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SEIBERG-WITTEN-LIKE EQUATIONS ON THE STRICTLY PSEUDOCONVEX *CR*-3 MANIFOLDS

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Abstract. In this paper, Seiberg-Witten-like equations are written down on 3-manifolds. Then, it has been proved that the L^2-solutions of these equations are trivial on $\mathbb{R}^3$. Finally, a global solution is obtained on the strictly pseudoconvex CR-3 manifolds for a given constant negative scalar curvature.

1. Introduction

The Seiberg-Witten equations, which are consisted of the curvature and Dirac equation, carries subtle information that can be used to investigate the topology and geometry of the manifolds [10, 22, 27]. Although these equations are defined in 4-manifolds, they are also handled by many authors on higher dimensional manifolds [4–6, 19, 22]. This paper is mainly interested with the Seiberg-Witten-like equations on 3-manifold. Seiberg-Witten-like equations on 3-manifold are studied to obtain the equations of motion of $U(1)$ Chern-Simons theory coupled to a massless spinor field and investigated their moduli space of the gauge equivalence classes of their solutions [3]. Also, these equations are studied on a compact 3-manifold with boundary to show that solution space of these equations is a Banach manifold [17]. As mentioned above, Seiberg-Witten-like equations have been investigated by many authors in three dimensions [10, 15, 18, 24, 26]. In this paper, these equations are investigated with a different perspective. Just as in the 4-manifolds, there is a need for a Spin^c-structure in order to be able to construct spinor bundle on 3-manifolds. With all these, the Dirac equation can be defined on the spinor bundle. However, the definition of the curvature equation differs from the curvature equation which are defined on 4-manifolds. On 4-manifolds the curvature equation is defined as being dependent on the self-duality concept and also in higher dimensions the curvature equation is defined with the help of the generalised self-duality

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concept [4–6]. Since in 3-manifolds self-duality concept is meaningless, the curvature equation is defined independently from the self-duality concept.

This paper is organized as follows. At first, some basic facts concerning contact metric manifold, Spin^c-structure and Dirac operator are introduced. Then, in Section 3, Seiberg-Witten-like equations on 3-manifolds are defined and some useful identities are obtained to determine Seiberg-Witten-like functional. Therefore, a bound to the solutions of these equations with the approach given in [16] is obtained. Furthermore, it is proved that L^2-solutions of these equations are trivial on $\mathbb{R}^3$. Finally, a global solution to these equations on the strictly pseudoconvex CR-3 manifolds is obtained for a given constant negative scalar curvature.

2. Some basic materials

2.1. Contact metric manifolds

Let M be a smooth 3-manifold. A smooth 1-form α on M is called a contact form if $\alpha \wedge (d\alpha) \neq 0$ everywhere on M. A hyperplane subbundle H of the tangent bundle TM, which is given by $H = ker\alpha$, is induced by contact form α. The Reeb vector field ξ is the unique vector field satisfying $\alpha(\xi) = 1$ and $d\alpha(\xi, \dot{}) = 0$. Then (M,α) is called a contact manifold. The tangent bundle TM splits into $TM = H \oplus \mathbb{R}\xi$. Let X be any vector field on M. Then, the decomposition of X can be written as

$$X = X_H + f\xi,$$

where $f \in C^\infty(M, \mathbb{R})$ and X_H is the horizontal part of X.

If (M, α) is a contact manifold, the pair $(H, d\alpha\big|_H)$ is a symplectic vector bundle and an almost complex structure J_H can be fixed on H, which is compatible with $d\alpha\big|_H$. Since $J_H^2 = -I_d$, the following eigenspaces decomposition can be given by:

$$\Lambda^1_H(M) = H \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}_H(M) \oplus \Lambda^{0,1}_H(M),$$

where

$$\Lambda^{1,0}_H(M) = \{Z \in H \otimes_{\mathbb{R}} \mathbb{C} \,|\, J_H Z = iZ\},$$
$$\Lambda^{0,1}_H(M) = \{Z \in H \otimes_{\mathbb{R}} \mathbb{C} \,|\, J_H Z = -iZ\}.$$

The complexification of $\Lambda^s_H(M)$ is decomposed as follows

$$\Lambda^s_H(M) = \sum_{q+r=s} \Lambda^{q,r}_H(M),$$

where $\Lambda^{q,r}(M)_H = span\{u \wedge v \,|\, u \in \Lambda^q\big(\Lambda^{1,0}_H(M)\big), v \in \Lambda^r\big(\Lambda^{0,1}_H(M)\big)\}$. Also, J_H can be extended to an endomorphism J of the tangent bundle TM by setting $J\xi = 0$. At this point $J^2 = -Id + \alpha \otimes \xi$ is satisfied. With this in mind, g_α defines a Riemannian metric on TM given by

$$g_\alpha(X, Y) = d\alpha(X, JY) + \alpha(X)\alpha(Y).$$

The metric g_α is called a Webster metric and is said to be associated to α. Moreover, g_α satisfies the following relations:

$$g_\alpha(X,Y) = \alpha(X),\ g_\alpha(JX,Y) = d\alpha(X,Y),$$
$$g_\alpha(JX,JY) = g_\alpha(X,Y) - \alpha(X)\alpha(Y),$$

for any $X, Y \in \chi(M)$. We call $(M, g_\alpha, \alpha, \xi, J)$ as a contact metric manifold. For more information see [1,2,21].

On the contact metric manifold $(M, g_\alpha, \alpha, \xi, J)$, the generalized Tanaka-Webster connection ∇^{TW} is given by:

$$\nabla_X^{TW} Y = \nabla_X Y - \big(\nabla_X \alpha\big)(Y)\xi - \alpha(X)\nabla_Y \xi - \alpha(X)\alpha(Y),$$

where ∇ is the Levi-Civita connection and $X, Y \in \chi(M)$ [25]. Also, the generalized Tanaka-Webster connection ∇^{TW} satisfies $\nabla^{TW}\alpha = 0$ and $\nabla^{TW} g_\alpha = 0$. Moreover, if J is integrable, i.e., $\nabla^{TW} J = 0$, then the contact metric manifold $(M, g_\alpha, \alpha, \xi, J)$ is called a strictly pseudoconvex CR manifold [20,21].

