Bull. Korean Math. Soc. **56** (2019), No. 6, pp. 1601–1615
https://doi.org/10.4134/BKMS.b190029
pISSN: 1015-8634 / eISSN: 2234-3016

ON THE ORBITAL STABILITY OF INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATIONS WITH SINGULAR POTENTIAL

Yonggeun Cho and Misung Lee

Abstract. We show the existence of ground state and orbital stability of standing waves of nonlinear Schrödinger equations with singular linear potential and essentially mass-subcritical power type nonlinearity. For this purpose we establish the existence of ground state in H^1. We do not assume symmetry or monotonicity. We also consider local and global well-posedness of Strichartz solutions of energy-subcritical equations. We improve the range of inhomogeneous coefficient in [5, 12] slightly in 3 dimensions.

1. Introduction

In this paper we consider the following Cauchy problem:

$$\begin{cases} i\partial_t\psi - \Delta\psi = N(x,\psi) \ \text{ in } \ \mathbb{R}^{1+n}, \\ \psi(0,x) = \psi_0(x) \ \text{ in } \ \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here $n \ge 1$, $\psi : \mathbb{R}^{1+n} \to \mathbb{C}$ and $N : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C}$.

To present our results let us set $N(x,\psi) = V(x)\psi + g(x)|\psi|^{p-1}\psi$ $(p > 1)$ and describe assumptions:

(A1) $V \in C(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ and for some constants $A > 0$ and $a \ge 0$,

$$|V(x)| \le A|x|^{-a}.$$

(A2) $g \in C(\mathbb{R}^n \setminus \{0\}, [0,+\infty))$ and for some constants $B > 0$ and $b \ge 0$

$$g(x) \le B|x|^{-b} \ \text{ for all } \ x \in \mathbb{R}^n \setminus \{0\}.$$

(A3) There exist $B_0, R > 0$ and $b_0 \ge b$ such that

$$g(x) \ge B_0|x|^{-b_0} \ \text{ if } \ |x| \ge R, \ \text{ and } \ \lim_{|x|\to\infty} g(x) = 0.$$

The model of (1.1) can be a dilute Bose-Einstein condensate when interactions of the condensate are considered to be inhomogeneous. For this see [2,18]

Received January 8, 2019; Revised March 26, 2019; Accepted April 12, 2019.
2010 *Mathematics Subject Classification.* 35Q40, 35Q55.
Key words and phrases. inhomogeneous NLS, singular potential, ground state, orbital stability, well-posedness, Strichartz solution.

and the references therein. Also it has been considered to study the laser guiding in an axially nonuniform plasma channel. For this see [11,15,17]. If $V = 0$ and $g = \gamma|x|^{-b}$ for a fixed $\gamma \in \mathbb{R} \setminus \{0\}$, then the equation has scaling invariant structure. That is, the scaled function $u_\lambda(t,x) = \lambda^{-\frac{2-b}{p-1}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ is also the solution of (1.1). If $p = 1 + \frac{2(2-b)}{n}$, then the space scaling is L^2-invariant. We call the equation with this p mass-critical one. If $p < 1 + \frac{2(2-b)}{n}$ (if $>$), then we say that it is mass-sub (super) critical.

We define energy functional E by

$$E(\psi) = \frac{1}{2}\|\nabla\psi\|_{L^2}^2 - \frac{1}{2}\int V(x)|\psi|^2\,dx - \frac{1}{p+1}\int g(x)|\psi|^{p+1}\,dx$$

and also mass m by $m(\psi) = \int |\psi|^2\,dx$. By a standing wave of (1.1) we mean a solution $\psi(t,x)$ of the form $e^{i\omega t}u$ for some $\omega \in \mathbb{R}$, where u is a solution of the equation

$$-\Delta u - \omega u = V(x)u + g(x)|u|^{p-1}u. \tag{1.2}$$

Many authors have studied the existence of u and (in)stability of standing waves under suitable conditions on V, g. For instance see [4,8,10] and references therein. For this purpose they showed that if (u_k) is a minimizing sequence of the problem

$$I_\mu = \inf\{E(u) : u \in S_\mu\}, \quad S_\mu = \{u \in H^1(\mathbb{R}^n, \mathbb{C}) : m(u) = \mu\}$$

with a prescribed positive number μ, then $u_k \to u$ in H^1 up to a subsequence, where u is a solution of (1.2) for some ω. Here H^1 denotes the usual L^2-Sobolev space with the norm $\|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}$. In this paper we will also use the L^r-Sobolev space $H_r^1 (1 \le r \le \infty)$ whose norm is defined by $\|u\|_{H_r^1} = \|u\|_{L^r} + \|\nabla u\|_{L^r}$.

Now by following the definition of Cazenave-Lions, we set

$$\mathcal{O}_\mu = \{u \in S_\mu : E(u) = I_\mu\}.$$

Our first result is the existence of ground states of case when $p < 1 + \frac{2(2-b)}{n}$, which is usually referred as mass-subcritical case.

Proposition 1.1. *Let $n \ge 1$, $0 < b \le b_0 < \min(n,2)$, $1 < p < \min(3, 1 + \frac{2(\min(n,2)-b_0)}{n})$, and $\frac{n(p-1)+2b_0}{2} < a < \min(n,2)$. Suppose that V, and g satisfy the assumptions* **(A1)**, *and* **(A2)** *and* **(A3)**, *respectively. Then $\mathcal{O}_\mu$ is not empty for any $\mu > 0$. If $b = 0$, then we have the same conclusion for $0 < b_0 < \min(n,2)$.*

For the proof we use the standard concentration-compactness argument of [14]. The difficulty is coming from the competition between the singularities of linear and nonlinear potentials. We find a room for singularity of linear potential to settle it. For this the lower bound of a is necessary. The case $b = 0$ seems new as far as we know. When $b_0 = b = 0$, it would be interesting to show the existence of ground state by assuming $\lim_{|x|\to\infty} g(x) = B_0 > 0$. One may try this issue with the argument of [1].

We say that $\mathcal{O}_\mu$ is stable if it is not empty and satisfies that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\psi_0 \in H^1$ with

$$\inf_{u\in\mathcal{O}_\mu} \|\psi_0 - u\|_{H^s} < \delta,$$

then

$$\inf_{u\in\mathcal{O}_\mu} \|\psi(t,\cdot) - u\|_{H^s} < \varepsilon$$

for all $t \in [-T_1, T_2]$. Here ψ is the unique solution to (1.1) in $C([-T_1, T_2]; H^1)$ with $m(\psi(t)) = m(\varphi)$ and $E(\psi(t)) = E(\psi_0)$ for all $t \in [-T_1, T_2]$.

Let us introduce our main result.

