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THREE GEOMETRIC CONSTANTS FOR MORREY SPACES

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Abstract. In this paper we calculate three geometric constants, namely the von Neumann-Jordan constant, the James constant, and the Dunkl-Williams constant, for Morrey spaces and discrete Morrey spaces. These constants measure uniformly nonsquareness of the associated spaces. We obtain that the three constants are the same as those for L^1 and L^∞ spaces.

1. Introduction

The *von Neumann-Jordan constant* $\mathrm{C_{NJ}}(X)$ (see [8]), the *James constant* $\mathrm{C}_J(X)$ (see [6]) and the *Dunkl-Williams constant* $\mathrm{C_{DW}}(X)$ (see [3]) for a Banach space X are given by

$$\mathrm{C_{NJ}}(X) := \sup\left\{\frac{\|x+y\|_X^2+\|x-y\|_X^2}{2(\|x\|_X^2+\|y\|_X^2)} \,:\, x,y\in X\setminus\{0\}\right\},$$

$$\mathrm{C_J}(X) := \sup\left\{\min\{\|x+y\|_X,\|x-y\|_X\} \,:\, x,y\in X,\ \|x\|_X=\|y\|_X=1\right\},$$

and

$$\mathrm{C_{DW}}(X) := \sup\left\{\frac{\|x\|_X+\|y\|_X}{\|x-y\|_X}\left\|\frac{x}{\|x\|_X}-\frac{y}{\|y\|_X}\right\|_X : x,y\in X,\ x,y,x-y\neq 0\right\},$$

respectively. It is well known that $1\le C_{\mathrm{NJ}}(X)\le 2$ for every Banach space X, and that $C_{\mathrm{NJ}}(X)=1$ if and only if X is a Hilbert space. Meanwhile, $\sqrt{2}\le C_{\mathrm{J}}(X)\le 2$ holds for every Banach space X, and $C_{\mathrm{J}}(X)=\sqrt{2}$ if (but not only if) X is a Hilbert space (see [2,4]). As for the Dunkl-Williams constant, we have $2\le \mathrm{C_{DW}}(X)\le 4$ and $\mathrm{C_{DW}}(X)=2$ if and only if X is a Hilbert space [3]. For Lebesgue spaces $L^p=L^p(\mathbb{R}^d)$ where $1\le p\le\infty$, we have $C_{\mathrm{NJ}}(L^p)=$

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$\max\{2^{2/p-1}, 2^{1-2/p}\}$ and $C_{\mathrm{J}}(L^p) = \max\{2^{1/p}, 2^{1-1/p}\}$ [9]. Meanwhile, we know that $C_{\mathrm{DW}}(L^1) = C_{\mathrm{DW}}(L^\infty) = 4$ [7].

In this paper, we shall calculate the three constants for Morrey spaces and discrete Morrey spaces. Let $1 \le p \le q < \infty$. The *Morrey space* $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^d)$ is the set of all the measurable functions f on $\mathbb{R}^d$ for which

$$\|f\|_{\mathcal{M}^p_q} := \sup_{B=B(a,r)} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty,$$

where $B(a,r)$ denotes the ball centered at $a \in \mathbb{R}^d$ having radius $r > 0$ and Lebesgue measure $|B|$ (see, e.g., [1]). Since $\mathcal{M}^p_q$ is a Banach space, it follows from [2–4] that

$$C_{\mathrm{NJ}}(\mathcal{M}^p_q),\ C_{\mathrm{J}}(\mathcal{M}^p_q) \le 2 \ \text{ and } \ C_{\mathrm{DW}}(\mathcal{M}^p_q) \le 4.$$

Our result for Morrey spaces is the following:

Theorem 1.1. *If $1 \le p < q < \infty$, then $C_{\mathrm{NJ}}(\mathcal{M}^p_q) = C_{\mathrm{J}}(\mathcal{M}^p_q) = 2$ and $C_{\mathrm{DW}}(\mathcal{M}^p_q) = 4$.*

Note that $\mathcal{M}^p_p = L^p$ holds and that their norms are identical. The above theorem tells us that the case where $q > p$ is quite different from the case where $q = p$. When $q > p$, the three constants $C_{\mathrm{J}}(\mathcal{M}^p_q)$, $C_{\mathrm{NJ}}(\mathcal{M}^p_q)$, and $C_{\mathrm{DW}}(\mathcal{M}^p_q)$ take the same value as those for L^1 and L^∞ spaces.

