Bull. Korean Math. Soc. **56** (2019), No. 6, pp. 1539–1550
https://doi.org/10.4134/BKMS.b181266
pISSN: 1015-8634 / eISSN: 2234-3016

MINIMAL AND CONSTANT MEAN CURVATURE SURFACES IN $\mathbb{S}^3$ FOLIATED BY CIRCLES

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Abstract. We classify minimal surfaces in $\mathbb{S}^3$ which are foliated by circles and ruled constant mean curvature (cmc) surfaces in $\mathbb{S}^3$. First we show that minimal surfaces in $\mathbb{S}^3$ which are foliated by circles are either ruled (that is, foliated by geodesics) or rotationally symmetric (that is, invariant under an isometric $\mathbb{S}^1$-action which fixes a geodesic). Secondly, we show that, locally, there is only one ruled cmc surface in $\mathbb{S}^3$ up to isometry for each nonnegative mean curvature. We give a parametrization of the ruled cmc surface in $\mathbb{S}^3$ (cf. Theorem 3).

1. Introduction

In this paper, we classify minimal surfaces and surfaces of constant mean curvature (cmc) in $\mathbb{S}^3$, which are foliated by circular arcs or circles. Let $\mathbb{S}^3$ be embedded as the unit sphere in $\mathbb{R}^4$ centered at the origin. We say that a smooth surface Σ in $\mathbb{S}^3$ is *foliated by circular arcs or circles,* if there is a smooth one parameter family of 2-planes $\{P_t\}$ in $\mathbb{R}^4$ such that, taking smoothly assigned orthonormal frames $\{e_1(t), e_2(t)\}$ in the 2-planes P_t, the expression

$$X(t, \theta) = c(t) + r(t)(\cos(\theta)e_1(t) + \sin(\theta)e_2(t))$$

gives a nonsingular parameterization of Σ. We note that $c(t)$ and $r(t)$ are the Euclidean center and radius of the circular arc on the plane $c(t) + P_t$.

A surface in $\mathbb{S}^3$ is said to be *rotationally symmetric* if it is invariant under an isometric $\mathbb{S}^1$-action which fixes a geodesic. The smooth complete rotationally symmetric surfaces of constant mean curvature are classified in [3]. In particular, there are 5 qualitative types of rotationally symmetric cmc surfaces analogous to the Delaunay surfaces in $\mathbb{R}^3$.

We first show that a minimal surface in $\mathbb{S}^3$ which is foliated by circular arcs is either part of a ruled minimal surface, that is, each circular arc of the foliation is part of a geodesic, or part of a rotationally symmetric minimal surface. Recently, Kutev and Milousheva classified minimal surfaces in $\mathbb{S}^3$ which

Received December 27, 2018; Revised May 7, 2019; Accepted May 30, 2019.
2010 *Mathematics Subject Classification.* 53A10, 53C12.
Key words and phrases. minimal and cmc surfaces, circle foliation.
This work was supported by Hankuk University of Foreign Studies Research Fund.

are foliated by circles [4]. They showed that there are two types of minimal surfaces in $\mathbb{S}^3$ which are foliated by circles. The first type is ruled, and the circles of the second type are principal lines. Our result shows that the second type surfaces are rotationally symmetric.

Helicoid is the only nonplanar ruled minimal surface in $\mathbb{R}^3$ up to homothety [1]. In $\mathbb{S}^3$, there is a one parameter family of ruled minimal surfaces [5]. On the other hand, round cylinder is the only ruled cmc surface of given nonzero constant mean curvature in $\mathbb{R}^3$. We show that there is only one ruled cmc surface in $\mathbb{S}^3$ for each nonzero mean curvature, whose parametrization is given in Theorem 3.

2. Minimal surfaces in $\mathbb{S}^3$ which are foliated by circles

In the following, $\mathbb{S}^3$ is embedded in $\mathbb{R}^4$ as the unit sphere centered at the origin. Let $\psi : M \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an isometric immersion of a Riemann surface M. Then ψ is a minimal immersion into $\mathbb{S}^3$ if and only if

$$\Delta\psi = -2\psi, \tag{1}$$

where Δ is the Laplace-Beltrami operator on M [5]. If (x_1, x_2) is a local coordinates of M, then

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial x^j}\right),$$

where $\sum g_{ij} dx^i dx^j$ is the metric of M, $\left(g^{ij}\right) = (g_{kl})^{-1}$ and $g = \det(g_{ij})$. In general, if ν is a unit normal vector field on $M \subset \mathbb{S}^3$, then the mean curvature H of M with respect to ν is given by

$$H = \frac{1}{2}\Delta\psi \cdot \nu.$$

Let Σ be a smooth surface in $\mathbb{S}^3$ which is foliated by circular arcs. Let $\{P_t\}$ be the smooth one parameter family of 2-planes in $\mathbb{R}^4$ containing the circular arcs of foliation of Σ. We recall the result of Frank and Giering [2]. It says that the frames $\{e_1(t), e_2(t)\}$ of P_t can be chosen in such a way, along with an extension to an orthonormal frame $\{e_1(t), e_2(t), e_3(t), e_4(t)\}$ of $\mathbb{R}^4$, that certain Frenet type equations hold. We give a proof for the completeness.