2.2. Spin^c-structure and Dirac operator

A complex vector bundle $\mathbb{S}$ can be constructed by a given Spin^c representation $\kappa_3 : \mathrm{Spin}^c(3) \to Aut(\Delta_3)$ and denoted by $\mathbb{S} = P_{\mathrm{Spin}^c(3)} \times_{\kappa_3} \Delta_3$. Also this complex vector bundle is called a spinor bundle for a given Spin^c-structure on M [8]. $\kappa : \mathbb{R}^3 \to End(\mathbb{S})$ is a linear map satisfying the following conditions:

$$\kappa(v)^* + \kappa(v) = 0, \quad \kappa(v)^*\kappa(v) = |v|^2\mathbb{I}$$

for every $v \in \mathbb{R}^3$. Then,

$$\rho : \Lambda^2(T^*M) \to End(\mathbb{S})$$

can be defined on the frames by extending map $\kappa : TM \to End(\mathbb{S})$ of κ. Let $\{e_1, e_2, e_3\}$ be an orthonormal frame on an open subset $U \subset M$. Then

$$\alpha = \sum_{i<j} \alpha_{ij} e^i \wedge e^j \to \rho(\alpha) = \sum_{i<j} \alpha_{ij}\kappa(e_i)\kappa(e_j).$$

ρ can be extended to complex valued 2-forms such that

$$\rho : \Lambda^2(T^*M) \otimes \mathbb{C} \to End(\mathbb{S}).$$

A connection ∇^A on $\mathbb{S}$, which is called a spinor covariant derivative operator, is obtained by using an $i\mathbb{R}$-valued connection 1-form $A \in \Omega(M, i\mathbb{R})$ and the Levi-Civita connection ∇ on M. At this point the definition of the Dirac operator $D_A : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ can be given by

$$D_A(\Psi) = \sum_{i=1}^{3} \kappa(e_i)\nabla_{e_i}^A \Psi,$$

where $\{e_1, e_2, e_3\}$ is any positively oriented local orthonormal frame of TM [14]. A Spin^c-structure is needed to describe the Dirac operator on contact metric manifold. It is known that any contact metric manifold admits a canonical Spin^c-structure. By using this canonical Spin^c-structure, an associated canonical spinor bundle can be constructed and described by the following isomorphism:

$$\mathbb{S} \cong \Omega^{0,*}(M),$$

where $\Omega^{0,*}(M)$ is the direct sum of $\Omega(M)^{0,0} \oplus \Omega(M)^{0,1}$. Furthermore, on this spinor bundle, the Clifford multiplication "$\cdot$" is given by:

$$V \cdot \Psi = \sqrt{2}\Big((V_H^{0,1})^* \wedge \Psi - \iota(V_H^{0,1})\Psi \Big) + i(-1)^{\deg \Psi + 1}\eta(V)\psi,$$

where V_H denotes the horizontal part of V. According to these multiplication one can easily obtain $\xi\psi = i(-1)^{\deg \psi + 1}\psi$.

The spinor bundle $\mathbb{S}$ carries a natural Hermitian metric, denoted by $(\ ,\ )$ and for any vector field X and spinor field Ψ, Φ satisfies [11]

$$(X \cdot \Psi, \Phi) = -(\Psi, X \cdot \Phi). \tag{1}$$

Also, the norm $\|\cdot\|$ in the Hilbert space L^2 is defined as [8, 22],

$$\|\Psi\|^2 = \sqrt{\int_M |\Psi|^2 dvol.} \tag{2}$$

On the $2n+1$-dimensional contact metric manifold $(M, g_\alpha, \alpha, \xi, J)$ equipped with a Spin^c-structure, each unitary connection A on L induces a spinorial connection ∇^A on $\mathbb{S}$ with the generalized Tanaka-Webster connection ∇^{TW}. Then according to ∇^A the Kohn-Dirac operator D_H^A is defined as follows [21]:

$$D_H^A = \sum_{i=1}^{2n} \kappa(e_i)(\nabla_{e_i}^A), \tag{3}$$

where $\{e_i\}$ is a local orthonormal frame of H. The Dirac operator D_A is defined by [21]

$$D_A = D_H^A + \xi \cdot \nabla_\xi^A. \tag{4}$$

Also, by considering strictly pseudoconvex CR manifolds with $\Omega_H^{0,*}(M)$ associated spinor bundle the Dirac type operator is defined as follows:

Let

$$\overline{\partial}_H : \Omega_H^{0,r}(M) \longrightarrow \Omega_H^{0,r+1}(M),\ \overline{\partial}_H^* : \Omega_H^{0,r}(M) \longrightarrow \Omega_H^{0,r-1} \tag{5}$$

respectively given by:

$$\overline{\partial}_H = \sum_{i=1}^{n} \overline{Z}_i^* \wedge \nabla_{\overline{Z_i}}^{TW},\ \overline{\partial}_H^* = -\sum_{i=1}^{n} \iota(\overline{Z_i})^* \wedge \nabla_{\overline{Z_i}}^{TW},$$

where ∇^{TW} is the extension of the generalized Webster-Tanaka connection to $\Omega_H^{0,*}(M)$ and ι is the contraction operator.

It follows from (3) that we have on $\Omega_H^{0,*}(M)$

$$\mathcal{H} = \sqrt{2}\sum_{r=0}^{n} \left(\overline{\partial}_H + \overline{\partial}_H^*\right) + \sum_{r=0}^{n} (-1)^{r+1}\sqrt{-1} \cdot \nabla_\xi^{TW}. \tag{6}$$

Since $\mathbb{S} \cong \Omega_H^{0,*}(M)$, (4) coincides with (6).

3. Seiberg-Witten-like equations on 3-manifolds

In this section, we write down Seiberg-Witten-like equation on 3-manifold M. Then, we get explicit forms of these equations on $\mathbb{R}^3$.