Theorem 1.2. *Let $n \ge 1$, $0 < b \le b_0 < \min(n, 2)$, $1 < p < \min(3, 1 + \frac{2(\min(n,2)-b_0)}{n})$, and $\frac{n(p-1)+2b_0}{2} < a < \min(n, 2)$. Suppose that V satisfies the assumption* **(A1)** *and g satisfy* **(A2)** *and* **(A3)***. Let ψ be a solution in $C([-T_1, T_2]; H^1)$ with $m(\psi(t)) = m(\psi_0)$ and $E(\psi(t)) = E(\psi_0)$ for all $t \in [-T_1, T_2]$. Then $\mathcal{O}_\mu$ is stable.*

In [8,10] the authors studied the stability when $p < 1+\frac{2(2-b)}{n}$ and instability when $1 + \frac{2(2-b)}{n} < p < 1 + \frac{2(2-b)}{n-2} (n \ge 3)$. We only considered the stability result because the approach of instability will be much different from the one used in this paper. We will treat the instability issue in a different place.

We now consider the well-posedness of Strichartz solutions of (1.1). By Duhamel's formula, (1.1) is written as an integral equation

$$u = U(t)\psi_0 - i \int_0^t U(t - t')N(x, \psi(t'))\, dt'. \tag{1.3}$$

Here we define the linear propagator $U(t)$ given by the linear problem $i\partial_t v = \Delta v$ with initial datum $v(0) = f$. It is formally given by

$$U(t)f = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi+t|\xi|^2)} \widehat{f}(\xi)\, d\xi, \tag{1.4}$$

where $\widehat{f} = \mathcal{F}(f)$ denotes the Fourier transform of f and $\mathcal{F}^{-1}$ the inverse Fourier transform such that

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \quad \mathcal{F}^{-1}(g)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\, d\xi.$$

The well-posedness can be shown by a classical argument of [3] based on the functional analysis. But in this paper we use the standard contraction principle via Strichartz estimates for future work about scattering and blowup.

If a pair (q, r) satisfies that $2 \le q, r \le \infty$, $\frac{2}{q}+\frac{n}{r} = \frac{n}{2}$, and $(n, q, r) \ne (2, 2, \infty)$, then it is said to be *admissible*. Let (q, r) and $(\widetilde{q}, \widetilde{r})$ be any admissible pair.

Then we have the following Strichartz estimates [13]

$$\|U(t)\varphi\|_{L^q(-T,T;L^r)} \le C\|\varphi\|_{L^2_x},$$
$$\|\int_0^t U(t-t')F\,dt'\|_{L^q(-T,T;L^r)} \le C\|F\|_{L^{\widetilde{q}'}(-T,T;L^{\widetilde{r}'})},$$

where the constant C does not depend on T.

To simplify our well-posedness result we define the following numbers

$$2_b = \begin{cases} \infty & \text{if } n = 1,2, \\ 1 + \frac{2(2-b)}{n-2} & \text{if } n \ge 3 \end{cases} \quad \text{and} \quad \widetilde{2} = \begin{cases} 1 & \text{if } n = 1,2, \\ \frac{3}{2} & \text{if } n = 3, \\ 2 & \text{if } n \ge 4. \end{cases}$$

Theorem 1.3. *Let $n \ge 1$, $0 \le a < \widetilde{2}$, $0 \le b < \widetilde{2}$ and $1 < p < 2_b$. Let us assume that $V, g \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfy the assumptions* **(A1)** *and* **(A2)** *and that $\nabla V, \nabla g \in L^\infty(|x| > 1)$. Suppose that there exist positive constants A', B' depending on n such that if $n = 1$, then for some $0 \le a', b' < 1$*

$$|\frac{d}{dx}V(x)| \le A'|x|^{-a'}, \quad |\frac{d}{dx}g(x)| \le B'|x|^{-b'}, \quad 0 < |x| \le 1,$$

and if $n \ge 2$, then

$$|\nabla V(x)| \le A'|x|^{-a-1}, \quad |\nabla g(x)| \le B'|x|^{-b-1}, \quad 0 < |x| \le 1.$$

Then for any $\psi_0 \in H^1$ there exists maximal time interval $I_* = (-T_*, T^*)$ for $T_*, T^* \in (0, +\infty]$ such that there exist a unique $\psi \in C(I_*; H^1)$ and $\psi \in L^q(-T_1, T_2; H^1_r)$ for any admissible pair (q, r) and for any $[-T_1, T_2] \subset I_*$ satisfying that $m(\psi(t)) = m(\psi_0)$ and $E(\psi(t)) = E(\psi_0)$ for all $t \in \mathbb{R}$. If $p < 1 + \frac{2(2-b)}{n}$, then $I_* = \mathbb{R}$.

Guzmán [12] and Dinh [5] considered well-posedness in H^1 when $V = 0$ and $g = |x|^{-b}$. When $n = 3$ they could get the well-posedness for $0 < b < 1$ and $1 < p < 2_b$, and $1 \le b < \frac{3}{2}$ and $p < \frac{5-2b}{2b-1}$, respectively. We improve the range of p up to 2_b when $1 < b < \frac{3}{2}$ by dividing ∇g in- and outside the unit ball. The global well-posedness for mass-critical and mass-supercritical case will be interesting. For the case $V = 0$, see [6, 7, 9].

Our paper is organized as follows. In Section 2 we will prove the existence of ground states by showing the compactness of the minimizing sequences of the constrained variational problem. This is a key step to show the orbital stability of standing waves. This goal is achieved in Theorem 1.2, which will be shown in Section 3. In the last section, we will discuss the Strichartz solutions of the Cauchy problem for a large class of nonlinearities.

2. Ground state

2.1. Proof of Proposition 1.1

If $1 < p < 1 + \frac{2(2-b)}{n}$ and $0 < a < 2$, then from Hardy-Sobolev's, Gagliardo-Nirenberg's, and then Young's inequalities it follows that for any $u \in S_\mu$ there exists a constant $C_0 > 0$ such that

$$
\begin{aligned}
E(u) &= \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2}\int V(x)|u|^2\,dx - \frac{1}{p+1}\int g(x)|u|^{p+1}\,dx \\
&\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2}A\int |x|^{-a}|u|^2\,dx - \frac{B}{p+1}\int |x|^{-b}|u|^{p+1}\,dx \\
&\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2}AC_a\|u\|_{L^2}^{2-a}\|\nabla u\|_{L^2}^a \\
&\quad - \frac{BC_{b,p}}{p+1}\|u\|_{L^2}^{p+1-\frac{n(p-1)+2b}{2}}\|\nabla u\|_{L^2}^{\frac{n(p-1)+2b}{2}} \\
&\geq \frac{1}{4}\|\nabla u\|_{L^2}^2 - C_0(\mu^2 + \mu^{\theta(p)}),
\end{aligned} \tag{2.1}
$$

where $\theta(p) = (p+1-\frac{n(p-1)+2b}{2})\cdot\frac{2}{4-n(p-1)-2b}$. Thus $I_\mu > -\infty$ for all $\mu > 0$.