Theorem A. *Let $\{P_t\}$ be a smooth one-parameter family of planes in $\mathbb{R}^4$ passing through the origin. There is a one-parameter family of orthonormal frame $\{e_1(t), e_2(t), e_3(t), e_4(t)\}$ of $\mathbb{R}^4$ such that $e_1(t)$ and $e_2(t)$ span P_t, and the following equations hold*

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}' = \begin{pmatrix} 0 & \beta & \kappa & 0 \\ -\beta & 0 & 0 & \tau \\ -\kappa & 0 & 0 & \eta \\ 0 & -\tau & -\eta & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}, \tag{2}$$

where $' = \partial/\partial t$ *and* $\kappa^2 \geq \tau^2$.

Proof. Let $\{f_1(t), f_2(t)\}$ be an orthonormal frame of $\{P_t\}$ smooth in t. For smooth $f(t) = \sum_{i=1,2} \gamma_i(t) f_i(t)$ with $\gamma_1(t)^2 + \gamma_2(t)^2 = 1$, let

$$\overset{\circ}{f}(t) = f'(t) - \sum_{i=1,2} \langle f'(t), f_i(t)\rangle f_i(t) \tag{3}$$

the projection of $f'(t)$ onto $P_t^{\perp}$. For simplicity, we omit t in the following. Clearly

$$\overset{\circ}{f_1} = f_1' - \sum_{j=1,2} \langle f_1', f_j\rangle f_j = f_1' - \langle f_1', f_2\rangle f_2,$$

and

$$\overset{\circ}{f_2} = f_2' - \langle f_2', f_1\rangle f_1.$$

We claim that there is an orthonormal frame $\{e_1, e_2\}$ of P_t such that

$$\| \overset{\circ}{e_1} \|^2 \geq \| \overset{\circ}{e_2} \|^2 \text{ and } \langle \overset{\circ}{e_1}, \overset{\circ}{e_2}\rangle = 0. \tag{4}$$

We have

$$\overset{\circ}{f} = f' - \sum_{i=1,2} \langle f', f_i\rangle f_i = \sum_{i=1,2} \gamma_i \left(f_i' - \sum_{j=1,2} \langle f_i', f_j\rangle f_j \right) = \sum_{i=1,2} \gamma_i \overset{\circ}{f_i} .$$

For each fixed $t \in I$, there exists a point $(\gamma_1(t), \gamma_2(t)) \in \mathbb{S}^1$ where

$$\|\overset{\circ}{f}(t)\|^2 = \left\langle \overset{\circ}{f}(t), \overset{\circ}{f}(t) \right\rangle = \sum_{i,j=1,2} \gamma_i(t)\gamma_j(t) \left\langle \overset{\circ}{f_i}(t), \overset{\circ}{f_j}(t) \right\rangle$$

attains minimum.

For a fixed $t_0 \in I$, we may assume that $(0,1) \in \mathbb{S}^1$ is the minimum point of $\|\overset{\circ}{f}(t_0)\|^2$. Hence $f_2(t_0)$ minimizes $\|\overset{\circ}{f}(t_0)\|^2$. Since $\|\overset{\circ}{f}(t_0)\|^2$ is quadratic in $\gamma_1(t_0)$ and $\gamma_2(t_0)$, we have $\langle \overset{\circ}{f_1}(t_0), \overset{\circ}{f_2}(t_0)\rangle = 0$ and $(1,0) \in \mathbb{S}^1$ is the maximum point of $\|\overset{\circ}{f}(t_0)\|^2$. Therefore $f_1(t_0)$ maximizes $\|\overset{\circ}{f}(t_0)\|^2$, and

$$\|\overset{\circ}{f}(t_0)\|^2 = \gamma_1^2(t_0)\|\overset{\circ}{f_1}(t_0)\|^2 + \gamma_2^2(t_0)\|\overset{\circ}{f_2}(t_0)\|^2$$

with $\|\overset{\circ}{f_1}(t_0)\|^2 \geq \|\overset{\circ}{f_2}(t_0)\|^2 \geq 0$.

We define an orthonormal frame $\{e_1(t), e_2(t)\}$ as follows. For each t, let $e_1(t)$ and $e_2(t)$ be the unit vectors on P_t corresponding to the maximizer and minimizer of $\|\overset{\circ}{f}(t)\|^2$ respectively. The above argument shows that $e_1(t)$ and $e_2(t)$ are smooth in t with $e_1 \perp e_2$ and (4) holds. We define $e_3(t)$ and $e_4(t)$ by

$$\| \overset{\circ}{e_1} \| e_3 := \overset{\circ}{e_1} = e_1' - \langle e_1', e_2\rangle e_2, \quad \| \overset{\circ}{e_2} \| e_4 := \overset{\circ}{e_2} = e_2' - \langle e_2', e_1\rangle e_1.$$