Definition. Let M be a 3-manifold endowed with a $\mathrm{Spin}^c(3)$-structure and A be the fixed connection on $U(1)$-principal bundle. Then, for any $\Psi \in \Gamma(\mathbb{S})$ Seiberg-Witten-like equations are defined by

$$D_A(\Psi) = 0,$$
$$F_A = \frac{1}{4}\sigma(\Psi),$$

where $F_A = dA$ is the imaginary-valued curvature 2-form of the connection A in the P_{S^1}-bundle associated with the Spin^c-structure.

Moreover, the well known formula called the Schrödinger-Lichnerowicz formula is given by [8, 22]

$$D_A^* D_A \Psi = (\nabla^A)^* \nabla^A \Psi + \frac{s}{4}\Psi + \frac{1}{2} d_A \cdot \Psi, \tag{7}$$

where s is the scalar curvature of M, $(\nabla^A)^*$ is the adjoint of the covariant derivative operator ∇^A and D_A^* is the adjoint of D_A.

3.1. Seiberg-Witten-like equations on $\mathbb{R}^3$

Let $\kappa : \mathbb{R}^3 \to End(\mathbb{C}^2)$ be the $\mathrm{Spin}^c(3)$-structure which is defined on generators $\{e_1, e_2, e_3\}$ by the followings:

$$\kappa(e_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \kappa(e_2) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$
$$\kappa(e_3) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \ \kappa(d\alpha) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

where $d\alpha = e^1 \wedge e^2$. Note that there is no decomposition of spinor space over $\mathbb{R}^3$ contrary to the case $\mathbb{R}^4$ [8]. The Spin^c-connection ∇^A on $\mathbb{R}^3$ is given by

$$\nabla_j^A \Psi = \frac{\partial \Psi}{\partial x_j} + \frac{1}{2} A_j \Psi,$$

where $A_j : \mathbb{R}^3 \to i\mathbb{R}$ for $j = 1, 2, 3$ and $\Psi : \mathbb{R}^3 \to \mathbb{C}^2$ are smooth maps. Then the associated connection on the line bundle P_{S^1} is the connection 1-form and represented by

$$A = \sum_{i=1}^{3} A_i dx^i \in \Omega(\mathbb{R}^3, i\mathbb{R})$$

and its curvature 2-form is given by

$$F_A = \sum_{i<j}^{3} F_{ij} dx^i \wedge dx^j \in \Omega^2(\mathbb{R}^3, i\mathbb{R})$$

where $F_{ij} = \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)$ for $i, j = 1, 2, 3$. Then the Dirac operator $D_A : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ on $\mathbb{R}^3$ can be written with respect to given Spin^c- connection ∇^A as follows:

$$D_A \Psi = \sum_{i=1}^{3} \kappa(e_i) \nabla^A_{e_i} \Psi.$$

Therefore, the Dirac equation in the flat case is given by

$$\begin{aligned} D_A(\Psi) &= \kappa(e_1)\nabla^A_{e_1}\Psi + \kappa(e_2)\nabla^A_{e_2}\Psi + \kappa(e_3)\nabla^A_{e_3}\Psi \\ &= \sum_{i=1}^{3} \kappa(e_i)\Big(\nabla^A_{e_i}\Psi\Big) \\ &= \sum_{i=1}^{3} \kappa(e_i) \begin{bmatrix} \frac{\partial \psi_1}{\partial x_i} + \frac{1}{2}A_i\psi_1 \\ \frac{\partial \psi_2}{\partial x_i} + \frac{1}{2}A_i\psi_2 \end{bmatrix} \\ &= \begin{bmatrix} i\Big(\frac{\partial \psi_2}{\partial x_2} + \frac{\partial \psi_1}{\partial x_3}\Big) + \frac{\partial \psi_2}{\partial x_1} + \frac{1}{2}\Big(A_1\psi_2 + i\big(A_3\psi_1 + A_2\psi_2\big)\Big) \\ i\Big(-\frac{\partial \psi_2}{\partial x_3} + \frac{\partial \psi_1}{\partial x_2}\Big) - \frac{\partial \psi_1}{\partial x_1} + \frac{1}{2}\Big(-A_1\psi_1 + i\big(A_2\psi_1 - A_3\psi_2\big)\Big) \end{bmatrix}. \end{aligned}$$

Let us consider the complexified space $\Lambda^2(\mathbb{R}^3) \otimes \mathbb{C}$ and F_A be the curvature form of the imaginary valued connection 1-form A. Then,

$$F_A = \sum_{i<j}^{3} F_{ij} dx^i \wedge dx^j.$$

The curvature equation is defined by

$$F_A = \frac{1}{4}\sigma(\Psi),$$

where $\sigma(\Psi)$ is an imaginary valued 2-form defined by the formula

$$\sigma(\Psi)(X, Y) = \big(X \cdot Y \cdot \Psi, \Psi\big) + \big\langle X, Y\big\rangle |\Psi|^2$$

for any $\Psi \in \Gamma(\mathbb{S})$. The map $\sigma : \Gamma(\mathbb{S}) \to \Omega^1(M, i\mathbb{R})$ is called a quadratic map.

The explicit form of the second equation can be expressed as follows:

$$\begin{aligned} F_{12} &= -\frac{i}{4}\Big(|\psi_1|^2 - |\psi_2|^2\Big), \\ F_{13} &= \frac{i}{4}\Big(\overline{\psi_2}\psi_1 + \overline{\psi_1}\psi_2\Big), \\ F_{23} &= \frac{1}{4}\Big(\overline{\psi_2}\psi_1 - \overline{\psi_1}\psi_2\Big). \end{aligned}$$

4. Seiberg-Witten-like functional

The Energy functional consistent with the 3-dimensional Seiberg-Witten-like equations is defined by

$$E(A,\Psi)=\int_M \Big(|D_A\Psi|^2+|F_A-\frac{1}{4}\sigma(\Psi)|^2\Big)dvol.$$

Note that solutions of 3-dimensional Seiberg-Witten-like equations are zeros of Energy functional. In this section, we obtain some useful identities related with spinors and their image under the quadratic map σ. With the aid of the following lemma we obtain another form of Seiberg-Witten-like functional and we get a bound for the solutions of Seiberg-Witten-like equations.