Now we show that

$$
I_\mu < 0 \quad \text{for all } \mu > 0. \tag{2.2}
$$

In fact, for $0 < \lambda \ll 1$ letting $\varphi_\lambda(x) = \lambda^{\frac{n}{2}}\varphi(\lambda x)$ for a nonnegative, rapidly decreasing radial smooth function φ in S_μ, we see that $\varphi_\lambda \in S_\mu$ and

$$
\begin{aligned}
E(\varphi_\lambda) &= \frac{1}{2}\|\nabla\varphi_\lambda\|_{L^2}^2 - \frac{1}{2}\int V(x)(\varphi_\lambda)^2\,dx - \frac{1}{p+1}\int g(x)(\varphi_\lambda)^{p+1}\,dx \\
&\leq \frac{1}{2}\|\nabla\varphi_\lambda\|_{L^2}^2 - \frac{1}{2}\int V(x)(\varphi_\lambda)^2\,dx - \frac{1}{p+1}\int_{|x|\geq\lambda^{-1}R} g(x)(\varphi_\lambda)^{p+1}\,dx \\
&\leq \frac{1}{2}\lambda^2\|\nabla\varphi\|_{L^2}^2 + \frac{AC_a}{2}\|\varphi_\lambda\|_{L^2}^{2-a}\|\nabla\varphi_\lambda\|_{L^2}^a \\
&\quad - \frac{B_0}{p+1}\lambda^{\frac{n(p+1)}{2}-n+b_0}\int_{|x|\geq R}|x|^{-b_0}(\varphi(x))^{p+1}\,dx.
\end{aligned}
$$

Since $0 < \lambda \ll 1$ and φ is smooth and rapidly decreasing, there exist constants $C_1, C_2 > 0$ such that

$$
E(\varphi_\lambda) \leq \lambda^a C_1 - \lambda^{\frac{n(p-1)+2b_0}{2}}C_2,
$$

which is strictly negative from the condition $a > \frac{n(p-1)+2b_0}{2}$ if λ is sufficiently small.

On the other hand, one can easily show that I_μ is continuous on $(0,\infty)$. The proof will be given in Section 2.2.

Using the continuity, we deduce that for each $\mu > 0$ and $\theta > 1$ there exist $\varepsilon < -I_\mu(1-\theta^{-\frac{p-1}{2}})$, and $v \in S_\mu$ such that $I_\mu < E(v) < I_\mu + \varepsilon$. Then it follows from the definition of E and I_μ that

$$
I_{\theta\mu} \leq E(\sqrt{\theta}v) \leq \theta^{\frac{p+1}{2}}E(v) \leq \theta^{\frac{p+1}{2}}(I_\mu+\varepsilon) < \theta I_\mu,
$$

which implies that

$$
I_\mu < I_\nu + I_{\mu-\nu} \quad \text{for all } 0 < \nu < \mu. \tag{2.3}
$$

For the general situation we refer the readers to Lemma II. 1 of [14].

Let $(u_j) \subset S_\mu$ be a minimizing sequence such that $E(u_j) \to I_\mu$. From (2.1) we deduce that (u_j) is bounded in H^1. To show $\mathcal{O}_\mu \neq \varnothing$ we will use the concentration-compactness (see [14]). Let the concentration function $\mathfrak{m}_j$ be defined by

$$\mathfrak{m}_j(r) = \sup_{y \in \mathbb{R}^n} \int_{|x-y|<r} |u_j(x)|^2\, dx \ \text{ for } \ r > 0.$$

Set

$$\nu = \lim_{r \to \infty} \liminf_{j \to \infty} \mathfrak{m}_j(r).$$

Then $0 \leq \nu \leq \mu$ and there exists a subsequence u_j (still denoted by u_j) satisfying the following properties (see [14] or [3]).

(1) If $\nu = 0$, then $\|u_j\|_{L^q} \to 0$ as $j \to \infty$ for all q with $2 < q < 2^*$, $2^* = \frac{2n}{n-2}$ if $n > 2$ and $2^* = \infty$ if $n = 1, 2$.

(2) If $\nu = \mu$, then there exists a sequence $(y_j) \subset \mathbb{R}^n$ and $u \in H^1$ such that for any q with $2 \leq q < 2^*$

$$u_j(\cdot + y_j) \to u \ \text{ as } \ j \to \infty \ \text{ in } \ L^q$$

and given $\varepsilon > 0$ there exist $j_0(\varepsilon)$ and $r(\varepsilon)$ such that

$$\int_{|x-y_j|<r(\varepsilon)} |u_j|^2\, dx \geq \mu - \varepsilon, \ \text{ whenever } \ j \geq j_0(\varepsilon).$$

(3) If $0 < \nu < \mu$, then there exist $(v_j), (w_j) \subset H^1$ such that

$$\operatorname{supp} v_j \cap \operatorname{supp} w_j = \varnothing, \tag{2.4}$$

$$\|v_j\|_{H^1} + \|w_j\|_{H^1} \leq C\|u_j\|_{H^1}, \tag{2.5}$$

$$\lim\nolimits_{j\to\infty} m(v_j) = \nu, \quad \lim\nolimits_{j\to\infty} m(w_j) = \mu - \nu, \tag{2.6}$$

$$\liminf\nolimits_{j\to\infty} \left(\|\nabla u_j\|_{L^2}^2 - \|\nabla v_j\|_{L^2}^2 - \|\nabla w_j\|_{L^2}^2 \right) \geq 0, \tag{2.7}$$

$$\lim\nolimits_{j\to\infty} \|u_j - v_j - w_j\|_{L^q} = 0, \ \ 2 \leq q < 2^*. \tag{2.8}$$

If $\nu = 0$, then using Hardy-Sobolev's and Gagliardo-Nirenberg's inequality near the origin as in (2.1), we have that for any $2 < q < \frac{2n}{n-a}$

$$\begin{aligned} &\left| \frac{1}{2} \int V(x)|u_j|^2\, dx + \frac{1}{p+1} \int g(x)|u_j|^{p+1}\, dx \right| \\ &\leq C\|u_j\|_{L^2(|x|\leq 1)}^{2-a} \|\nabla u_j\|_{L^2}^a + C\||x|^{-a}\|_{L^{\frac{q}{q-2}}(|x|>1)} \|u_j\|_{L^q}^2 \\ &\quad + C\|u_j\|_{L^2(|x|\leq 1)}^{p+1-\frac{n(p-1)+2b}{2}} \|\nabla u_j\|_{L^2}^{\frac{n(p-1)+2b}{2}} + C\|u_j\|_{L^{p+1}(|x|>1)}^{p+1} \to 0 \text{ as } j \to \infty. \end{aligned} \tag{2.9}$$

This implies $I_\mu = \lim_{j\to\infty} E(u_j) \geq \frac{1}{2} \liminf \|\nabla u_j\|_{L^2}^2 \geq 0$ and contradicts (2.2).

If $0 < \nu < \mu$, then from the support condition (2.4) it follows that

$$
\begin{aligned}
&E(u_j) - E(v_j) - E(w_j) \\
&= \frac{1}{2}\left(\|\nabla u_j\|_{L^2}^2 - \|\nabla v_j\|_{L^2}^2 - \|\nabla w_j\|_{L^2}^2\right) \\
&\quad - \frac{1}{2}\int V(x)(|u_j|^2 - |v_j + w_j|^2)\,dx - \frac{1}{p+1}\int g(x)(|u_j|^{p+1} - |v_j + w_j|^{p+1})\,dx.
\end{aligned}
$$

From (2.7), (2.8), and estimates in (2.9) we deduce that

$$
\liminf_{j\to\infty}(E(u_j) - E(v_j) - E(w_j)) \geq 0
$$

and thus

$$
I_\mu = \lim_{j\to\infty} E(u_j) \geq \liminf_{j\to\infty} E(v_j) + \liminf_{j\to\infty} E(w_j).
$$

Since $m(v_j) \to \nu$ and $m(w_j) \to \mu - \nu$, by the continuity of I_μ on $(0, \infty)$ we get

$$
I_\mu \geq I_\nu + I_{\mu-\nu},
$$

which contradicts (2.3).