From (3) and (4), we have $\langle e_i, e_j\rangle = \delta_{ij}$ for $1 \le i, j \le 4$. Hence

$$\begin{aligned} e_1' &= \langle e_1', e_2\rangle e_2 + \| \overset{\circ}{e}_1 \| e_3, \\ e_2' &= \langle e_2', e_1\rangle e_1 + \| \overset{\circ}{e}_2 \| e_4, \\ e_3' &= -\| \overset{\circ}{e}_1 \| e_1 + \langle e_3', e_4\rangle e_4, \\ e_4' &= -\| \overset{\circ}{e}_2 \| e_2 + \langle e_4', e_3\rangle e_3. \end{aligned}$$

This completes the proof. □

Using the above orthonormal frame of $\mathbb{R}^4$, a circle-foliated surface Σ is parametrized by

$$X(t,\theta) = c(t) + r(t)(\cos\theta e_1 + \sin\theta e_2). \tag{5}$$

We note that $c(t)$ and $r(t)$ are the Euclidean center and the Euclidean radius of the circle of foliation on the plane $c(t) + P_t$, where

$$r^2 + \|c(t)\|^2 = 1.$$

We define $\alpha_1, \dots, \alpha_4$ to satisfy

$$c'(t) = \sum_{i=1}^{4} \alpha_i(t) e_i(t).$$

Let $c_i = \langle c(t), e_i(t)\rangle$ for $i = 1, \dots, 4$. Clearly, $c_1 = c_2 = 0$. The following are straightforward.

$$\begin{aligned} X_t = &(\alpha_1 + r'\cos\theta - r\beta\sin\theta)e_1 + (\alpha_2 + r'\sin\theta + r\beta\cos\theta)e_2 \\ &+ (\alpha_3 + r\kappa\cos\theta)e_3 + (\alpha_4 + r\tau\sin\theta)e_4, \\ X_\theta = &- r\sin\theta e_1 + r\cos\theta e_2. \end{aligned}$$

Let

$$h : \textstyle\bigwedge^3 \mathbb{R}^4 \to \mathbb{R}^4$$

be the canonical isomorphism. A normal vector N of X is perpendicular to X_t, X_θ and X. Hence N is parallel to $h(X_t \wedge X_\theta \wedge X)$. Direct computation shows that

$$\begin{aligned} h(X_t \wedge X_\theta \wedge X) = &\, r\cos\theta \left[c_3(\alpha_4 + r\tau\sin\theta) - c_4(\alpha_3 + r\kappa\cos\theta)\right] e_1 \\ &+ r\sin\theta \left[c_3(\alpha_4 + r\tau\sin\theta) - c_4(\alpha_3 + r\kappa\cos\theta)\right] e_2 \\ &+ \left[rc_4(r' + \alpha_1\cos\theta + \alpha_2\sin\theta) - r^2(\alpha_4 + r\tau\sin\theta)\right] e_3 \\ &- \left[rc_3(r' + \alpha_1\cos\theta + \alpha_2\sin\theta) - r^2(\alpha_3 + r\kappa\cos\theta)\right] e_4. \end{aligned}$$

For the local classification of minimal surface foliated by circular arcs, it suffices to assume that t is in an open interval I. We consider two cases: I) $c(t) \ne 0$ except for finitely many $t \in I$, or II) $c(t) \equiv 0$ on I. If $c(t) \ne 0$, then $c(t) = c_3\, e_3 + c_4\, e_4$ and, from (2), we have

$$c'(t) = c_3'\, e_3 + c_3(-\kappa e_1 + \eta e_4) + c_4'\, e_4 + c_4(-\tau e_2 - \eta e_3).$$

Hence

$$\begin{aligned}\alpha_1 = -\kappa c_3,\ \alpha_2 = -\tau c_4,\\ \alpha_3 = c_3' - \eta c_4,\ \alpha_4 = c_4' + \eta c_3.\end{aligned} \tag{6}$$

Let

$$N = \frac{h(X_t \wedge X_\theta \wedge X)}{r\,[c_3(\alpha_4 + r\tau\sin\theta) - c_4(\alpha_3 + r\kappa\cos\theta)]}, \tag{7}$$

unless the denominator is 0. Note that $\|N\| \neq 1$.

If II) holds, then the circles of the foliation are geodesics and $r = 1$, $c_i = \alpha_i = 0$ for $i = 1, \ldots, 4$. Then

$$h(X_t \wedge X_\theta \wedge X) = -\tau\sin\theta\, e_3 + \kappa\cos\theta\, e_4.$$

In this case, we let

$$N = -\tau\sin\theta\, e_3 + \kappa\cos\theta\, e_4. \tag{8}$$

For simplicity, we let

$$N = \epsilon\cos\theta e_1 + \epsilon\sin\theta e_2 + \gamma e_3 + \delta e_4, \tag{9}$$

where $\epsilon = 1$ if $c(t) \neq 0$ and $\epsilon = 0$ if II) holds.

B. Lawson classified ruled minimal surfaces in $\mathbb{S}^3$ (Proposition 7.2 of [5]).

Proposition 1. *Every ruled minimal surface in $\mathbb{S}^3$ is an open submanifold of one of the surfaces $\mathcal{M}_\alpha$, given by*

$$T(x, y) = (\cos\alpha x\cos y, \sin\alpha x\cos y, \cos x\sin y, \sin x\sin y) \tag{10}$$

for some $\alpha \geq 0$.

We give a proof of the proposition later.