Moving on to discrete Morrey spaces, let $\omega := \mathbb{N} \cup \{0\}$. For $m := (m_1, \ldots, m_d) \in \mathbb{Z}^d$ and $N \in \omega$, let

$$S_{m,N} := \{k \in \mathbb{Z}^d \colon \|k - m\|_\infty \le N\},$$

where $\|(m_1, \ldots, m_d)\|_\infty := \max\{|m_i| \colon 1 \le i \le d\}$ for $(m_1, \ldots, m_d) \in \mathbb{Z}^d$. The cardinality of $S_{m,N}$, denoted by $|S_{m,N}|$, is $(2N+1)^d$, for every $m \in \mathbb{Z}^d$ and $N \in \omega$. Given $1 \le p \le q < \infty$, we define the *discrete Morrey space* $\ell^p_q = \ell^p_q(\mathbb{Z}^d)$ to be the space of all functions (sequences) $x \colon \mathbb{Z}^d \to \mathbb{R}$ for which

$$\|x\|_{\ell^p_q} := \sup_{m \in \mathbb{Z}^d, N \in \omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in S_{m,N}} |x(k)|^p \right)^{\frac{1}{p}} < \infty.$$

We note that ℓ^p_q, equipped with the above norm, is a Banach space (see [5]). Our result for discrete Morrey spaces is the following:

Theorem 1.2. *If $1 \le p < q < \infty$, then $C_{\mathrm{NJ}}(\ell^p_q) = C_{\mathrm{J}}(\ell^p_q) = 2$ and $C_{\mathrm{DW}}(\ell^p_q) = 4$.*

This theorem also tells us that the case where $q > p$ is quite different from the case where $q = p$ (where $\ell^p_p = \ell^p$).

2. Proof of Theorems

We prove both theorems by finding two elements in the space such that the associated expressions are equal to two, two, and four, respectively.

2.1. Proof of Theorem 1.1

Proof. Let $1 \le p < q < \infty$, and let $f(x) := |x|^{-d/q}$, $x \in \mathbb{R}^d$, where $|x|$ denotes the Euclidean norm of x. Then $f \in \mathcal{M}^p_q(\mathbb{R}^d)$ (see [10, §2]). Define $g(x) := \chi_{(0,1)}(|x|)f(x)$, $h(x) := f(x) - g(x)$, and $k(x) := -f(x) + 2g(x)$, for $x \in \mathbb{R}^d$. By a change of variables, we see that

$$\|t^{d/q}g(t\cdot)\|_{\mathcal{M}^p_q} = \|g\|_{\mathcal{M}^p_q}$$

and

$$\|t^{d/q}h(t\cdot)\|_{\mathcal{M}^p_q} = \|h\|_{\mathcal{M}^p_q}$$

for all $t > 0$. Since

$$t^{d/q}g(tx) = \chi_{(0,1)}(t|x|)f(x)$$

and

$$t^{d/q}h(tx) = \chi_{(0,1)}(t|x|)f(x) - \chi_{[1,\infty)}(t|x|)f(x)$$

for $t > 0$ and $x \in \mathbb{R}^d$, by the monotone convergence property of Morrey spaces we have

$$\|f\|_{\mathcal{M}^p_q} = \|g\|_{\mathcal{M}^p_q} = \|h\|_{\mathcal{M}^p_q} = \|k\|_{\mathcal{M}^p_q} \in (0,\infty).$$

This implies that

$$\|f+k\|^2_{\mathcal{M}^p_q} + \|f-k\|^2_{\mathcal{M}^p_q} = 4(\|f\|^2_{\mathcal{M}^p_q} + \|k\|^2_{\mathcal{M}^p_q})$$

and

$$\min\{\|f+k\|_{\mathcal{M}^p_q}, \|f-k\|_{\mathcal{M}^p_q}\} = \min\{\|2g\|_{\mathcal{M}^p_q}, \|2h\|_{\mathcal{M}^p_q}\} = 2\|f\|_{\mathcal{M}^p_q} = 2\|k\|_{\mathcal{M}^p_q}.$$

By definition and the fact that both $C_{\mathrm{NJ}}(\mathcal{M}^p_q)$, $C_{\mathrm{J}}(\mathcal{M}^p_q) \le 2$, we conclude that

$$C_{\mathrm{NJ}}(\mathcal{M}^p_q) = C_{\mathrm{J}}(\mathcal{M}^p_q) = 2,$$

as desired.

