(z) - f(0)|^p \frac{(1 - |z|^2)^\alpha}{|1 - \langle z, w \rangle|^\beta} dv(z).$$

In particular,

$$(11) \quad \int_{\mathbb{B}_n} |Rf(z)|^p \frac{(1 - |z|^2)^{p+\alpha}}{|1 - \langle z, w \rangle|^\beta} dv(z) \lesssim \int_{\mathbb{B}_n} |f(z) - f(0)|^p \frac{(1 - |z|^2)^\alpha}{|1 - \langle z, w \rangle|^\beta} dv(z).$$

Proof. We only need to verify (10), since (11) can be easily obtained from (5) and (10).

Replacing f by $f - f(0)$ in (9), we have

$$|\tilde{\nabla} f(z)|^p \lesssim (1 - |z|^2)^{n+1+\beta} \int_{\mathbb{B}_n} \frac{|f(w) - f(0)|^p dv_\beta(w)}{|1 - \langle z, w \rangle|^{2(n+1+\beta)}}.$$

This implies that

$$\begin{aligned} & \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^p \frac{(1 - |z|^2)^\alpha}{|1 - \langle z, w \rangle|^\beta} dv(z) \\ & \lesssim \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^{n+1+\beta+\alpha}}{|1 - \langle z, w \rangle|^\beta} \int_{\mathbb{B}_n} \frac{|f(u) - f(0)|^p dv_\beta(u)}{|1 - \langle z, u \rangle|^{2(n+1+\beta)}} dv(z) \\ & = \int_{\mathbb{B}_n} |f(u) - f(0)|^p \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^{n+1+\beta+\alpha} dv(z)}{|1 - \langle z, w \rangle|^\beta |1 - \langle z, u \rangle|^{2(n+1+\beta)}} dv_\beta(u) \\ & \lesssim \int_{\mathbb{B}_n} |f(u) - f(0)|^p \frac{1}{|1 - \langle u, w \rangle|^\beta (1 - |u|^2)^{\beta-\alpha}} dv_\beta(u) \\ & = \int_{\mathbb{B}_n} |f(u) - f(0)|^p \frac{(1 - |u|^2)^\alpha}{|1 - \langle u, w \rangle|^\beta} dv(u). \end{aligned}$$

The proof is completed. \square

Theorem 3.4. *Let $1 \leq p < \infty$, $\alpha > -1 - p$, $\alpha \geq -n - 2$, and let f be holomorphic on \mathbb{B}_n . Let $s, t > -1$ such that $\min(s, t) > \alpha$. Then*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{n+1+s+t-\alpha}} (1 - |w|^2)^s (1 - |z|^2)^t dv(z) dv(w)$$

is comparable to

$$\int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^\alpha dv(z).$$

Proof. It suffices to consider the case $s = t$ since $s \leq t$ implies

$$\frac{2^{s-t}(1-|z|^2)^t(1-|w|^2)^t}{|1-\langle z, w \rangle|^{n+1+2t-\alpha}} \leq \frac{(1-|z|^2)^s(1-|w|^2)^t}{|1-\langle z, w \rangle|^{n+1+s+t-\alpha}} \leq \frac{(1-|z|^2)^s(1-|w|^2)^s}{2^{s-t}|1-\langle z, w \rangle|^{n+1+2s-\alpha}}.$$

Let $\zeta = \varphi_w(z)$ and recall that

$$(12) \quad 1 - \langle \varphi_w(\zeta), \varphi_w(a) \rangle = \frac{(1-|w|^2)(1-\langle \zeta, a \rangle)}{(1-\langle \zeta, w \rangle)(1-\langle w, a \rangle)}, \quad w, a, \zeta \in \mathbb{B}_n.$$