Let E, F and G be the coefficients of the first fundamental form of X:

$$\begin{aligned} E = |X_t|^2 &= \sum_{i=1}^4 \alpha_i^2 + {r'}^2 + r^2\beta^2 + 2r'\alpha_1\cos\theta + 2r'\alpha_2\sin\theta\\ &\quad - 2r\alpha_1\beta\sin\theta + 2r\alpha_2\beta\cos\theta + 2r\alpha_3\kappa\cos\theta\\ &\quad + r^2\kappa^2\cos^2\theta + 2r\alpha_4\tau\sin\theta + r^2\tau^2\sin^2\theta,\\ F = X_t \cdot X_\theta &= -r\alpha_1\sin\theta + r\alpha_2\cos\theta + r^2\beta,\\ G = |X_\theta|^2 &= r^2.\end{aligned}$$

Let

$$l = X_{tt}\cdot N,\ m = X_{t\theta}\cdot N,\ n = X_{\theta\theta}\cdot N,$$

where N is given by (7) or (8). Let

$$\mathcal{H} := lG + nE - 2mF.$$

Direct computation shows that

$$
\text{(11)}\quad \mathcal{H} = \begin{array}{l} r^2 \left\{ \begin{array}{l} \epsilon\left[\alpha_1' \cos\theta + r'' - \alpha_2\beta\cos\theta + \alpha_1\beta\sin\theta + \alpha_2'\sin\theta - r\beta^2 \right. \\ \left. -\alpha_3\kappa\cos\theta - r\kappa^2\cos^2\theta - \alpha_4\tau\sin\theta - r\tau^2\sin^2\theta\right] \\ +\gamma\left(\begin{array}{l} \alpha_3' + 2r'\kappa\cos\theta + \alpha_1\kappa + r\kappa'\cos\theta \\ -r\beta\kappa\sin\theta - \alpha_4\eta - r\tau\eta\sin\theta \end{array}\right) \\ +\delta\left(\begin{array}{l} \alpha_4' + 2r'\tau\sin\theta + \alpha_2\tau + r\tau'\sin\theta \\ +r\beta\tau\cos\theta + \alpha_3\eta + r\kappa\eta\cos\theta \end{array}\right) \end{array} \right\} \\ -\epsilon\, r \left\{ \begin{array}{l} \sum_{i=1}^4 \alpha_i^2 + {r'}^2 + r^2\beta^2 + 2r'\alpha_1\cos\theta + 2r'\alpha_2\sin\theta \\ -2r\alpha_1\beta\sin\theta + 2r\alpha_2\beta\cos\theta + 2r\alpha_3\kappa\cos\theta \\ +r^2\kappa^2\cos^2\theta + 2r\alpha_4\tau\sin\theta + r^2\tau^2\sin^2\theta \end{array} \right\} \\ -2(-\epsilon r\beta - r\gamma\kappa\sin\theta + r\delta\tau\cos\theta)(-r\alpha_1\sin\theta + r\alpha_2\cos\theta + r^2\beta). \end{array}
$$

Proof of Proposition 1. Let Σ be a ruled minimal surface in $\mathbb{S}^3$. Then $c(t) \equiv 0$ and $r \equiv 1$ in (5) and $\mathcal{H} = 0$. Clearly, $\alpha_i = 0$ for $i = 1, \dots, 4$. From (8), we have $\kappa \neq 0$. Let $\epsilon = 0$, $\gamma = -\tau\sin\theta$ and $\delta = \kappa\cos\theta$ in (9). Substituting these into (11), we have

$$
\text{(12)}\qquad \mathcal{H} = \eta\kappa^2\cos^2\theta + \eta\tau^2\sin^2\theta - (\kappa'\tau - \kappa\tau')\cos\theta\sin\theta - \beta\kappa\tau.
$$

Hence $\kappa'\tau - \kappa\tau' = 0$, $\eta\left(\kappa^2 - \tau^2\right) = 0$ and $\eta\tau^2 - \beta\kappa\tau = 0$. From $\kappa'\tau - \kappa\tau' = 0$, we have either i) $\tau = 0$ and $|\kappa| > 0$ or ii) $|\kappa| = a|\tau| > 0$ for some constant $a \geq 1$.

If i) holds, then $\eta = 0$. Hence $e_4' = 0$, and Σ lies in a 3-dimensional subspace. Therefore Σ is part of a great sphere. If ii) holds with $a > 1$, then $\eta = 0$ and $\beta = 0$. Upon a change of the variable t, we may assume that $\kappa = 1$. In (2), we may let $e_1 = (\cos t, \sin t, 0, 0)$ and $e_2 = (0, 0, \cos\tau t, \sin\tau t)$. Letting $\alpha = \tau$, Σ is parametrized by (10).