Finally, we calculate the Dunkl–Williams constant using the same ideas as in [7]. We consider f and $(1+r)g + (1-r)h$ for $r \in (0,1)$. We calculate

$$\begin{aligned}
&\frac{\|f\|_{\mathcal{M}^p_q} + \|(1+r)g+(1-r)h\|_{\mathcal{M}^p_q}}{\|f-(1+r)g-(1-r)h\|_{\mathcal{M}^p_q}} \left\| \frac{f}{\|f\|_{\mathcal{M}^p_q}} - \frac{(1+r)g+(1-r)h}{\|(1+r)g+(1-r)h\|_{\mathcal{M}^p_q}} \right\|_{\mathcal{M}^p_q} \\
&= \frac{\|f\|_{\mathcal{M}^p_q} + (1+r)\|f\|_{\mathcal{M}^p_q}}{r\|f\|_{\mathcal{M}^p_q}} \left\| \frac{f}{\|f\|_{\mathcal{M}^p_q}} - \frac{(1+r)g+(1-r)h}{(1+r)\|f\|_{\mathcal{M}^p_q}} \right\|_{\mathcal{M}^p_q} \\
&= \frac{\|f\|_{\mathcal{M}^p_q} + (1+r)\|f\|_{\mathcal{M}^p_q}}{r\|f\|_{\mathcal{M}^p_q}} \left\| \frac{2rh}{(1+r)\|f\|_{\mathcal{M}^p_q}} \right\|_{\mathcal{M}^p_q} \\
&= \frac{4+2r}{1+r}.
\end{aligned}$$

If we let $r \downarrow 0$, we obtain $C_{\mathrm{DW}}(\mathcal{M}^p_q) = 4$, as required. □

Before we conclude this subsection, a remark may be in order. Let $1 \leq p \leq q < \infty$. The *local Morrey space* $\mathrm{L}\mathcal{M}^p_q = \mathrm{L}\mathcal{M}^p_q(\mathbb{R}^d)$ is the set of all the measurable functions f on $\mathbb{R}^d$ for which

$$\|f\|_{\mathrm{L}\mathcal{M}^p_q} := \sup_{B=B(0,r)} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty.$$

Arguing similarly as before, we see that $C_{\mathrm{NJ}}(\mathrm{L}\mathcal{M}^p_q) = C_{\mathrm{J}}(\mathrm{L}\mathcal{M}^p_q) = 2$ and $C_{\mathrm{DW}}(\mathrm{L}\mathcal{M}^p_q) = 4$ whenever $1 \leq p < q < \infty$.

2.2. Proof of Theorem 1.2

Proof. Let $1 \leq p < q < \infty$, and let us first consider the case where $d = 1$. Let $n \in \mathbb{Z}$ be an even number with $n > 2^{\frac{q}{q-p}} - 1$, or equivalently

$$(n+1)^{\frac{1}{q}-\frac{1}{p}} < 2^{-\frac{1}{p}}.$$

Consider the sequence $(x_k)_{k \in \mathbb{Z}}$ defined by

$$x_0 = x_n = 1, \text{and } x_k = 0 \text{ for all } k \notin \{0, n\}$$

and the sequence $(y_k)_{k \in \mathbb{Z}}$ defined by

$$y_0 = 1,\ y_n = -1, \text{and } y_k = 0 \text{ for all } k \notin \{0, n\}.$$

Then, we have

$$\begin{aligned}
\|x\|_{\ell^p_q} &= \sup_{m \in \mathbb{Z}, N \in \omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} \\
&= \max \left\{ 1, |S_{\frac{n}{2},\frac{n}{2}}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in S_{\frac{n}{2},\frac{n}{2}}} |x_k|^p \right)^{1/p} \right\} \\
&= \max \left\{ 1, (n+1)^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{1}{p}} \right\}.
\end{aligned}$$