Lemma 4.1. *Let $\kappa : TM \to End(\mathbb{S})$ be a $Spin^c$-structure on a compact oriented smooth 3-dimensional Riemannian manifold M. Then, the following equalities hold*

$$\big(\sigma(\Psi)\Psi,\ \Psi\big)=\big(\sigma(\Psi),\ \sigma(\Psi)\big)=|\Psi|^4, \tag{8}$$

where $\Psi\in\Gamma(\mathbb{S})$ and $\sigma(\Psi)\in\Omega^2(M,i\mathbb{R})$.

Lemma 4.1 can be proved with an easy computation.

In the following, we give a bound to the negative constant scalar curvature of (M,g) by using the usual Laplacian on defined as follows [8, 22],

$$\Delta|\Psi|^2=2\big((\nabla^A)^*\nabla^A\Psi,\Psi\big)-2\big(\nabla^A\Psi,\nabla^A\Psi\big), \tag{9}$$

where $\Psi\in\Gamma(\mathbb{S})$ and $(\nabla^A)^*$ is the adjoint of the covariant derivative operator ∇^A.

Lemma 4.2. *Let (A,Ψ) be a solution of $D_A\Psi=0,\ F_A=\frac{1}{4}\sigma(\Psi)$ over a compact smooth 3-dimensional Riemannian manifold (M,g) with a negative constant scalar curvature s. Then, at each point*

$$\frac{|\Psi(x)|^2}{2}\le -s_{\min},\ \ \textit{where } s_{\min}=\min\{s(m): m\in M\}.$$

Proof. At a point x where $|\Psi(x)|^2$ attains its maximum we have $0\le\Delta|\Psi|^2$. Then

$$\begin{aligned}0\le\Delta|\Psi|^2&=2\big((\nabla^A)^*\nabla^A\Psi,\Psi\big)-2\big(\nabla^A\Psi,\nabla^A\Psi\big)\\&\le 2\big((\nabla^A)^*\nabla^A\Psi,\Psi\big)\\&=2\big(D_A^*D_A\Psi-\frac{s}{4}\Psi-\frac{1}{2}dA\cdot\Psi,\Psi\big)\\&=\big(-\frac{s}{2}\Psi-dA\cdot\Psi,\ \Psi\big),\\&=-\frac{s}{2}|\Psi|^2-\big(dA\cdot\Psi,\ \Psi\big)\\&=-\frac{s}{2}|\Psi|^2-\frac{1}{4}\big(\sigma(\Psi)\Psi,\ \Psi\big)\end{aligned}$$

$$= -\frac{s}{2}|\Psi|^2 - \frac{1}{4}|\Psi|^4.$$

Now, if $|\Psi(x)|^2 > 0$, then $0 \le -\frac{s}{2}|\Psi|^2 - \frac{1}{4}|\Psi|^4_{\max}$ and $\frac{1}{2}|\Psi(x_{\max})|^2 \le -s_{\min}$. □

Lemma 4.3. *Under the same conditions as in Lemma* 4.2*, the following inequality is satisfied*

$$|F_A| \le \tfrac{1}{2}|s|.$$

Proof.

$$\begin{aligned} |F_A|^2 = |\frac{1}{4}\sigma(\Psi)|^2 &= \big(\frac{1}{4}\sigma(\Psi),\ \frac{1}{4}\sigma(\Psi)\big) \\ &= \frac{1}{16}\big(\sigma(\Psi),\ \sigma(\Psi)\big) \\ &= \frac{1}{16}|\Psi|^4. \end{aligned}$$

As a result, $|F_A| = \frac{1}{4}|\Psi|^2 \le \frac{-s}{2} \le \frac{1}{2}|s|$. □

Since 3-dimensional Hyperbolic space is a Riemannian manifold with negative constant curvature and it satisfies Lemma 4.2 and Lemma 4.3 [9]. In addition, manifolds of negative constant curvature are given in [13].

Lemma 4.4. *On the compact oriented smooth* 3*-dimensional Riemannian manifold* (M, g)*, by considering Seiberg-Witten-like equation*:

$$D_A\Psi = 0,\ F_A = \frac{1}{4}\sigma(\Psi), \tag{10}$$

the Seiberg-Witten-like functional is obtained as follows

$$E(A,\Psi) = \int_M \Big(|F_A|^2 + |\nabla^A\Psi|^2 + \frac{s}{4}|\Psi|^2 + \frac{1}{16}|\Psi|^4\Big) dvol.$$

Proof. Using the Schrodinger-Lichnerowicz formula given in (7), we have

$$\int_M |D_A\Psi|^2 dvol = \int_M \Big[|\nabla^A\Psi|^2 + \frac{s}{4}|\Psi|^2 + \big(\frac{1}{2}dA\cdot\Psi, \Psi\big)\Big] dvol. \tag{11}$$

Since F_A and $\sigma(\Psi)$ are 2-forms with purely imaginary values, calculating their length amounts to

$$|F_A - \frac{1}{4}\sigma(\Psi)|^2 = |F_A|^2 - \frac{1}{2}\big(dA\cdot\Psi, \Psi\big) + \frac{1}{16}|\sigma(\Psi)|^2.$$

This implies

$$\begin{aligned} E(A,\Psi) &= \int_M \Big[|F_A - \frac{1}{4}\sigma(\Psi)|^2 + |D_A\Psi|^2\Big] dvol \\ &= \int_M \Big[|F_A|^2 + |\nabla^A\Psi|^2 + \frac{s}{4}|\Psi|^2 + \frac{1}{16}|\Psi|^4\Big] dvol. \qquad (12) \end{aligned}$$

□

In 3-dimensional case i.e., $M = \mathbb{R}^3$, the following lemma shows that L^2-solutions of these equations are trivial.