If ii) holds with $a = 1$, we may assume that $\kappa = \tau$. From $\mathcal{H} = 0$, we have $\eta - \beta = 0$. Let ϕ satisfy $\phi' = \beta$. For

$$
\begin{aligned}
\tilde{e}_1 &= \cos\phi\, e_1 - \sin\phi\, e_2, \quad \tilde{e}_2 = \sin\phi\, e_1 + \cos\phi\, e_2, \\
\tilde{e}_3 &= \cos\phi\, e_3 - \sin\phi\, e_4, \quad \tilde{e}_4 = \sin\phi\, e_3 + \cos\phi\, e_4,
\end{aligned}
$$

we have $\tilde{e}_1' = \kappa\,\tilde{e}_3$, $\tilde{e}_2' = \kappa\,\tilde{e}_4$, $\tilde{e}_3' = -\kappa\tilde{e}_1$ and $\tilde{e}_4' = -\kappa\tilde{e}_2$. By changing the variable t, we may assume that $\kappa = 1$. Then Σ is the Clifford torus, which is M_1. □

3. Local classification

We first consider the local case. Suppose that $\Sigma \subset \mathbb{S}^3$ is foliated by circular arcs and let e_1, e_2, e_3 and e_4 be as in Theorem A. We note that a great sphere admits various foliations by circles.

Theorem 1. *A minimal surface in $\mathbb{S}^3$ which is foliated by circular arcs is locally either foliated by geodesic arcs, that is, part of a ruled minimal surface, or part of a rotationally symmetric minimal surface, or part of a great sphere.*

Proof. Let Σ be a minimal surface in $\mathbb{S}^3$ which is foliated by circular arcs and let (5) be a parametrization of Σ with e_1 and e_2 satisfying (2). Let I be an interval for which Theorem A holds.

(I) First we assume that $\kappa \equiv 0$ on I. Then we have $\tau \equiv 0$, and $\alpha_1 \equiv 0$ and $\alpha_2 \equiv 0$ by (6). If $\beta \equiv 0$, then e_1 and e_2 are constant. This implies that the plane P_t of Theorem A is fixed. Then the plane $P^\perp$, spanned by e_3 and e_4, is also fixed, and $c(t)$ lies on $P^\perp$. If $c(t) \equiv 0$, then Σ is a great circle, which is impossible. Otherwise, the planes $c(t) + P_t$ are parallel. We may assume that $\eta = 0$. Then $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = c_3'$ and $\alpha_4 = c_4'$. Let $C(t)$ be the circular arc of the foliation on P_t. Clearly $C(t)$ lies on the hyperplane spanned by e_1, e_2 and $c(t)$, and the spherical center of $C(t)$ is on the geodesic $P^\perp \cap \mathbb{S}^3$. It follows that each circular arc of Σ is part of circle invariant under the rotation of $\mathbb{S}^3$ fixing $P^\perp \cap \mathbb{S}^3$. Hence Σ is part of a rotationally symmetric minimal surface.

If $\beta \neq 0$, then we define ϕ to satisfy $\phi' = \beta$. For

$$\tilde{e}_1 = \cos\phi\, e_1 - \sin\phi\, e_2,\ \ \tilde{e}_2 = \sin\phi\, e_1 + \cos\phi\, e_2,$$

we have $\tilde{e}_1' = \tilde{e}_2' = 0$. Since $\tilde{e}_1$ and $\tilde{e}_2$ also spans P_t, P_t is fixed and Σ is part of a rotationally symmetric surface as above.

(II) Now we assume that $\kappa \neq 0$ on I. (If necessary, we replace I with a subinterval to assume that $\kappa \neq 0$ on I.) (II-1) First assume that $\alpha_1 \equiv 0$ and $\alpha_2 \equiv 0$ on I. From (6), we have two cases:

i) $c_3 \equiv 0$ and $c_4 \equiv 0$ on I,
ii) $c_3 \equiv 0$ and $c_4 \neq 0$ on I.

In the first case, we have $c(t) \equiv 0$ on I. Hence the circular foliating Σ are geodesic arcs and Σ is part of a ruled surface.

If the second case holds, then we have $\tau \equiv 0$ on I from (6) and $c(t) = c_4\, e_4$. From $X_t \cdot N = X \cdot N = 0$, we have, in (9),

$$\gamma = \frac{r\alpha_4 - c_4 r'}{c_4(\alpha_3 + r\kappa\cos\theta)},\ \ \delta = -\frac{r}{c_4}.$$

The trigonometric polynomial $c_4(\alpha_3 + r\kappa\cos\theta)\mathcal{H}$ is of degree 3. The coefficient of $\cos 3\theta$ of $c_4(\alpha_3 + r\kappa\cos\theta)\mathcal{H}$ is $(1/2)r^4 c_4\kappa^3$. Since $\mathcal{H} \equiv 0$, we have $\kappa = 0$. This is a contradiction. Therefore if $\kappa \neq 0$, $\alpha_1 \equiv 0$ and $\alpha_2 \equiv 0$, then $c(t) \equiv 0$ and Σ is part of a ruled surface.

(II-2) Suppose that $\kappa \neq 0$, $\alpha_1 \neq 0$ and $\alpha_2 \equiv 0$ on I. From (6), we have $c_3 \neq 0$ and $\alpha_2 = -\tau c_4 \equiv 0$. If $\tau \equiv 0$, $c_4 \equiv 0$ and $\alpha_4 \equiv 0$, then $\eta \equiv 0$ from (6). It follows that $e_4' = 0$ and $c(t) = c_3\, e_3$. Hence Σ lies in a 3-dimensional subspace, and Σ is part of a great sphere.