Therefore $\nu = \mu$. Set $\widetilde{u}_j(x) = u_j(x + y_j)$. Then $u, \widetilde{u}_j \in S_\mu$ and $\widetilde{u}_j \to u$ in L^q for all $2 \leq q < 2^*$. On the other hand, (u_j) is bounded in H^1. Hence there is a subsequence (still denoted by u_j) converging to v weakly in H^1 and strongly in L^q_{loc} for any $1 \leq q < 2^*$. If (y_j) are unbounded, then up to subsequence we may assume that $|y_j| \to \infty$. Since $\widetilde{u}_j \to u$ in L^2, $u_j - u(\cdot - y_j) \to 0$ in the sense of distributions. But $u(\cdot - y_j) \to 0$ and $u_j \to v$ in the sense of distributions and thus $v = 0$.

Now for any $\varepsilon > 0$ we can find $R_0, j_0 > 1$ such that if $j \geq j_0$, then

$$
\begin{aligned}
&\int_{|x|>R_0} |V(x)|(|u_j|^2 + |v|^2)\,dx \leq CR_0^{-a} < \frac{\varepsilon}{4}, \\
&\int_{|x|\leq R_0} |V(x)|(|u_j| + |v|)|u_j - v|\,dx \\
&\leq C\||x|^{-a}\|_{L^{\frac{q}{q-2}}(|x|\leq R_0)} \||u_j| + |v|\|_{L^q} \|u_j - v\|_{L^q(|x|\leq R_0)} < \frac{\varepsilon}{4},
\end{aligned}
$$

where $\frac{2n}{n-a} < q < 2^*$, and also such that

(1) Case: $b > 0$

$$
\begin{aligned}
&\int_{|x|>R_0} g(x)(|u_j|^{p+1} + |v|^{p+1})\,dx \leq CR_0^{-b} < \frac{\varepsilon}{4}, \\
&\int_{|x|\leq R_0} g(x)(|u_j|^p + |v|^p)|u_j - v|\,dx \\
&\leq C\||x|^{-b}\|_{L^{\frac{q}{q-(p+1)}}(|x|\leq R_0)} \||u_j| + |v|\|_{L^q}^p \|u_j - v\|_{L^q(|x|\leq R_0)} \\
&< \frac{\varepsilon}{4} \text{ for } \frac{(p+1)n}{n-b} < q < 2^*.
\end{aligned}
$$

(2) Case: $b = 0$ and $|y_j| \le R_1$

$$\begin{aligned}&\int_{\{|x-y_j|>R_0\}\cap\{|x|>R_0\}} g(x)(|u_j|^{p+1} + |v|^{p+1})\,dx\\ &\le C\int_{\{|x|>R_0\}} |\widetilde{u}_j|^{p+1}\,dx + C\int_{\{|x|>R_0\}} |v|^{p+1}\,dx < \frac{\varepsilon}{4},\\ &\int_{\{|x-y_j|\le R_0\}\cup\{|x|\le R_0\}} g(x)(|u_j|^p + |v|^p)|u_j - v|\,dx\\ &\le C\||u_j| + |v|\|_{L^{pq'}}^p \|u_j - v\|_{L^q(|x|\le R_0+R_1)} < \frac{\varepsilon}{4} \ \text{ for } \ 2 \le pq' \le 2^*.\end{aligned}$$

(3) Case: $b = 0$ and (y_j) are unbounded

$$\begin{aligned}&\int_{\{|x-y_j|>R_0\}\cap\{|x|>R_0\}} g(x)(|u_j|^{p+1}+|v|^{p+1})\,dx \le C\int_{\{|x|>R_0\}} |\widetilde{u}_j|^{p+1}\,dx < \frac{\varepsilon}{4},\\ &\int_{\{|x-y_j|\le R_0\}\cup\{|x|\le R_0\}} g(x)(|u_j|^p + |v|^p)|u_j - v|\,dx\\ &\le \int_{\{|x-y_j|\le R_0\}} g(x)|u_j|^{p+1}\,dx + C\int_{\{|x|\le R_0\}} |u_j|^{p+1}\,dx\\ &\le C\int_{\{|x|\le R_0\}} g(x+y_j)|\widetilde{u}_j - u|^{p+1}dx + C\int_{\{|x|\le R_0\}} g(x+y_j)|u|^{p+1}dx + \frac{\varepsilon}{8} < \frac{\varepsilon}{4}\end{aligned}$$

due to the fact $\widetilde{u}_j \to u$ in L^2 and $g(x + y_j) \to 0$.

Set $P(w) := E(w) - \frac{1}{2}\|\nabla w\|_{L^2}^2$. Then $P(u_j) \to P(v)$ as $j \to \infty$. Suppose that (y_j) is unbounded. Then $v = 0$ and hence $P(u_j) \to 0$ as $j \to \infty$. This implies that $I_\mu = \lim_{j\to\infty} E(u_j) \ge 0$, which contradicts (2.2). Thus (y_j) is bounded. Now let $R_1 = \sup_{j\ge1} |y_j|$. Then for any $\varepsilon > 0$ we have

$$\int_{|x|<R_1+r(\varepsilon)} |u_j|^2\,dx \ge \int_{|x-y_j|<r(\varepsilon)} |u_j|^2\,dx \ge \mu - \varepsilon, \ \text{ if } \ j \ge j_0,$$

and thus

$$m(v) \ge \int_{|x|<R_1+r(\varepsilon)} |v|^2\,dx \ge \lim_{j\to\infty}\int_{|x|<R_1+r(\varepsilon)} |u_j|^2\,dx \ge \mu - \varepsilon.$$

This means $m(v) \ge \mu$, while the semi-continuity of weak limit implies $m(v) \le \mu$. Then $v \in S_\mu$. Since $P(u_j) \to P(v)$, we have

$$I_\mu \le E(v) \le \liminf \frac{1}{2}\|\nabla u_j\|_{L^2}^2 + P(v) = \liminf(E(u_j)) = I_\mu. \tag{2.10}$$

Therefore $E(v) = I_\mu$. This completes the proof of Proposition 1.1.