With the choice of n above, we see that

$$(n+1)^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{1}{p}} < 1.$$

Therefore $\|x\|_{\ell^p_q} = 1$. Similarly, one may verify that $\|y\|_{\ell^p_q} = 1$. Moreover, we may observe that

$$\|x+y\|_{\ell^p_q} = 2 \quad \text{and} \quad \|x-y\|_{\ell^p_q} = 2.$$

Hence, we obtain

$$\frac{\|x+y\|^2_{\ell^p_q} + \|x-y\|^2_{\ell^p_q}}{2(\|x\|^2_{\ell^p_q} + \|y\|^2_{\ell^p_q})} = \frac{2^2+2^2}{2(1^2+1^2)} = 2.$$

Hence we conclude that $C_{\mathrm{NJ}}(\ell_q^p) = 2$. With the same choices of x and y, we have

$$C_{\mathrm{J}}(\ell_q^p) = \sup\{\min\{\|x+y\|_{\ell_q^p}, \|x-y\|_{\ell_q^p}\}\colon x, y \in X, \|x\|_{\ell_q^p} = \|y\|_{\ell_q^p} = 1\} = 2.$$

We shall now consider the general case where $d \geq 1$. Let $n \in \mathbb{Z}$ be an even number with $n > 2^{\frac{q}{d(q-p)}} - 1$, or equivalently

$$(n+1)^{d(\frac{1}{q}-\frac{1}{p})} < 2^{-\frac{1}{p}}.$$

Let $x \in \ell_q^p$ be the function $x\colon \mathbb{Z}^d \to \mathbb{R}$ where

$$x(k) := \begin{cases} 1, & \text{if } k = (0,0,\ldots,0), (n,0,\ldots,0), \\ 0, & \text{otherwise,} \end{cases}$$

and $y \in \ell_q^p$ be the function $y\colon \mathbb{Z}^d \to \mathbb{R}$ where

$$y(k) := \begin{cases} 1, & \text{if } k = (0,0,\ldots,0), \\ -1, & \text{if } k = (n,0,\ldots,0), \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned} \|x\|_{\ell_q^p} &= \sup_{m \in \mathbb{Z}^d, N \in \omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} \\ &= \max\left\{ 1, |S_{\frac{n}{2},\frac{n}{2}}|^{d(\frac{1}{q}-\frac{1}{p})} \left(\sum_{k \in S_{\frac{n}{2},\frac{n}{2}}} |x_k|^p \right)^{1/p} \right\} \\ &= \max\left\{ 1, (n+1)^{d(\frac{1}{q}-\frac{1}{p})} 2^{\frac{1}{p}} \right\}. \end{aligned}$$

Note that with the choice of n above, we have

$$(n+1)^{d(\frac{1}{q}-\frac{1}{p})} 2^{\frac{1}{p}} < 1,$$

whence $\|x\|_{\ell_q^p} = 1$. Similarly $\|y\|_{\ell_q^p} = 1$. Moreover, we also have

$$\|x+y\|_{\ell_q^p} = 2 \quad \text{and} \quad \|x-y\|_{\ell_q^p} = 2.$$

Therefore, we obtain

$$\frac{\|x+y\|_{\ell_q^p}^2 + \|x-y\|_{\ell_q^p}^2}{2(\|x\|_{\ell_q^p}^2 + \|y\|_{\ell_q^p}^2)} = \frac{2^2+2^2}{2(1^2+1^2)} = 2,$$

whence $C_{\mathrm{NJ}}(\ell_q^p) = 2$. The same choices of x and y give

$$C_{\mathrm{J}}(\ell_q^p) = \sup\{\min\{\|x+y\|_{\ell_q^p}, \|x-y\|_{\ell_q^p}\}\colon x, y \in X, \|x\|_{\ell_q^p} = \|y\|_{\ell_q^p} = 1\} = 2.$$

Finally, for the Dunkl–Williams constant, we use the couple $x+y$ and $(1+r)x + (1-r)y$ for $0 < r < 1$ and argue similarly to the case of Morrey spaces. $\square$

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