If $\tau \equiv 0$, $c_4 \equiv 0$ and $\alpha_4 \neq 0$. Then $c_3\alpha_4\mathcal{H}$ is of degree 2. The coefficient of $\cos 2\theta$ of $c_3\alpha_4\mathcal{H}$ is $(1/2)r^3\kappa^2\eta$ from (6) and $r^2 + \|c(t)\|^2 = 1$. Hence $\eta = 0$, which contradicts $\alpha_4 \neq 0$.

Suppose now that $c_4 \neq 0$ and $\tau \equiv 0$ or $c_4 \equiv 0$ and $\tau \neq 0$ on I. From $X_t \cdot N = X \cdot N = 0$, we have

$$\gamma = \frac{c_4(r' + \alpha_1 \cos\theta) - r(\alpha_4 + r\tau \sin\theta)}{c_3(\alpha_4 + r\tau \sin\theta) - c_4(\alpha_3 + r\kappa \cos\theta)},$$
$$\delta = -\frac{c_3(r' + \alpha_1 \cos\theta) - r(\alpha_3 + r\kappa \cos\theta)}{c_3(\alpha_4 + r\tau \sin\theta) - c_4(\alpha_3 + r\kappa \cos\theta)}.$$

For the above γ and δ, the trigonometric polynomial

$$(c_3(\alpha_4 + r\tau \sin\theta) - c_4(\alpha_3 + r\kappa \cos\theta))\,\mathcal{H}$$

is of degree 3, and the coefficient of $\cos(3\theta)$ is $(1/2)r^2 c_4 \kappa \left[r^2(\kappa^2 - \tau^2) + \alpha_1^2\right]$ by (6) and $r^2 + \|c(t)\|^2 = 1$. Since $\kappa^2 \geq \tau^2$ and $\mathcal{H} \equiv 0$, we have $\kappa^2 = \tau^2$ and $\alpha_1^2 = 0$, which is a contradiction.

(II-3) Now suppose that $\kappa \neq 0$, $\alpha_1 \equiv 0$ and $\alpha_2 \neq 0$ on I. Then $c_3 \equiv 0$, $\tau \neq 0$, $c_4 \neq 0$ and $r^2 + c_4^2 = 1$. From (6), we have $\alpha_2 = -\tau c_4$, $\alpha_3 = -\eta c_4$ and $\alpha_4 = c_4'$. Then

$$\gamma = \frac{r(\alpha_4 + r\tau \sin\theta) - c_4(r' + \alpha_2 \sin\theta)}{c_4(\alpha_3 + r\kappa \cos\theta)} = \frac{r\alpha_4 - r'c_4 + \tau \sin\theta}{c_4(\alpha_3 + r\kappa \cos\theta)},$$
$$\delta = -\frac{r}{c_4}.$$

The trigonometric polynomial $c_4(\alpha_3 + r\kappa \cos\theta)\mathcal{H}$ is of degree 3, and the coefficient of $\cos 3\theta$ is $(1/2)r^2 c_4 \kappa(\tau^2 - r^2\kappa^2)$. Therefore we have $r^2\kappa^2 = \tau^2$. We may assume that $\tau = r\kappa$. Direct computation shows that the coefficient of $\cos 2\theta$ of $c_4(\alpha_3 + r\kappa \cos\theta)\mathcal{H}$ is $(5/2)r^2 c_4^2 \kappa^2 \eta$. Hence we have $\eta = 0$, and $\alpha_3 = 0$ from (6). Then the coefficient of $\sin 2\theta$ of $c_4(\alpha_3 + r\kappa \cos\theta)\mathcal{H}$ is $(5/2)r^2\kappa^2 r'$. Hence $r' = 0$. Since $rr' + c_4 c_4' = 0$ from $r^2 + c_4^2 = 1$, r and c_4 are constant and $\alpha_4 = 0$. It follows that the coefficients of $\cos\theta$ is $-3rc_4\tau^3$. Therefore $\tau = 0$, which is a contradiction.

(II-4) Finally, suppose that $\kappa \neq 0$ and $\alpha_1\alpha_2 \neq 0$ on I. From (7), we have

$$\gamma = \frac{c_4(r' + \alpha_1 \cos\theta + \alpha_2 \sin\theta) - r(\alpha_4 + r\tau \sin\theta)}{c_3(\alpha_4 + r\tau \sin\theta) - c_4(\alpha_3 + r\kappa \cos\theta)},$$
$$\delta = -\frac{c_3(r' + \alpha_1 \cos\theta + \alpha_2 \sin\theta) - r(\alpha_3 + r\kappa \cos\theta)}{c_3(\alpha_4 + r\tau \sin\theta) - c_4(\alpha_3 + r\kappa \cos\theta)}.$$

The trigonometric polynomial $(c_3(\alpha_4 + r\tau \sin\theta) - c_4(\alpha_3 + r\kappa \cos\theta))\mathcal{H}$ is of degree 3, and the coefficients of $\cos 3\theta$ and $\sin 3\theta$ are

$$\frac{1}{2} r^2 c_4 \kappa \left[r^2(\kappa^2 - \tau^2) + (\alpha_1^2 - \alpha_2^2) + 2c_3^2\tau^2\right]$$

and

$$-\frac{1}{2} r^2 c_3 \tau \left[r^2(\kappa^2 - \tau^2) + (\alpha_1^2 - \alpha_2^2) - 2c_4^2\kappa^2\right],$$

which must be 0. Therefore, we have

$$c_3^2\tau^2 + c_4^2\kappa^2 = 0.$$

This contradicts $\alpha_1\alpha_2 \neq 0$.