Lemma 4.5. *Let $A \in \Omega^1(\mathbb{R}^3,\ i\mathbb{R})$ and $\Psi \in C^\infty(\mathbb{R}^3, \mathbb{C}^2)$ the following equations are satisfied:*

$$\nabla_1^A \Psi + \nabla_2^A \Psi + \nabla_3^A \Psi = 0,$$

$$\begin{aligned} F_{12} &= -\frac{i}{4}(|\psi_1|^2 - |\psi_2|^2) = -\frac{1}{4}\Psi^* K \Psi, \\ F_{13} &= \frac{i}{4}(\overline{\psi_2}\psi_1 + \overline{\psi_1}\psi_2) = \frac{1}{4}\Psi^* J \Psi, \\ F_{23} &= \frac{1}{4}(\overline{\psi_2}\psi_1 - \overline{\psi_1}\psi_2) = -\frac{1}{4}\Psi^* I \Psi, \end{aligned} \tag{13}$$

where

$$I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad K = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Then

(1) *If $\Psi \in L^2$, then $\Psi \equiv 0$.*
(2) *If $E(A, \Psi) < \infty$, then $\Psi \equiv 0$ and $F_A \equiv 0$.*

Proof. Let

$$\Delta = -\sum_{i=1}^{3} \frac{\partial^2}{(\partial x^i)^2} \tag{14}$$

be the usual Laplacian on $\mathbb{R}^3$. At first we claim that

$$\Delta|\Psi|^2 = -2\sum_{i=1}^{3} \frac{\partial}{\partial x^i} Re\big(\Psi,\ \nabla_i^A \Psi\big). \tag{15}$$

To proof our claim we compute

$$\begin{aligned} \frac{\partial}{\partial x^i}|\Psi|^2 &= \partial_i(\overline{\psi_1}\psi_1 + \overline{\psi_2}\psi_2) \\ &= \overline{\psi_1}\partial_i\psi_1 + \psi_1\partial_i\overline{\psi_1} + \overline{\psi_2}\partial_i\psi_2 + \psi_2\partial_i\overline{\psi_2}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \big(\Psi,\ \nabla_i^A\Psi\big) &= \big(\Psi,\ \partial_i\Psi + \frac{1}{2}A_i\Psi\big) \\ &= \overline{\psi_1}(\partial_i\psi_1 + \frac{1}{2}A_i\psi_1) + \overline{\psi_2}(\partial_i\psi_2 + \frac{1}{2}A_i\psi_2) \\ &= \overline{\psi_1}\partial_i\psi_1 + \overline{\psi_2}\partial_i\psi_2 + \frac{1}{2}A_i(|\psi_1|^2 + |\psi_2|^2). \end{aligned} \tag{17}$$

By inserting (17) into the following equality

$$\begin{aligned} 2Re\big(\Psi,\ \nabla_i^A\Psi\big) &= \big(\Psi,\ \nabla_i^A\Psi\big) + \overline{\big(\Psi,\ \nabla_i^A\Psi\big)} \\ &= \overline{\psi_1}\partial_i\psi_1 + \psi_1\partial_i\overline{\psi_1} + \overline{\psi_2}\partial_i\psi_2 + \psi_2\partial_i\overline{\psi_2} \end{aligned} \tag{18}$$

is obtained. At the end, by comparing (16) with (18), one gets the following equality:

$$\frac{\partial}{\partial x^i}|\Psi|^2 = 2Re\big(\Psi,\ \nabla_i^A\Psi\big).$$

Also, this equality can be written as in the following

$$-\frac{\partial^2}{(\partial x^i)^2}|\Psi|^2 = -2\frac{\partial}{\partial x^i}Re\big(\Psi,\ \nabla_i^A\Psi\big),$$

which means that (14) equals to (15).

Moreover, one can show that

$$\frac{\partial}{\partial x^i}\Big(Re\big(\Psi,\ \nabla_i^A\Psi\big)\Big) = |\nabla_i^A\Psi|^2 + Re\big(\Psi,\ \nabla_i^A\nabla_i^A\Psi\big). \tag{19}$$

To obtain this, firstly, the right side of (19) is computed:

$$\begin{aligned}
2\frac{\partial}{\partial x^i}Re\big(\Psi,\ \nabla_i^A\Psi\big) &= \frac{\partial}{\partial x^i}(\overline{\psi_1}\partial_i\psi_1 + \psi_1\partial_i\overline{\psi_1} + \overline{\psi_2}\partial_i\psi_2 + \psi_2\partial_i\overline{\psi_2}) \\
&= \overline{\psi_1}\partial_i\partial_i\psi_1 + \partial_i\overline{\psi_1}\partial_i\psi_1 + \psi_1\partial_i\partial_i\overline{\psi_1} + \partial_i\psi_1\partial_i\overline{\psi_1} \\
&\quad + \overline{\psi_2}\partial_i\partial_i\psi_2 + \partial_i\overline{\psi_2}\partial_i\psi_2 + \psi_2\partial_i\partial_i\overline{\psi_2} + \partial_i\psi_2\partial_i\overline{\psi_2} \\
&= \overline{\psi_1}\partial_i\partial_i\psi_1 + \psi_1\partial_i\partial_i\overline{\psi_1} + \overline{\psi_2}\partial_i\partial_i\psi_2 + \psi_2\partial_i\partial_i\overline{\psi_2} \\
&\quad + \partial_i\psi_1\partial_i\overline{\psi_1} + \partial_i\psi_2\partial_i\overline{\psi_2}.
\end{aligned} \tag{20}$$

