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UNICITY OF MERMORPHIC FUNCTIONS CONCERNING SHARED FUNCTIONS WITH THEIR DIFFERENCE

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Abstract. In this paper, we investigate the uniqueness of meromorphic functions of finite order concerning sharing small functions and prove that if $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z), \infty$ CM, where $a(z), b(z) (\not\equiv \infty)$ are two distinct small functions of $f(z)$, then $f(z) \equiv \Delta_c f(z)$. The result improves the results due to Li et al. ([9]), Cui et al. ([1]) and Lü et al. ([12]).

1. Introduction

Throughout this paper, a meromorphic function always means a function which is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory (see [7, 15, 16]).

In addition, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. We say that $a(z)$ is a small function of $f(z)$ if $T(r, a) = S(r, f)$.

We use $\rho(f)$ to denote the order of f. We say that two meromorphic functions f and g share a IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that they share a CM (counting multiplicities), where a is a small function of f and g.

For a meromorphic function $f(z)$, we define its shift by $f_c(z) = f(z + c)$ and its difference operator by $\Delta_c f(z) = f(z + c) - f(z)$.

In 1929, Nevanlinna [13] proved the following famous five-value theorem.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let a_j $(j = 1, 2, 3, 4, 5)$ be five distinct values in the extended complex plane. If $f(z)$ and $g(z)$ share a_j $(j = 1, 2, 3, 4, 5)$ IM, then $f(z) \equiv g(z)$.*

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In 2000, Li and Qiao [10] proved that Theorem 1.1 is still valid for five small functions, they proved:

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $a_j(z)$ $(j = 1,2,3,4,5)$ (one of them can be ∞) be five distinct small functions of $f(z)$ and $g(z)$. If $f(z)$ and $g(z)$ share $a_j(z)$ $(j = 1,2,3,4,5)$ IM, then $f(z) \equiv g(z)$.*

Recently, value distribution in difference analogue of meromorphic functions has become a subject of some interests, see ([1–6, 8, 17]).

In 2014, Zhang and Liao [17], Liu et al. [11] proved the following result independently.

Theorem 1.3. *Let $f(z)$ be a transcendental entire function of finite order, and let a, b be two distinct constants. If $f(z)$ and $\Delta_c f(z)$ share a, b CM, then $f(z) \equiv \Delta_c f(z)$.*

In fact, Liu et al. [11] proved the following more general case.

Theorem 1.4. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(z)(\not\equiv 0), b(z)(\not\equiv 0)$ be two distinct small functions of $f(z)$. If $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z)$ CM, then $f(z) \equiv \Delta_c f(z)$.*

More recently, Li et al. [9], Cui et al. [1], Lü et al. [12], proved that Theorem 1.3 still holds for meromorphic functions of finite order if $f(z)$ and $\Delta_c f(z)$ sharing ∞ CM. They proved:

Theorem 1.5. *Let $f(z)$ be a transcendental meromorphic function of finite order, and let a, b be two distinct constants. If $f(z)$ and $\Delta_c f(z)$ share a, b, ∞ CM, then $f(z) \equiv \Delta_c f(z)$.*

A nature problem arise: Does Theorem 1.5 still hold if $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z)$ and ∞ CM, where $a(z), b(z)$ are two distinct small functions of $f(z)$?

In this paper, we study the problem and give a positive answer to the question.

Theorem 1.6. *Let $f(z)$ be a transcendental meromorphic function of finite order, and let $a(z)(\not\equiv \infty), b(z)(\not\equiv \infty)$ be two distinct small functions of $f(z)$. If $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z), \infty$ CM, then $f(z) \equiv \Delta_c f(z)$.*

Example 1.7. Let $f(z) = \frac{e^z}{e^{2z}+1}$, $c = \pi i$, $a(z) \equiv 0$. Then $\Delta_c f(z) = f(z + c) - f(z) = -2\frac{e^z}{e^{2z}+1}$. Obviously, $\Delta_c f(z)$ and $f(z)$ share $a(z), \infty$, but $\Delta_c f(z) \not\equiv f(z)$. This example shows that the number of shared functions can not be reduce to two.