2.2. Proof of continuity of I_μ

For any $\mu > 0$ let us take sequences $\mu_j \in (0,\infty)$ and $u_j \in S_{\mu_j}$ such that $\mu_j \to \mu$ and $I_{\mu_j} < E(u_j) < I_{\mu_j} + \frac{1}{j}$. From (2.1) it follows that $\|u_j\|_{H^1} \le M$ for some constant $M > 0$. Then $\|u_j - \frac{\mu}{\mu_j}u_j\|_{H^1} \le M|1 - \frac{\mu}{\mu_j}|$ and hence there exists j_0 such that $\|u_j - \frac{\mu}{\mu_j}u_j\|_{H^1} \le M$ for $j \ge j_0$. On the other hand, $E \in C^1(H^1, \mathbb{R}), E' \in C(H^1, H^{-1})$ and for any $v, h \in H^1$

$$\langle E'(v), h\rangle = \langle \nabla v, \nabla h\rangle - \langle Vv, h\rangle - \mathrm{Re}\langle g|v|^{p-1}v, h\rangle.$$

Thus for any $v \in H^1$ with $\|v\|_{H^1} \le 2M$ we have from Hardy-Sobolev inequality that

(2.11)

$$\begin{aligned}
&|\langle E'(v), h\rangle| \\
&\le M\|h\|_{H^1} + A\||x|^{-a/2}v\|_{L^2}\||x|^{-a/2}h\|_{L^2} + B\||x|^{-\frac{b}{p+1}}v\|^p_{L^{p+1}}\||x|^{-\frac{b}{p+1}}h\|_{L^{p+1}} \\
&\le CM\|h\|_{H^1} + \|v\|_{L^2}^{p(1-\frac{n(p-1)+2b}{2(p+1)})}\|\nabla u\|_{L^2}^{\frac{p(n(p-1)+2b)}{2(p+1)}}\|h\|_{L^2}^{1-\frac{n(p-1)+2b}{2(p+1)}}\|\nabla h\|_{L^2}^{\frac{n(p-1)+2b}{2(p+1)}} \\
&\le C(M + CM^p)\|h\|_{H^1}
\end{aligned}$$

and therefore $\|E'(v)\|_{H^{-1}} \le C(M+M^p)$. Using this and Mean Value Theorem we get

$$|E(u_j) - E(\frac{\mu}{\mu_j}u_j)| \le C(M + M^p)M\left|1 - \frac{\mu}{\mu_j}\right|$$

if $j \ge j_0$. This implies that $I_\mu \le \liminf_{j\to\infty} E(\frac{\mu}{\mu_j}u_j) \le \liminf_{j\to\infty} I_{\mu_j}$.

Now we choose a sequence $(v_j) \subset S_\mu$ such that $E(v_j) \to I_\mu$. By (2.1) we deduce that there exists $K > 0$ such that $\|v_j\|_{H^1} \le K$. Thus from (2.11) it follows that

$$I_{\mu_j} \le E(\frac{\mu_j}{\mu}v_j) \le |E(\frac{\mu_j}{\mu}v_j) - E(v_j)| + E(v_j) \le C(K + K^p)K|1 - \frac{\mu_j}{\mu}| + E(v_j).$$

This implies that $\limsup_{j\to\infty} I_{\mu_j} \le I_\mu$. This concludes the proof.

3. Proof of Theorem 1.2

The proof proceeds by contradiction. Suppose that $\mathcal{O}_\mu$ is not stable, then either $\mathcal{O}_\mu$ is empty or there exist $w \in \mathcal{O}_\mu$ and a sequence $\psi_0^j \in H^1$ such that

$$\|\psi_0^j - w\|_{H^1} \to 0 \text{ as } j \to \infty$$

but

$$\inf_{v\in\mathcal{O}_\mu} \|\psi^j(t_j, \cdot) - v\|_{H^1} \ge \varepsilon_0 \tag{3.1}$$

for some sequence $t_j \in [-T_1, T_2]$ and ε_0, where $\psi^j(t, \cdot)$ is the solution of (1.1) corresponding to the initial data ψ_0^j. Let $w_j = \psi^j(t_j, \cdot)$. Since $w \in S_\mu$ and $E(w) = I_\mu$, it follows from the continuity of L^2 norm and E in H^1 that

$$\|\psi_0^j\|^2_{L^2} \to \mu \text{ and } E(\psi_0^j) \to I_\mu.$$

Thus we deduce from the conservation laws that

$$\|w_j\|^2_{L^2} = \|\psi_0^j\|^2_{L^2} \to \mu, \quad E(w_j) = E(\psi_0^j) \to I_\mu.$$

Therefore (w_j) has a subsequence converging to an element $v' \in H^1$ such that $\|v'\|_{L^2}^2 = \mu$ and $E(v) = I_\mu$. This shows that $v' \in \mathcal{O}_\mu$ but

$$\inf_{v\in\mathcal{O}_\mu} \|\psi^j(t_j,\cdot) - v\|_{H^1} \le \|w_j - v'\|_{H^1},$$

which contradicts (3.1). Since $\mathcal{O}_\mu$ is not empty, to show the orbital stability of $\mathcal{O}_\mu$ one has to prove that any sequence $(w_j) \subset H^1$ with

$$\|w_j\|_{L^2}^2 \to \mu \ \text{ and } \ E(w_j) \to I_\mu \tag{3.2}$$

is relatively compact in H^1. Let $\mu_j = \|w_j\|_{L^2}^2$ and $u_j = \frac{\mu}{\mu_j} w_j$. Then $u_j \in S_\mu$, and since I_μ is finite for all $\mu \in (0,\infty)$ and $p < 1 + \frac{2(2-b)}{n}$, by the arguments in the proof of Proposition 1.1 we may assume that (u_j) is bounded in H^1 and also verify from all argument around (2.10) that by passing to a subsequence there exists $v \in H^1$ such that

$$u_j \rightharpoonup v \ \text{ in } \ H^1 \ \text{ and } \ \lim_{j\to\infty} \|\nabla u_j\|_{L^2} = \|\nabla v\|_{L^2}. \tag{3.3}$$

This implies $w_j \to v$ in H^1 and thus the relative compactness.

4. Well-posedness

In this section we prove Theorem 1.3. Let us first consider the local well-posednss on $[-T,T]$. Let (X_T^ρ, d_X) be a metric space with metric d_X defined by

$$X_T^\rho = \{\psi \in C([-T,T];H^1) \cap L_T^{q_0} H_{r_0}^1 : \|\psi\|_{L_T^\infty H^1 \cap L_T^{q_0} H_{r_0}^1} \le \rho\},$$
$$d_X(\psi,\psi') = \|\psi - \psi'\|_{L_T^\infty H^1 \cap L_T^{q_0} L^{r_0}},$$

where L_T^q denotes $L^q([-T,T])$ and (q_0, r_0) is an admissible pair, which will be chosen later. Then X_T^ρ is clearly complete metric space. We define a mapping Φ on X_T^ρ by

$$\Phi(\psi)(t) = U(t)\psi_0 - i\int_0^t U(t-t')[N(\cdot,\psi)](t')\,dt'. \tag{4.1}$$