Explicit form of $|\nabla_i^A\Psi|^2$ is obtained as in the following:

$$\begin{aligned}
2|\nabla_i^A\Psi|^2 &= 2\big(\nabla_i^A\Psi,\ \nabla_i^A\Psi\big) \\
&= 2\big(\partial_i\Psi,\ \partial_i\Psi\big) + \big(\partial_i\Psi,\ A_i\Psi\big) + \big(A_i\Psi,\ \partial_i\Psi\big) + \frac{1}{2}\big(A_i\Psi,\ A_i\Psi\big) \\
&= 2\partial_i\overline{\psi_1}\partial_i\psi_1 + 2\partial_i\overline{\psi_2}\partial_i\psi_2 + 2Re\big(\partial_i\Psi,\ A_i\Psi\big) + \frac{1}{2}|A_i\Psi|^2.
\end{aligned} \tag{21}$$

Also,

$$\begin{aligned}
\nabla_i^A\nabla_i^A\Psi &= (\partial_i + \frac{1}{2}A_i)(\partial_i\Psi + \frac{1}{2}A_i\Psi) \\
&= \partial_i\partial_i\Psi + \frac{1}{2}\partial_i(A_i\Psi) + \frac{1}{2}A_i\partial_i\Psi + \frac{1}{4}A_i^2\Psi \\
&= \partial_i\partial_i\Psi + \frac{1}{2}A_i\partial_i\Psi + \frac{1}{2}\Psi\partial_iA_i + \frac{1}{2}A_i\partial_i\Psi + \frac{1}{4}A_i^2\Psi \\
&= \partial_i\partial_i\Psi + A_i\partial_i\Psi + \frac{1}{2}\Psi\partial_iA_i + \frac{1}{4}A_i^2\Psi \\
&= \partial_i\partial_i\Psi + A_i\partial_i\Psi + \frac{1}{2}\Psi\partial_iA_i - \frac{1}{4}|A_i|^2\Psi.
\end{aligned} \tag{22}$$

Hermitian inner product Ψ with (22) is calculated by

$$\big(\Psi,\ \nabla_i^A\nabla_i^A\Psi\big) = \big(\Psi,\ \partial_i\partial_i\Psi\big) + A_i\big(\Psi,\ \partial_i\Psi\big) + \frac{1}{2}\partial_iA_i\big(\Psi,\ \Psi\big) - \frac{1}{4}|A_i|^2\big(\Psi,\ \Psi\big)$$

$$= \overline{\psi_1}\partial_i\partial_i\Psi_1 + \overline{\psi_2}\partial_i\partial_i\psi_2 + A_i\big(\Psi,\ \partial_i\Psi\big) + \frac{1}{2}\partial_i A_i|\Psi|^2 - \frac{1}{4}|A_i\Psi|^2. \tag{23}$$

The real part of (23) is

$$\begin{aligned} 2Re\big(\Psi,\ \nabla_i^A\nabla_i^A\Psi\big) &= \big(\Psi,\nabla_i^A\nabla_i^A\Psi\big) + \overline{\big(\Psi,\ \nabla_i\nabla_i\Psi\big)} \\ &= \overline{\psi_1}\partial_i\partial_i\Psi_1 + \overline{\psi_2}\partial_i\partial_i\psi_2 + A_i\big(\Psi,\partial_i\Psi\big) + \frac{1}{2}\partial_i A_i|\Psi|^2 \\ &\quad - \frac{1}{4}|A_i\Psi|^2 + \psi_1\partial_i\partial_i\overline{\Psi_1} + \psi_2\partial_i\partial_i\overline{\psi_2} - A_i\overline{\big(\Psi,\partial_i\Psi\big)} \\ &\quad - \frac{1}{2}\partial_i A_i|\Psi|^2 - \frac{1}{4}|A_i\Psi|^2. \end{aligned} \tag{24}$$

Since

$$\begin{aligned} A_i\big(\Psi,\ \partial_i\Psi\big) &= \big(\overline{A_i}\Psi,\partial_i\Psi\big) = \big(-A_i\Psi,\partial_i\Psi\big) \\ &= -\big(A_i\Psi,\partial_i\Psi\big) \\ &= -\overline{\big(\partial_i\Psi,\ A_i\Psi\big)}, \end{aligned} \tag{25}$$

$$\begin{aligned} Re\big(A_i\big(\Psi,\partial_i\Psi\big)\big) &= -Re\overline{\big(\partial_i\Psi,A_i\Psi\big)} \\ &= -Re\big(\partial_i\Psi,A_i\Psi\big) \end{aligned} \tag{26}$$

are obtained. Inserting (26) in (24), one has

$$\begin{aligned} 2Re\big(\Psi,\ \nabla_i\nabla_i\Psi\big) = \overline{\psi_1}\partial_i\partial_i\psi_1 + \psi_1\partial_i\partial_i\overline{\psi_1} + \overline{\psi_2}\partial_i\partial_i\psi_2 + \psi_2\partial_i\partial_i\overline{\psi_2} \\ - 2Re\big(\partial_i\Psi,A_i\Psi\big) - \frac{1}{2}|A_i\Psi|^2. \end{aligned}$$

Since (20) is the sum of (21) and (24), (19) is proved.

Considering the scalar curvature $s=0$ and Dirac equation $D_A\Psi=0$ in (7), we get

$$(\nabla_i^A)^*\nabla_i^A\Psi + \frac{1}{2}dA\cdot\Psi = 0.$$

Since $(\nabla^A)^* = -\nabla^A$ [8,22], we obtain

$$\nabla_i^A\nabla_i^A\Psi = \frac{1}{2}F_A\cdot\Psi. \tag{27}$$

Inserting (27) in (19), we get the following equation:

$$\begin{aligned} \Delta|\Psi|^2 &= -2\sum_{i=1}^{3}|\nabla_i^A\Psi|^2 + Re\big(\Psi,\rho(F_A)\Psi\big) \\ &= -2\sum_{i=1}^{3}|\nabla_i^A\Psi|^2 - 2Re\big(\Psi,F_{12}K\Psi\big) - Re\big(\Psi,F_{13}J\Psi\big) - Re\big(\Psi,F_{23}I\Psi\big). \end{aligned} \tag{28}$$