(III) Suppose that $\kappa = 0$ at an isolated point $t_0 \in I$. The above result about case (II) shows that $c(t) \equiv 0$ on $I' \setminus \{t_0\}$ for some subinterval I' of I. Hence Σ is ruled. □

4. Global classification

Now we give the global classification. For the following lemma, we recall some results about rotationally symmetric minimal surfaces in $\mathbb{S}^3$ from [3]. For now, we use the stereographic projection of $\mathbb{S}^3$, that is, $\mathbb{R}^3$ equipped with the metric $ds^2 = \frac{4(dx^2+dy^2+dz^2)}{(1+x^2+y^2+z^2)^2}$. For an immersed surface $M \subset \mathbb{R}^3$, let N_e be the Euclidean unit normal of M. Let H_s be the mean curvature of M with respect to ds^2, and let H_e be the mean curvature of M with respect to the Euclidean metric $dx^2 + dy^2 + dz^2$ in the direction N_e. Then we have [3]

$$H_s = \frac{(1 + |X|^2)}{2} H_e + X \cdot N_e, \tag{13}$$

where X is the position vector of M and $\cdot$ denotes the Euclidean inner product.

Let Σ be a rotationally symmetric surface in $\mathbb{S}^3$, whose axis of rotational symmetry is the great circle on the xy-plane centered at the origin O of $\mathbb{R}^3$. Then the circles of Σ are perpendicular to the xy-plane and the great sphere centered at O. Let Π_θ be the plane containing the z-axis whose angle with the xz-plane is θ.

Locally, we consider two cases: i) $\Sigma \cap \Pi_\theta = \emptyset$ except for a fixed θ_0, or ii) $\Sigma \cap \Pi_\theta \neq \emptyset$ for θ varying in some interval of $[0, 2\pi)$. If i) holds, then we may let $\theta_0 = 0$. Hence Σ lies on the plane Π_0 locally.

When ii) holds and $\Sigma \cap \Pi_\theta \neq \emptyset$, let C_θ and ρ be the Euclidean center and radius of the circle $\Sigma \cap \Pi_\theta$. For a point $P \in \Sigma \cap \Pi_\theta$, let ϕ be the angle between $\overrightarrow{C_\theta P}$ and the ray $\overrightarrow{C_\theta O}$. Then Σ is parametrized as follows:

$$X(\theta, \phi) = (\sqrt{1 + \rho^2} - \rho \cos\phi)\,(\cos\theta, \sin\theta, 0) + (0, 0, \rho \sin\phi).$$

Straightforward computations using (13) show that [3]

$$H_s = \frac{\sqrt{\rho^2 + 1}(\rho(\rho^2 + 1)\rho'' - \rho'^2 + \rho^4 - 1)}{2\rho(\rho'^2 + \rho^2 + 1)^{3/2}}.$$

Lemma 1. *Let Σ be a complete rotationally symmetric minimal surface in $\mathbb{S}^3$. Then Σ is either a great sphere or foliated by non-geodesic circles.*

Proof. We show that, if Σ is not a great sphere, then no circle of Σ is a geodesic. Suppose that the above condition ii) holds. Then ρ satisfies

$$\rho(\rho^2 + 1)\rho'' - \rho'^2 + \rho^4 - 1 = 0, \tag{14}$$

where $' = \frac{d}{d\theta}$ (cf. equation (25) of [3]). The solution of (14) is bounded and periodic [3]. Therefore no circle of Σ is a great circle.

The first integral of (14) is

$$\frac{\rho}{\sqrt{(\rho'^2+\rho^2+1)(\rho^2+1)}} = c,$$

where c is a constant. Hence ρ' is also bounded, and the graph of the polar equation $\rho = \rho(\theta)$ on the xy-plane is transversal to each ray $\theta = constant$. It follows that Σ is transversal to each plane containing the circle of foliation.

If i) holds, then the corresponding circles of Σ foliates the xz-plane. From the above argument, we see that the circles of Σ should stay in the xz-plane. Hence Σ is a great sphere. □

In the following theorem, the parametrization X is allowed to be singular at isolated t or θ as long as Σ is regular. As an example, one may consider the foliation of $\mathbb{S}^2 \subset \mathbb{R}^3$ induced by the rotation of $\mathbb{R}^3$ about the z-axis.

Theorem 2. *Let Σ be a complete minimal surface in $\mathbb{S}^3$ foliated by circles. Then Σ is either ruled or rotationally symmetric, or a great sphere.*

Proof. Let $X(t,\theta)$ be regular for $t \in I$ and θ in some interval of $[0, 2\pi)$. By Theorem 1, open part Σ_o of Σ, where $X(t,\theta)$ is regular, is part of either i) a ruled minimal surface, or ii) a rotationally symmetric minimal surface, or iii) a great sphere with non-rotationally symmetric foliation.