It follows from Corollary 3.2 that

$$\begin{aligned} & \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|1-\langle z, w \rangle|^{n+1+2s-\alpha}} (1-|w|^2)^s (1-|z|^2)^s dv(z) dv(w) \\ &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(\varphi_w(\zeta)) - f(\varphi_w(0))|^p (1-|\varphi_w(\zeta)|^2)^s}{|1-\langle \varphi_w(\zeta), \varphi_w(0) \rangle|^{n+1+2s-\alpha} (1-|w|^2)^{-s}} dv(\varphi_w(\zeta)) dv(w) \\ &= \int_{\mathbb{B}_n} (1-|w|^2)^\alpha \int_{\mathbb{B}_n} \frac{|f(\varphi_w(\zeta)) - f(\varphi_w(0))|^p (1-|\zeta|^2)^s}{|1-\langle \zeta, w \rangle|^{n+1+\alpha}} dv(\zeta) dv(w) \\ &\lesssim \int_{\mathbb{B}_n} (1-|w|^2)^\alpha \int_{\mathbb{B}_n} |\tilde{\nabla}(f \circ \varphi_w)(\zeta)|^p \frac{(1-|\zeta|^2)^s}{|1-\langle \zeta, w \rangle|^{n+1+\alpha}} dv(\zeta) dv(w) \\ &= \int_{\mathbb{B}_n} (1-|w|^2)^\alpha \int_{\mathbb{B}_n} |\tilde{\nabla}(f) \circ \varphi_w(\zeta)|^p \frac{(1-|\zeta|^2)^{s+n+1}}{|1-\langle \zeta, w \rangle|^{n+1+\alpha}} d\lambda(\zeta) dv(w) \\ &= \int_{\mathbb{B}_n} (1-|w|^2)^\alpha \int_{\mathbb{B}_n} |\tilde{\nabla}(f)(z)|^p \frac{(1-|\varphi_w(z)|^2)^{s+n+1}}{|1-\langle \varphi_w(z), w \rangle|^{n+1+\alpha}} d\lambda(z) dv(w) \\ &= \int_{\mathbb{B}_n} (1-|w|^2)^s \int_{\mathbb{B}_n} |\tilde{\nabla}(f)(z)|^p \frac{(1-|z|^2)^s}{|1-\langle z, w \rangle|^{n+1+2s-\alpha}} dv(z) dv(w) \\ &= \int_{\mathbb{B}_n} |\tilde{\nabla}(f)(z)|^p (1-|z|^2)^s \int_{\mathbb{B}_n} \frac{(1-|w|^2)^s}{|1-\langle z, w \rangle|^{n+1+2s-\alpha}} dv(w) dv(z) \\ &= \int_{\mathbb{B}_n} |\tilde{\nabla}(f)(z)|^p (1-|z|^2)^\alpha dv(z). \end{aligned}$$

For the converse direction, recall that if $f \in H(\mathbb{B}_n)$, then (see [10, page 87])

$$(13) \quad |\tilde{\nabla}(f)(z)|^p \lesssim \int_{\mathbb{B}_n} |f \circ \varphi_z(w) - f(z)|^p dv_\beta(w)$$

for all $z \in \mathbb{B}_n$ and $\beta > -1$. Choose $\beta = 1 + s$ and let $w = \varphi_z(\eta)$ in (13) we get

$$\begin{aligned} & \int_{\mathbb{B}_n} |\tilde{\nabla}(f)(z)|^p dv_\alpha(z) \lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |f \circ \varphi_z(w) - f(z)|^p dv_\beta(w) dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |f(\eta) - f(z)|^p (1-|\varphi_z(\eta)|^2)^{\beta+n+1} d\lambda(\varphi_z(\eta)) dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |f(\eta) - f(z)|^p \frac{(1-|z|^2)^{\beta+n+1} (1-|\eta|^2)^{\beta+n+1}}{|1-\langle z, \eta \rangle|^{2(\beta+n+1)}} d\lambda(\eta) dv_\alpha(z) \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(\eta) - f(z)|^p (1 - |z|^2)^s (1 - |\eta|^2)^s}{|1 - \langle z, \eta \rangle|^{n+1+2s-\alpha}} \frac{(1 - |z|^2)^{\alpha+n+2} (1 - |\eta|^2)}{|1 - \langle z, \eta \rangle|^{n+3+\alpha}} dv(\eta) dv(z) \\
&\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(\eta) - f(z)|^p (1 - |z|^2)^s (1 - |\eta|^2)^s}{|1 - \langle z, \eta \rangle|^{n+1+2s-\alpha}} dv(\eta) dv(z),
\end{aligned}$$

where the last inequality follows from the assumption $\alpha \geq -n-2$, which implies that

$$\frac{(1 - |z|^2)^{\alpha+n+2} (1 - |\eta|^2)}{|1 - \langle z, \eta \rangle|^{n+3+\alpha}} \lesssim 1.$$

This gives the desired result. \square

Proof of Theorem 1.1. Put Theorem 3.4 and Lemma 2.1 together, we get Theorem 1.1. \square

A slight modification of the proof of Theorem 3.4 can give the following corollary.

Corollary 3.5. (1) Let $1 \leq p < \infty$ and f be holomorphic on \mathbb{B}_n . Let $s, t > -1$, and $\gamma > t + n + 1$. Then

$$\begin{aligned}
&\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^\gamma} (1 - |w|^2)^s (1 - |z|^2)^t dv(z) dv(w) \\
&\lesssim \begin{cases} \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{t+s+n+1-\gamma} dv(z), & \text{if } \gamma > n + 1 + s; \\ \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^t \log \frac{1}{1-|z|^2} dv(z), & \text{if } \gamma = n + 1 + s; \\ \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^t dv(z), & \text{if } \gamma < n + 1 + s. \end{cases}
\end{aligned}$$

(2) Let $0 < p < \infty$ and f be holomorphic on \mathbb{B}_n . Let $\beta > -1$ and α is real. Then

$$\begin{aligned}
&\int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^\alpha dv(z) \\
&\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |f(z) - f(w)|^p \frac{(1 - |z|^2)^{\alpha+\beta+n+1} (1 - |w|^2)^\beta}{|1 - \langle z, w \rangle|^{2(\beta+n+1)}} dv(z) dv(w) \\
&\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^\gamma} (1 - |z|^2)^t (1 - |w|^2)^s dv(z) dv(w)
\end{aligned}$$

for all $s \leq \beta$, $t \leq \alpha + \beta + n + 1$ and $\gamma \geq n + 1 + t + s - \alpha$.