By using (13) in the following Hermitian inner product

$$\begin{aligned}(1)\ \big(\Psi, F_{12}K\Psi\big) &= \Big(\Psi, \Big(-\frac{1}{4}\Psi^*K\Psi\Big)K\Psi\Big)\\ &= \Big(-\frac{1}{4}\Psi^*K\Psi\Big)\Big(\Psi, K\Psi\Big)\\ &= -\frac{i}{4}(|\psi_1|^2-|\psi_2|^2)i(|\psi_1|^2-|\psi_2|^2)\\ &= \frac{1}{4}|\Psi^*K\Psi|^2\end{aligned}$$

is obtained. Also the following holds

$$-Re\big(\Psi, F_{12}K\Psi\big) = -\frac{1}{4}|\Psi^*K\Psi|^2. \tag{29}$$

Similarly, by using (13), one can obtain

$$\begin{aligned}(2)\ \big(\Psi, F_{13}J\Psi\big) &= \Big(\Psi, \Big(\frac{1}{4}\Psi^*J\Psi\Big)J\Psi\Big)\\ &= \Big(\frac{1}{4}\Psi^*J\Psi\Big)\Big(\Psi, J\Psi\Big)\\ &= \frac{i}{4}(\overline{\psi_2}\psi_1+\overline{\psi_1}\psi_2)i(\overline{\psi_1}\psi_2+\overline{\psi_2}\psi_1)\\ &= -\frac{1}{4}|\Psi^*J\Psi|^2\end{aligned}$$

and then

$$-\ Re\big(\Psi, F_{13}J\Psi\big) = \frac{1}{4}|\Psi^*J\Psi|^2. \tag{30}$$

At the end, with the aid of (13) the following identity holds

$$\begin{aligned}(3)\ \big(\Psi, F_{23}I\Psi\big) &= \Big(\Psi, \Big(-\frac{1}{4}\Psi^*I\Psi\Big)I\Psi\Big)\\ &= -\Big(\frac{1}{4}\Psi^*I\Psi\Big)\Big(\Psi, I\Psi\Big)\\ &= \frac{1}{4}(\overline{\psi_2}\psi_1-\overline{\psi_1}\psi_2)(\overline{\psi_1}\psi_2-\overline{\psi_2}\psi_1)\\ &= \frac{1}{4}|\Psi^*I\Psi|^2.\end{aligned}$$

Then,

$$-\ Re\big(\Psi, F_{23}I\Psi\big) = -\frac{1}{4}|\Psi^*I\Psi|^2 \tag{31}$$

is obtained. By inserting (29), (30), (31) into (28), one has

$$\Delta|\Psi|^2 = -2\sum_{i=1}^{3}|\nabla_i^A\Psi|^2 - \frac{1}{4}|\Psi^*K\Psi|^2 + \frac{1}{4}|\Psi^*J\Psi|^2 - \frac{1}{4}|\Psi^*I\Psi|^2. \tag{32}$$

Accordingly, the last three terms are obtained as:

$$\begin{aligned}&|\Psi^*K\Psi|^2 - |\Psi^*J\Psi|^2 + |\Psi^*I\Psi|^2\\ &= (|\psi_1|^2-|\psi_2|^2)^2 + (\overline{\psi_2}\psi_1+\overline{\psi_1}\psi_2)^2 + (\overline{\psi_2}\psi_1-\overline{\psi_1}\psi_2)^2\end{aligned}$$

$$\begin{aligned} &= |\psi_1|^4 - 2|\psi_1|^2|\psi_2|^2 + |\psi_2|^4 - \overline{\psi_2}^2\psi_1^2 + 2|\psi_2|^2|\psi_1|^2 - \overline{\psi_1}^2\psi_2^2 \\ &\quad + \overline{\psi_1}^2\psi_2^2 + 2|\psi_1|^2|\psi_2|^2 + \overline{\psi_2}^2\psi_1^2 \\ &= (|\psi_1|^2 + |\psi_2|^2)^2 \\ &= |\Psi|^4. \end{aligned} \tag{33}$$

After inserting (33) in (32), we get $\Delta|\Psi|^2 \leq 0$ which means that the function

$$x \longrightarrow |\Psi(x)|^2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

is subharmonic on $\mathbb{R}^3$ [22]. As a result, $|\Psi(x)|^2$ satisfies the mean value inequality for subharmonic functions. According to the Mean Value Theorem for subharmonic function, the following inequality is satisfied for any $r > 0$ and any $x \in \mathbb{R}^3$,

$$|\Psi(x)|^2 \leq \frac{3}{4\pi r^3}\int_{B_r(x)} |\Psi(x)|^2 dvol,$$

where $B_r(x)$ is the closed ball of radius r about x [7, 12, 23]. If $\Psi \in L^2$, $\int_{\mathbb{R}^3} |\Psi(x)|^2 dvol < \infty$. Hence, L^2-norm of Ψ is finite. Denoting the value of this integral by κ, we obtain

$$|\Psi(x)|^2 \leq \frac{3\kappa}{4\pi r^3}.$$

Since the L^2-norm of Ψ is finite it follows, by taking the limit $r \to \infty$, that $\Psi(x) = 0$ for all $x \in \mathbb{R}^3$.