Suppose that a circle C in the foliation of Σ is not a great circle. Clearly circles close to C are not great circles either. By Theorem 1, either ii) or iii) holds. If Σ_o is part of a rotationally symmetric minimal surface and not part of a geodesic, then Σ is rotationally symmetric by Lemma 1.

If Σ_o is part of a rotationally symmetric minimal surface and part of a great sphere or iii) holds, then all the circles of Σ should lie in the great sphere by Lemma 1. Hence Σ is a great sphere. □

5. Ruled cmc surfaces in $\mathbb{S}^3$

Let $\psi : M \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an isometric immersion of a Riemann surface M with constant mean curvature H with respect to some unit normal ν of $\psi(M) \subset \mathbb{S}^3$. Then ψ satisfies

$$\Delta\psi = 2H\nu - 2\psi. \tag{15}$$

We use the results of §2 to classify ruled cmc surfaces in $\mathbb{S}^3$. Let Σ be a surface in $\mathbb{S}^3$ which is foliated by geodesics and has nonzero constant mean curvature H. Let (5) be a parametrization of Σ. Let $\nu = N/\|N\|$. Since $\mathcal{H}$ of (11) is computed with respect to N, the mean curvature H satisfies

$$2H(EG - F^2)\|N\| = \mathcal{H}. \tag{16}$$

Theorem 3. *Ruled surface of constant mean curvature H in $\mathbb{S}^3$ is an open submanifold of*

$$X(t,\theta) = \cos\theta e_1 + \sin\theta e_2,$$

where e_1 *and* e_2 *are part of an orthonormal frame* $e_1, \dots, e_4$ *of* $\mathbb{R}^4$ *satisfying*

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2H \\ 0 & -1 & -2H & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \tag{17}$$

Proof. Suppose that Σ is parametrized by (5). Since Σ is ruled, $c(t) \equiv 0$ and $r = 1$. Then $EG - F^2 = \kappa^2 \cos^2\theta + \tau^2 \sin^2\theta$ with $\kappa \neq 0$. One may let $\epsilon = 0$, $\gamma = -\tau \sin\theta$ and $\delta = \kappa \cos\theta$ in (9). Hence $\|N\|^2 = \kappa^2 \cos^2\theta + \tau^2 \sin^2\theta$. From (11) and (16), we have

$$\begin{aligned} 2H(EG - F^2)\|N\| - \mathcal{H} &= 2H\left(\kappa^2 \cos^2\theta + \tau^2 \sin^2\theta\right)^{\frac{3}{2}} \\ &\quad - \left(\eta\kappa^2 \cos^2\theta + \eta\tau^2 \sin^2\theta + (-\kappa'\tau + \kappa\tau')\cos\theta\sin\theta - \beta\kappa\tau\right) = 0. \end{aligned}$$

Since this equation holds for all θ, we have $\kappa^2 = \tau^2$ and

$$\kappa^2\left(2H|\kappa| - \eta + \beta\right) = 0.$$

We may assume that $\kappa = \tau > 0$. Straightforward computation shows that

$$\Delta X = \frac{1}{\kappa}\left\{X_{tt} - \left(\frac{\beta}{\kappa}\right)_t X_\theta - 2\frac{\beta}{\kappa}X_{t\theta} + \frac{\kappa^2 + \beta^2}{\kappa}X_{\theta\theta}\right\}.$$

Since $\nu = N/\|N\| = -\sin\theta\, e_3 + \cos\theta\, e_4$ in (15), the coefficient of e_1 of $\Delta X - 2H\nu + 2X$ is

$$-\beta^2\cos\theta - \kappa^2\cos\theta - \beta'\sin\theta + \left(\frac{\beta}{\kappa}\right)'\sin\theta + \frac{\beta^2}{\kappa}\cos\theta + \kappa\cos\theta,$$

which is 0. From the coefficient of $\cos\theta$, it follows that $\kappa = 1$. Hence

$$\eta = 2H + \beta.$$

Let ϕ satisfy $\phi' = \beta$. For

$$\begin{aligned} \tilde{e}_1 &= \cos\phi\, e_1 - \sin\phi\, e_2,\ \tilde{e}_2 = \sin\phi\, e_1 + \cos\phi\, e_2, \\ \tilde{e}_3 &= \cos\phi\, e_3 - \sin\phi\, e_4,\ \tilde{e}_4 = \sin\phi\, e_3 + \cos\phi\, e_4, \end{aligned}$$

we have

$$\tilde{e}_1' = \tilde{e}_3,\ \tilde{e}_2' = \tilde{e}_4,\ \tilde{e}_3' = -\tilde{e}_1 + 2H\tilde{e}_4 \text{ and } \tilde{e}_4' = -\tilde{e}_2 - 2H\tilde{e}_3.$$

Then

$$X(t,\theta) = \cos\theta\,\tilde{e}_1 + \sin\theta\,\tilde{e}_2$$

is the desired parametrization of a ruled cmc surface in $\mathbb{S}^3$ with mean curvature H and $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ and $\tilde{e}_4$ satisfy (17). □

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