4. The Möbius invariant subspace of A_α^p

Proof of Theorem 1.3. For $a \in \mathbb{B}_n$, an easily application of (12) and (6) gives that

$$\begin{aligned}
&\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f \circ \varphi_a(z) - f \circ \varphi_a(w)|^p}{|1 - \langle z, w \rangle|^{n+1+s+t-\alpha}} (1 - |w|^2)^s (1 - |z|^2)^t dv(z) dv(w) \\
&= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(u) - f(\xi)|^p (1 - |\varphi_a(\xi)|^2)^{s+n+1} (1 - |\varphi_a(u)|^2)^{t+n+1}}{|1 - \langle \varphi_a(u), \varphi_a(\xi) \rangle|^{n+1+s+t-\alpha}} d\lambda(u) d\lambda(\xi)
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(u) - f(\xi)|^p (1 - |a|^2)^{n+1+\alpha} (1 - |\xi|^2)^{n+1+s} (1 - |u|^2)^{n+1+t} d\lambda(u) d\lambda(\xi)}{|1 - \langle u, \xi \rangle|^{n+1+s+t-\alpha} |1 - \langle a, \xi \rangle|^{n+1+\alpha+s-t} |1 - \langle a, u \rangle|^{n+1+\alpha+t-s}} \\
&= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(u) - f(\xi)|^p (1 - |a|^2)^{n+1+\alpha} (1 - |\xi|^2)^s (1 - |u|^2)^t dv(u) dv(\xi)}{|1 - \langle u, \xi \rangle|^{n+1+s+t-\alpha} |1 - \langle a, \xi \rangle|^{n+1+\alpha+s-t} |1 - \langle a, u \rangle|^{n+1+\alpha+t-s}}.
\end{aligned}$$

The proof is completed by taking supremum over $a \in \mathbb{B}_n$. \square

The holomorphic function spaces Q_s on the unit ball is introduced in [6]. For $s > 0$, Q_s is defined by

$$Q_s = \left\{ f \in H(\mathbb{B}_n) : \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 G(z, a)^s d\lambda(z) < \infty \right\},$$

where $G(z, a)$ is invariant Green's function of \mathbb{B}_n defined by $G(z, a) = g(\varphi_a(z))$, and

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} dt.$$

It is shown in [6, Proposition 3.4] that when $0 < s \leq 1$, a holomorphic function f on \mathbb{B}_n is belonging to Q_s if and only if

$$\sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 (1 - |\varphi_a(z)|^2)^{ns} d\lambda(z) < \infty.$$

Moreover, when $1 < s < \frac{n}{n-1}$, $f \in Q_s$ if and only if f is in the Bloch space, or equivalently,

$$\sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 (1 - |\varphi_a(z)|^2)^{nq} d\lambda(z) < \infty$$

for all $q > 1$. Therefore, when $0 < s \leq 1$, Q_s can be viewed as the Möbius invariant subspace of a holomorphic function space on \mathbb{B}_n satisfying

$$\int_{\mathbb{B}_n} |\tilde{\nabla} f(z)|^2 (1 - |z|^2)^{ns} d\lambda(z) < \infty.$$

On the other hand, it is well known that Q_s is nontrivial (i.e., contains all polynomials) if and only if $\frac{n-1}{n} < s < \frac{n}{n-1}$. Thus, we have the following characterization of Q_s on \mathbb{B}_n .

Corollary 4.1. *Let $\frac{n-1}{n} < s < \frac{n}{n-1}$ and let f be holomorphic on \mathbb{B}_n . Let $\beta, \gamma > -1$ such that $\min(\beta, \gamma) > ns - n - 1$. Then $f \in Q_s$ if and only if*

$$\sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^2 (1 - |a|^2)^{ns} (1 - |w|^2)^\beta (1 - |z|^2)^\gamma dv(z) dv(w)}{|1 - \langle z, w \rangle|^{2(n+1)+\beta+\gamma-ns} |1 - \langle a, w \rangle|^{ns+\beta-\gamma} |1 - \langle a, z \rangle|^{ns-\beta+\gamma}} < \infty.$$

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