2. Some lemmas

Lemma 2.1. *Let $A(\neq -1)$ be a nonzero constant. Suppose that $f(z) \not\equiv 0$ is a meromorphic solution of finite order to the following difference equation*

$$Af(z) + f(z + c) \equiv 0. \tag{1}$$

Then there exists a real number $B > 0$ such that $T(r, f) \geq Br$.

Proof. We consider three cases.

Case 1. $f(z) \neq 0, \infty$. Then $f(z) = e^{p(z)}$, where $p(z)$ is a polynomial. It follows from (1) that $p(z)$ is a nonconstant polynomial. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, where $a_n \neq 0$, $n \geq 1$. Hence there exists $B > 0$ such that

$$T(r, f) \geq \frac{1}{2}T(r, e^{a_n z^n}) \geq \frac{T(r, e^z)}{2} \geq Br.$$

Case 2. There exists z_0 such that $f(z_0) = 0$. Without loss of generality, let $z_0 = 0$. Then it follows from (1) that nc $(n = 0, 1, 2, \ldots)$ are also zeros of $f(z)$. For $|2nc| \leq r < |(2n+1)c|$, we have

$$\begin{aligned}
T(r, f) &\geq N(r, \frac{1}{f}) + O(1) = \int_0^r \frac{n(t, \frac{1}{f}) - n(0, \frac{1}{f})}{t} dt + n(0, \frac{1}{f}) \log r + O(1) \\
&\geq \sum_{j=1}^{2n-1} j \int_{|jc|}^{|(j+1)c|} \frac{1}{t} dt + n(0, \frac{1}{f}) \log r + O(1) \\
&\geq \log \frac{(2n)^{2n-1}}{(2n-1)!} + n(0, \frac{1}{f}) \log r + O(1) \\
&\geq n \log 2 + n(0, \frac{1}{f}) \log r + O(1) \\
&\geq \frac{r + |c|}{2|c|} \log 2 + n(0, \frac{1}{f}) \log r + O(1) \\
&\geq \frac{r}{4|c|}.
\end{aligned}$$

Case 3. There exists z_1 such that $f(z_1) = \infty$. Without loss of generality, let $z_1 = 0$. Then it follows from (1) that nc $(n = 0, 1, 2, \ldots)$ are also poles of $f(z)$. For $|2nc| \leq r < |(2n+1)c|$, we have

$$\begin{aligned}
T(r, f) \geq N(r, f) &= \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r \\
&\geq \sum_{j=1}^{2n-1} j \int_{jc}^{(j+1)c} \frac{1}{t} dt + n(0, f) \log r \\
&\geq \log \frac{(2n)^{2n-1}}{(2n-1)!} + n(0, f) \log r \\
&\geq n \log 2 + n(0, f) \log r + O(1) \\
&\geq \frac{r + |c|}{2|c|} \log 2 + n(0, f) \log r + O(1) \\
&\geq \frac{r}{4|c|}.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2 ([4,5]). *Let $f(z)$ be a meromorphic function of finite order, and let c be a nonzero complex constant. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.3 ([4,5]). *Let $c \in \mathbb{C}$, let k be a positive integer, and let $f(z)$ be a meromorphic function of finite order. Then*

$$m\left(r, \frac{\Delta_c^k f(z)}{f(z)}\right) = S(r, f).$$

Lemma 2.4 (see [15] Theorem 1.51). *Suppose that $f_i(z)$ $(i = 1, 2, \ldots, n)$ and $g_i(z)$ $(i = 1, 2, \ldots, n)$ $(n \geq 2)$ are entire functions satisfying*

(i) $\sum_{i=1}^{n} f_i(z) e^{g_i(z)} \equiv 0$.

(ii) *$g_j(z) - g_k(z)$ are not constants for $1 \leq j < k < n$.*

(iii) *For $1 \leq i \leq n$, $1 \leq k < l \leq n$,*

$$T(r, f_i) = o\left\{T(r, e^{g_k - g_l})\right\} \quad (r \to \infty,\ r \notin E).$$

Then $f_i(z) \equiv 0$ $(i = 1, 2, \ldots, n)$.