We have from Strichartz estimates with admissible pairs (q_i, r_i), $i = 0,1,\ldots,8$ that

$$\|\Phi(\psi)\|_{L_T^\infty H^1 \cap L_T^{q_0} H_{r_0}^1} \le C(\|\varphi\|_{H^1} + \sum_{i=1}^{8} \mathcal{N}_i), \tag{4.2}$$

where

$$\mathcal{N}_1 = \|V(\psi,\nabla\psi)\|_{L_T^{q_1'}L^{r_1'}(|x|\le 1)}, \qquad \mathcal{N}_2 = \|V(\psi,\nabla\psi)\|_{L_T^{q_2'}L^{r_2'}(|x|>1)},$$
$$\mathcal{N}_3 = \|\nabla V\psi\|_{L_T^{q_3'}L^{r_3'}(|x|\le 1)}, \qquad \mathcal{N}_4 = \|\nabla V\psi\|_{L_T^{q_4'}L^{r_4'}(|x|>1)},$$
$$\mathcal{N}_5 = \|g|\psi|^{p-1}(\psi,\nabla\psi)\|_{L_T^{q_5'}L^{r_5'}(|x|\le 1)}, \qquad \mathcal{N}_6 = \|g|\psi|^{p-1}(\psi,\nabla\psi)\|_{L_T^{q_5'}L^{r_5'}(|x|>1)},$$
$$\mathcal{N}_7 = \|\nabla g|\psi|^{p}\|_{L_T^{q_7'}L^{r_7'}(|x|\le 1)}, \qquad \mathcal{N}_8 = \|\nabla g|\psi|^{p}\psi\|_{L_T^{q_8'}L^{r_8'}(|x|>1)}.$$

Here we used the notation $\|(f,F)\|_{L^r} = \|f\|_{L^r} + \|F\|_{L^r}$ for $F = (f_1,\ldots,f_n)$.

Let $\nu = 1$ or p. Set $s_\nu = 1$ if $n = 1$, $1 < s_\nu < \min(\frac{1}{a},\frac{1}{b})$ if $n = 2$, and $s_\nu = \frac{2n}{n+2-(n-2)\nu}$ if $n \ge 3$. We proceed by dividing the sum $\sum_{i=1}^{8}\mathcal{N}_i$ into two parts: linear part ($i = 1,2,3,4$) and nonlinear part ($i = 5,6,7,8$).

4.1. Linear part

Here we set $(q_0,r_0) = (\frac{4s_1}{n},\frac{2s_1}{s_1-1})$ and take $(q_i,r_i) = (q_0,r_0)$ for $i = 1,3$ and $(q_i,r_i) = (\infty,2)$ for $i = 2,4$. Let $\psi \in X_T^\rho$. Then we have that for $n = 1,2$

$$\begin{aligned}\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_4 &\le C(T^{1-\frac{2}{q_0}}\|V\|_{L^{s_1}(|x|\le 1)}\|\psi\|_{L_T^{q_0}H^1_{r_0}} + T\|V\|_{L^\infty(|x|>1)}\|\psi\|_{L_T^\infty H^1} \\ &\qquad + T\|\nabla V\|_{L^\infty(|x|>1)}\|\psi\|_{L_T^\infty L^2}) \\ &\le C(T+T^{1-\frac{2}{q_0}})\|\psi\|_{L_T^\infty H^1\cap L_T^{q_0}H^1_{r_0}} \le C(T+T^{1-\frac{2}{q_0}})\rho.\end{aligned}$$

If $n \ge 3$, then we choose $\frac{2n}{n+2-2a} < r < r_0 = 2^*$ and let (q,r) be the corresponding admissible pair. Then we get

$$\begin{aligned}\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_4 &\le C(T^{\frac{1}{2}-\frac{1}{q}}\||x|^{-a}\|_{L^{\frac{2nr}{r(n+2)-2n}}(|x|\le 1)}\|\psi\|_{L_T^qH^1_r} \\ &\qquad + T\|V\|_{L^\infty(|x|>1)}\|\psi\|_{L_T^\infty H^1} + T\|\nabla V\|_{L^\infty(|x|>1)}\|\psi\|_{L_T^\infty L^2}) \\ &\le C(T+T^{\frac{1}{2}-\frac{1}{q}})\|\psi\|_{L_T^\infty H^1\cap L_T^{q_0}H^1_{r_0}} \le C(T+T^{\frac{1}{2}-\frac{1}{q}})\rho.\end{aligned}$$

To treat $\mathcal{N}_3$ we need to restrict the range of a. If $n = 1$, then

$$\mathcal{N}_3 \le CT^{\frac{1}{2}}\||x|^{-a'}\|_{L^1(|x|\le 1)}\|\psi\|_{L_T^4L^\infty} \le CT^{\frac{1}{2}}\rho.$$

If $n = 2$, then since $s_1 < \frac{1}{a}$ for $n = 2$

$$\begin{aligned}\mathcal{N}_3 &\le CT^{1-\frac{2}{q_0}}\||x|^{-a-1}\|_{L^{\frac{2s_1}{s_1+1}}(|x|\le 1)}\|\psi\|_{L_T^{q_0}L^\infty} \\ &\le CT^{1-\frac{1}{s_1}}\|\psi\|_{L_T^{q_0}H^1_{r_0}} \le CT^{1-\frac{1}{s_1}}\rho.\end{aligned}$$

If $n = 3$, then since $a < \frac{3}{2}$, we can choose r such that $\frac{6}{3-2a} < r < \infty$ (and hence $\frac{6r(a+1)}{5r-6}$) to get

$$\mathcal{N}_3 \le CT^{\frac{1}{12}}\||x|^{-a-1}\|_{L^{\frac{6r}{5r-6}}(|x|\le 1)}\|\psi\|_{L_T^{\frac{12}{5}}L^r} \le CT^{\frac{1}{12}}\|\psi\|_{L_T^{\frac{12}{5}}H^1_3} \le CT^{\frac{1}{12}}\rho.$$

If $n=4$, then we choose $\frac{4}{3}<r<\min(2,\frac{4}{a+1})$ and get

$$\begin{aligned}\mathcal{N}_3 &\le T^{\frac{3s-4}{2s}}\||x|^{-a-1}\|_{L^r(|x|\le 1)}\|\psi\|_{L_T^{\frac{2s}{4-s}}L^{\frac{4s}{3s-4}}}\\ &\le CT^{\frac{3s-4}{2s}}\|\psi\|_{L_T^{\frac{s}{2-s}}H^1_{\frac{s}{s-1}}} \le CT^{\frac{3s-4}{2s}}\rho.\end{aligned}$$

If $n\ge 5$, then we choose $\frac{n}{2}<r<\min(\frac{n}{a},n)$ (hence $\widetilde{r}:=\frac{2rn}{(n+2)r-2n}<\frac{2n}{n-2}$) to get

$$\begin{aligned}\mathcal{N}_3 &\le CT^{\frac{nr-n+r}{2nr}}\||x|^{-a}\|_{L^r(|x|\le 1)}\||x|^{-1}\psi\|_{L_T^{\frac{2nr}{n-r}}L^{\widetilde{r}}}\\ &\le CT^{\frac{nr-n+r}{2nr}}\|\psi\|_{L_T^{\frac{2nr}{n-r}}H^1_{\widetilde{r}}} \le CT^{\frac{nr-n+r}{2nr}}\rho.\end{aligned}$$