To prove second part, similar way is used. By inserting (33) into (32), one can provide

$$\Delta|\Psi|^2 = -2\sum_{i=1}^{3} |\nabla_i^A \Psi|^2 - \frac{1}{4}|\Psi|^4. \tag{34}$$

By means of standard identity from vector calculus, one has

$$\Delta\big(f \cdot g\big) = f \cdot \Delta g - 2\nabla f \cdot \nabla g + g \cdot \Delta f.$$

Taking $g = f$, one can provide

$$\Delta\big(f^2\big) = -2\nabla f \cdot \nabla f + 2f \cdot \Delta f,$$

so

$$\begin{aligned} \Delta|\Psi|^4 &= \Delta\Big(|\Psi|^2\Big)^2 \\ &= -2\Big(\nabla|\Psi|^2\Big).\Big(\nabla|\Psi|^2\Big) + 2|\Psi|^2\Delta|\Psi|^2 \end{aligned}$$

then

$$\Delta|\Psi|^4 = -2\Big(\nabla|\Psi|^2\Big).\Big(\nabla|\Psi|^2\Big) - 4|\Psi|^2\sum_{i=1}^{3} |\nabla_i^A \Psi|^2 - \frac{1}{2}|\Psi|^6.$$

Consequently, $\Delta|\Psi|^4 \leq 0$ on $\mathbb{R}^3$ so $x \longrightarrow |\Psi(x)|^4$ is subharmonic on $\mathbb{R}^3$. Thus, for every $r > 0$ and every $x \in \mathbb{R}^3$,

$$|\Psi(x)|^4 \leq \frac{3}{4\pi r^3} \int_{B_r(x)} |\Psi(x)|^4 dvol. \tag{35}$$

The assumptions $E(A, \Psi) < \infty$ can be written as in the following

$$E(A, \Psi) = \int_M \Big(|F_A|^2 + |\nabla^A \Psi|^2 + \frac{R}{4}|\Psi|^2 + \frac{1}{16}|\Psi|^4 \Big) dvol < \infty. \tag{36}$$

From (36), one has

$$\int_{\mathbb{R}^3} |\Psi|^4 dvol < \infty. \tag{37}$$

This means that $\Psi \equiv 0$ on $\mathbb{R}^3$. Consequently under the assumption $E(A, \Psi) < \infty$, $F_A \equiv 0$ since $\Psi \equiv 0$. $\square$

5. A non-trivial solution to Seiberg-Witten-like equations on 3-dimensional contact metric manifolds

In this section, Seiberg-Witten-like equations on the 3-dimensional strictly pseudoconvex CR-3 manifolds are written and a global solution to these equations is given.

On the 3-dimensional strictly pseudoconvex CR-3 manifolds, the spinor bundle can be decomposed as follows:

$$\mathbb{S} \cong \Lambda_H^{0,1}(M) \oplus \Lambda_H^{0,0}(M),$$

where $\Lambda_H^{0,1}(M)$ is the eigenspace corresponding to the eigenvalue i of the mapping $\kappa(d\alpha) : \mathbb{S} \to \mathbb{S}$ and has dimension 1, $\Lambda_H^{0,0}(M)$ is the eigenspace corresponding to the eigenvalue $-i$ of the mapping $\kappa(d\alpha) : \mathbb{S} \to \mathbb{S}$ and has dimension 1. If $\Psi_0 \in \mathbb{S}$, isomorphic to the constant function $1 \in \Lambda_H^{0,0}(M)$, then Ψ_0 denotes the spinor corresponding to the constant function 1 in the chosen coordinates

$$\Psi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

By using the expression of $\sigma_H(\Psi)$ in the local coordinates and $d\alpha \cdot \Psi_0 = -i\Psi_0$, the following identity is obtained:

$$\sigma_H(\Psi_0) = id\alpha.$$

On the subbundle H,the Ricci form ρ_H is defined by

$$\rho_H(X, Y) = Ric(X, J_H Y) = g_\alpha(X, J_H Ric Y) \tag{38}$$

for any $X, Y \in \Gamma(H)$. Since on the strictly pseudoconvex CR manifold, the almost complex structure J_H is complex,

$$Ric(X, Y) = i\rho_H(X, Y) \tag{39}$$

for any $X, Y \in \Gamma(H)$.

Proposition 5.1. *Suppose that ρ_H be a Ricci form on the subbundle H and s_H be a scalar curvature of H. Then, one can satisfy the following identity:*

$$\rho = -\frac{s_H}{2} d\alpha. \tag{40}$$

Proof. According to the local coordinates, the almost complex structure J is given as follows:

$$J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $J \circ Ric = Ric \circ J$ commutative, the reduced form of the Ric is

$$Ric = \begin{bmatrix} R_{11} & 0 & 0 \\ 0 & R_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By using (38) the explicit form of ρ_H is:

$$\rho_H = -R_{11} e^1 \wedge e^2 = -\frac{s_H}{2} d\alpha. \tag{41}$$

□

In the following theorem, a special solution of Seiberg-Witten-like equations is given on the 3-dimensional strictly pseudoconvex contact metric manifold.

Theorem 5.2. *Let $(M, g_\alpha, \alpha, \xi, J)$ be a strictly pseudoconvex CR-3 manifold. Then, for a given negative and constant scalar curvature s_H, $(A, \Psi = \sqrt{-2s_H}\Psi_0)$ is the solution of Seiberg-Witten-like equations.*

Proof. By using Ψ we get $\sigma_H(\Psi) = id\alpha$. Also it can be written as

$$\sigma_H(\Psi) = -2is_H d\alpha. \tag{42}$$

By using (39) and (42), one can satisfy,

$$F_A = Ric = i\rho_H = -i\frac{s_H}{2} d\alpha = \frac{1}{4}\sigma_H(\Psi). \tag{43}$$

Since $\sigma_H(\Psi) = \sigma(\Psi)$,

$$F_A = \frac{1}{4}\sigma(\Psi) \tag{44}$$

is satisfied.

The following is easily hold. By using the spinor field Ψ_0 corresponding to the constant function 1, one can obtain

$$\mathcal{H}(1) = \sqrt{2}\sum_{r=0}^{n} \left(\overline{\partial}_H + \overline{\partial}_H^*\right)(1) + \sum_{r=0}^{n} (-1)^{r+1}\sqrt{-1} \cdot \nabla_\xi^{TW}(1) = 0. \tag{45}$$

This means that $D_{A_0}\Psi = D_H^{A_0}\Psi + \xi \cdot \nabla_\xi^{A_0}\Psi = 0$.

As a result, $(A, \Psi = \sqrt{-2s_H}\Psi_0)$ is the solution of Seiberg-Witten-like equations on the strictly pseudoconvex CR-3 manifold. □

A 3-dimensional Hiperbolic space with a negative and constant scalar curvature can be given for the Theorem (5.2) (see [9]).

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