Therefore any $\psi\in X_T^\rho$ we obtain that for some $\theta>0$

$$\sum_{i=1}^{4}\mathcal{N}_i \le C(T+T^\theta)\rho. \tag{4.3}$$

4.2. Nonlinear part

Now we move onto $\mathcal{N}_i, i=5,6,8$. Let $(q_0,r_0)=(\frac{4s_p}{n},\frac{2s_p}{s_p-1})$ if $n=1,2$ and $(2,2^*)$ if $n\ge 3$. Since $g\le B|x|^{-b}$ with $0<b<\min(2,n)$ and $p<2_b$. For $i=5,6,8$ we take $(q_5,r_5)=(q_0,r_0)$, and $(q_6,r_6)=(q_8,r_8)=(\infty,2)$ for $n=1,2$ and $(\frac{4(p+1)}{n(p-1)},p+1)$ for $n\ge 3$. If $n=1$, then

$$\begin{aligned}\mathcal{N}_5+\mathcal{N}_6+\mathcal{N}_8 &\le C(T^{\frac12}+T)(\|g\|_{L^{s_p}(|x|\le 1)}+\|(g,g')\|_{L^\infty(|x|>1)})\\ &\qquad\times(\|\psi\|^{p-1}_{L_T^\infty H^1}\|\psi\|_{L_T^4H^1_\infty}+\|\psi\|^p_{L_T^{q_6}H^1_{r_6}})\\ &\le C(T^{\frac12}+T)\rho^p.\end{aligned}$$

If $n=2$, then we choose ε with $\frac{s_p}{1-\varepsilon s_p}<\frac{2}{b}$ (this is possible because $0<b<1$ and $s_p<\frac{1}{b}$). Let $r=\frac{p-1}{\varepsilon}$. Then we get

$$\begin{aligned}\mathcal{N}_5+\mathcal{N}_6+\mathcal{N}_8 &\le C(T^{1-\frac{2}{q_0}}+T^{1-\frac{1}{q_0}})(\|g\|_{L^{\frac{s_p}{1-\varepsilon s_p}}(|x|\le 1)}+\|(g,\nabla g)\|_{L^\infty(|x|>1)})\\ &\qquad\times(\|\psi\|^{p-1}_{L_T^\infty L^r}+\|\psi\|^{p-1}_{L_T^\infty L^{2(p-1)r_0 r_0-2}})\|\psi\|_{L_T^{q_0}H^1_{r_0}})\\ &\le C(T^{1-\frac{2}{q_0}}+T^{1-\frac{1}{q_0}})\rho^p.\end{aligned}$$

If $n\ge 3$, then let us invoke $s_p=\frac{2n}{n+2-(n-2)p}$ and choose a small ε with $p+\varepsilon 2^*<2_b$. Let $\frac{1}{r}=\frac{n-2}{2n}+\varepsilon$ and (q,r) be corresponding admissible pair. Then we have

$$\mathcal{N}_5\le CT^{\frac{\varepsilon n}{2}}\|g\|_{\frac{s_p}{1-\varepsilon s_p}}\|\psi\|^{p-1}_{L_T^\infty L^{r_0}}\|\psi\|_{L_T^qH^1_r}\le CT^{\frac{\varepsilon n}{2}}\rho^p.$$

As for $i = 6, 8$ let $(q, r) = (\frac{4(p+1)}{n(p-1)}, p+1)$. Then $q > 2$ and we have that

$$\mathcal{N}_6 + \mathcal{N}_8 \leq C(T^{1-\frac{2}{q}})\|(g, \nabla g)\|_{L^\infty(|x|>1)}(\|\psi\|^{p-1}_{L^\infty_T L^r}\|\psi\|_{L^q_T H^1_r}) \leq CT^{1-\frac{2}{q}}\rho^p.$$

Let us now consider $\mathcal{N}_7$. We take $(q_7, r_7) = (q_0, r_0)$. If $n = 1$, then we get

$$\mathcal{N}_7 \leq CT^{\frac{1}{2}}\||x|^{-b'}\|_{L^1(|x|\leq 1)}\|\psi\|^p_{L^4_T L^\infty} \leq CT^{\frac{1}{2}}\rho^p.$$

If $n = 2$, then we choose $s_p > 1$ and very close to 1. For a small $\varepsilon > 0$ so that $\frac{s_p(b+\varepsilon)}{1-2\varepsilon s_p} < 1$ we get that

$$\begin{aligned}\mathcal{N}_7 &\leq CT^{1-\frac{1}{q_0}}\||x|^{-b-\varepsilon}\|_{L^{\frac{2s_0}{1-\varepsilon s_p}}(|x|\leq 1)}\|p\|^{p-1}_{L^\infty_T L^{\frac{p-1}{\varepsilon}}}\||x|^{-1+\varepsilon}p\|_{L^\infty_T L^\infty}\\ &\leq CT^{1-\frac{1}{q_0}}\|\psi\|^p_{L^\infty_T H^1} \leq CT^{1-\frac{1}{q_0}}\rho^p.\end{aligned}$$

If $n = 3$, then we divide the range of p into two parts, (i) $1 < p < 2$ and (ii) $2 \leq p < 2_b = 5 - 2b$. For both cases we need the condition $b < \frac{3}{2}$.
Case (i): We take a small $\varepsilon > 0$ with $\frac{6(b+1)}{5-\varepsilon} < 3$. Then we get

$$\begin{aligned}\mathcal{N}_7 &\leq CT^{\frac{2-p}{4}}\||x|^{-b-1}\|_{L^{\frac{6}{5-\varepsilon}}(|x|\leq 1)}\|\|\psi\|^p_{L^{\frac{6p}{\varepsilon}}}\|_{L^{\frac{4}{p}}_T} \leq CT^{\frac{2-p}{4}}\|\|\psi\|_{H^1_3}\|_{L^{\frac{4}{p}}_T}\\ &\leq CT^{\frac{2-p}{4}}\|\|\psi\|^{\frac{p}{2}}_{H^1}\|\psi\|^{\frac{p}{2}}_{H^1_6}\|_{L^{\frac{4}{p}}_T}\\ &\leq CT^{\frac{2-p}{4}}\|\psi\|^{\frac{p}{2}}_{L^\infty_T H^1}\|\psi\|^{\frac{p}{2}}_{L^2_T H^1_6}\\ &\leq CT^{\frac{2-p}{4}}\rho^p.\end{aligned}$$

Case (ii): We take a small $\varepsilon > 0$ with $\frac{6(b+1)}{7-p-\varepsilon} < 3$ (hence $p + \varepsilon < 2_b$). Then we get for $\frac{1}{r} = \frac{\varepsilon}{12} - \frac{1}{3}$

$$\begin{aligned}\mathcal{N}_7 &\leq C\||x|^{-b-1}\|_{L^{\frac{6}{7-p-\varepsilon}}(|x|\leq 1)}\|\|\psi\|^{p-2}_{L^6}\|\psi\|^2_{L^{\frac{12}{\varepsilon}}}\|_{L^2_T} \leq C\|\|\psi\|^{p-2}_{H^1}\|\psi\|^2_{H^1_r}\|_{L^2_T}\\ &\leq C\|\|\psi\|^{p-2}_{H^1}\|\psi\|^{\frac{2+\varepsilon}{2}}_{H^1}\|\psi\|^{\frac{2-\varepsilon}{2}}_{H^1_6}\|_{L^2_T}\\ &\leq CT^{\frac{\varepsilon}{4}}\rho^p.\end{aligned}$$

If $n \geq 4$, then we choose ε with $\frac{2n(b+\varepsilon)}{4-(n-2)(p-1)} < n$. We have from Hardy-Sobolev's inequality that

$$\begin{aligned}\mathcal{N}_7 &\leq C\||x|^{-b-\varepsilon}\|_{L^{\frac{2n}{4-(n-2)(p-1)}}}\|\|\psi\|^{p-1}_{r_0}\||x|^{-1+\varepsilon}\psi\|_{L^{r_0}}\|_{L^2_T}\\ &\leq C\|\|\psi\|^{p-1}_{r_0}\|\psi\|^{1-\varepsilon}_{L^{r_0}}\|\nabla\psi\|^{\varepsilon}_{L^{r_0}}\|_{L^2_T}\\ &\leq CT^{\frac{1-\varepsilon}{2}}\|\psi\|^{p-\varepsilon}_{L^\infty_T H^1}\|\psi\|^{\varepsilon}_{L^{q_0}_T H^1_{r_0}}\\ &\leq CT^{\frac{1-\varepsilon}{2}}\rho^p.\end{aligned}$$

Therefore we can find $\theta' > 0$ such that

$$\sum_{i=5}^{8} \mathcal{N}_i \leq C(T + T^{\theta'})\rho^p. \tag{4.4}$$

Taking ρ, T such that $\rho \geq 2C\|\psi_0\|_{H^1}$ and $2C(2T + T^{\theta} + T^{\theta'})\rho^p \leq \rho$, from (4.3) and (4.4) we deduce that Φ is self-mapping on X_T^{ρ}.

4.3. Contraction

By direct calculation we have that for $p \geq 2$

$$\begin{aligned} |\nabla(|u|^{p-1}u) - \nabla(|v|^{p-1}v)| &\leq C(|u|^{p-2} + |v|^{p-2})(|\nabla u| + |\nabla v|)|u - v| \\ &\quad + C(|u|^{p-1} + |v|^{p-1})|\nabla u - \nabla v|, \end{aligned}$$

and for $1 < p < 2$

$$\begin{aligned} |\nabla(|u|^{p-1}u) - \nabla(|v|^{p-1}v)| &\leq C(|u|^{p-1} + |v|^{p-1})|\nabla u - \nabla v| \\ &\quad + C(|\nabla u| + |\nabla v|)|u - v|^{p-1}. \end{aligned}$$

Applying the above estimates of self-mapping one can easily obtain the contraction

$$d(\Phi(u), \Phi(v)) \leq \frac{1}{2} d(u, v), \tag{4.5}$$

provided T is a little smaller than one of self-mapping. This shows the local well-posedness of (1.1). One can also show the conservation laws by the argument for Strichartz solutions of [16] or classical argument of [3]. The global well-posedness follows easily from the conservations and (2.1), which give us uniform bound of $\|\nabla\psi\|_{L^2}$. We omit the detail.

Acknowledgments. The authors would like to thank the anonymous referee for his/her careful reading and valuable comments for this paper. This work was supported by NRF-2018R1D1A3B07047782.

References

[1] A. Bahri and Y. Y. Li, *On a min-max procedure for the existence of a positive solution for certain scalar field equations in* $\mathbb{R}^N$, Rev. Mat. Iberoamericana **6** (1990), no. 1-2, 1–15. `https://doi.org/10.4171/RMI/92`

[2] J. Belmonte-Beitia, V. M. Pérez-García, V. Vekslerchik, and P. J. Torres, *Lie symmetries, qualitative analysis and exact solutions of nonlinear Schrödinger equations with inhomogeneous nonlinearities*, Discrete Contin. Dyn. Syst. Ser. B **9** (2008), no. 2, 221–233. `https://doi.org/10.3934/dcdsb.2008.9.221`

[3] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, **10**, New York University, Courant Institute of Mathematical Sciences, New York, 2003. `https://doi.org/10.1090/cln/010`

[4] Y. Cho, H. Hajaiej, G. Hwang, and T. Ozawa, *On the orbital stability of fractional Schrödinger equations*, Commun. Pure Appl. Anal. **13** (2014), no. 3, 1267–1282. `https://doi.org/10.3934/cpaa.2014.13.1267`

[5] V. D. Dinh, *Scattering theory in a weighted* L^2 *space for a class of the defocusing inhomogeneous nonlinear Schrödinger equation*, in preprint (arXiv:1710.01392)

[6] L. G. Farah, *Global well-posedness and blow-up on the energy space for the inhomogeneous nonlinear Schrödinger equation*, J. Evol. Equ. **16** (2016), no. 1, 193–208. https://doi.org/10.1007/s00028-015-0298-y

[7] L. G. Farah and C. M. Guzmán, *Scattering for the radial 3D cubic focusing inhomogeneous nonlinear Schrödinger equation*, J. Differential Equations **262** (2017), no. 8, 4175–4231. https://doi.org/10.1016/j.jde.2017.01.013

[8] R. Fukuizumi and M. Ohta, *Instability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities*, J. Math. Kyoto Univ. **45** (2005), no. 1, 145–158. https://doi.org/10.1215/kjm/1250282971

[9] F. Genoud, *An inhomogeneous, L^2-critical, nonlinear Schrödinger equation*, Z. Anal. Anwend. **31** (2012), no. 3, 283–290. https://doi.org/10.4171/ZAA/1460

[10] F. Genoud and C. A. Stuart, *Schrödinger equations with a spatially decaying nonlinearity: existence and stability of standing waves*, Discrete Contin. Dyn. Syst. **21** (2008), no. 1, 137–186. https://doi.org/10.3934/dcds.2008.21.137

[11] T. S. Gill, *Optical guiding of laser beam in nonuniform plasma*, Pramana J. Phys. **55** (2000), 842–845.

[12] C. M. Guzmán, *On well posedness for the inhomogeneous nonlinear Schrödinger equation*, Nonlinear Anal. Real World Appl. **37** (2017), 249–286. https://doi.org/10.1016/j.nonrwa.2017.02.018

[13] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980.

[14] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 2, 109–145.

[15] C. S. Liu and V. K. Tripathi, *Laser guiding in an axially nonuniform plasma channel*, Phys. Plasma **1** (9) (1994), 3100–3103.

[16] T. Ozawa, *Remarks on proofs of conservation laws for nonlinear Schrödinger equations*, Calc. Var. Partial Differential Equations **25** (2006), no. 3, 403–408. https://doi.org/10.1007/s00526-005-0349-2

[17] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger Equation*, Applied Mathematical Sciences, **139**, Springer-Verlag, New York, 1999.

[18] X.-Y. Tang and P. K. Shukla, *Solution of the one-dimensional spatially inhomogeneous cubic-quintic nonlinear Schrödinger equation with an external potential*, Physical Review A **76** (2007), 013612–1–10.

Yonggeun Cho
Department of Mathematics and
Institute of Pure and Applied Mathematics
Chonbuk National University
Jeonju 54896, Korea
Email address: changocho@jbnu.ac.kr

Misung Lee
Department of Mathematics
Chonbuk National University
Jeonju 54896, Korea
Email address: voiceof1217@jbnu.ac.kr