Lemma 2.5 ([14]). *Suppose that $f(z)$ is a meromorphic function in the complex plane, and that $F(z) = a_n(z) f^n(z) + a_{n-1}(z) f^{n-1}(z) + \cdots + a_0(z)$, where $a_0(z), a_1(z), \ldots, a_n(\not\equiv 0)$ are small functions of $f(z)$. Then*

$$T(r, F) = nT(r, f) + S(r, f). \tag{2}$$

3. Proof of Theorem 1.6

Proof. Since $f(z)$ and $\Delta_c f(z)$ share $a(z), b(z)$ and ∞ CM, and f is a transcendental meromorphic function with finite order, combing that with Lemma 2.3, we have

$$\frac{\Delta_c f(z) - a(z)}{f(z) - a(z)} = e^{\alpha(z)}, \quad \frac{\Delta_c f(z) - b(z)}{f(z) - b(z)} = e^{\beta(z)}, \tag{3}$$

where $\alpha(z)$ and $\beta(z)$ are two polynomials such that

$$\max\{\deg \alpha(z), \deg \beta(z)\} \leq \rho(f). \tag{4}$$

It follows from (3) that

$$(e^{\alpha(z)} - e^{\beta(z)}) f(z) = a(z) e^{\alpha(z)} - b(z) e^{\beta(z)} - [a(z) - b(z)]. \tag{5}$$

If $e^{\alpha(z)} \equiv e^{\beta(z)}$, then from (5) we obtain

$$(a(z) - b(z))(e^{\alpha(z)} - 1) = 0.$$

Since $a(z) \not\equiv b(z)$, we get $e^{\alpha(z)} \equiv 1$, hence $f(z) \equiv \Delta_c f(z)$.

Next, we consider the case of $e^{\alpha(z)} \not\equiv e^{\beta(z)}$.

It follows from (5) and the first equation in (3) that

$$f(z) = \frac{a(z) e^{\alpha(z)} - b(z) e^{\beta(z)} - [a(z) - b(z)]}{e^{\alpha(z)} - e^{\beta(z)}}, \tag{6}$$

$$\Delta_c f(z) = \frac{e^{\alpha(z)}\left\{a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - [a(z)-b(z)]\right\}}{e^{\alpha(z)} - e^{\beta(z)}} - a(z)[e^{\alpha(z)} - 1]. \tag{7}$$

It follows from (6) that

$$\begin{aligned}\Delta_c f(z) &= f(z+c) - f(z)\\ &= \frac{a(z+c)e^{\alpha(z+c)} - b(z+c)e^{\beta(z+c)} - [a(z+c)-b(z+c)]}{e^{\alpha(z+c)} - e^{\beta(z+c)}}\\ &\quad - \frac{a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - [a(z)-b(z)]}{e^{\alpha(z)} - e^{\beta(z)}}.\end{aligned} \tag{8}$$

By (7) and (8), we obtain

$$\begin{aligned}&[a(z)-b(z)]e^{\alpha(z)+\alpha(z+c)+\beta(z)} + [a(z)-a(z+c)+b(z)]e^{\alpha(z)+\alpha(z+c)}\\ &- [a(z)-b(z)]e^{\alpha(z)+\beta(z)+\beta(z+c)} - [a(z)+b(z)-b(z+c)]e^{\alpha(z)+\beta(z+c)}\\ &- [a(z)-a(z+c)+b(z)]e^{\alpha(z+c)+\beta(z)}\\ &+ [a(z)+b(z)-b(z+c)]e^{\beta(z)+\beta(z+c)}\\ &+ [a(z+c)-b(z+c)]e^{\alpha(z)} - [a(z+c)-b(z+c)]e^{\beta(z)}\\ &- [a(z)-b(z)]e^{\alpha(z+c)} + [a(z)-b(z)]e^{\beta(z+c)} \equiv 0.\end{aligned} \tag{9}$$

It follows from (6) and $T(r,a) + T(r,b) = S(r,f)$ that

$$\begin{aligned}T(r,f(z)) &= T(r, \frac{a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - (a(z)-b(z))}{e^{\alpha(z)} - e^{\beta(z)}})\\ &\le 2(T(r,e^{\alpha}) + T(r,e^{\beta})) + S(r,f).\end{aligned}$$

Hence, we have

$$T(r,a) + T(r,b) = o(T(r,e^{\alpha}) + T(r,e^{\beta})). \tag{10}$$

By (10) and Lemma 2.2, we also obtain

$$T(r,a(z+c)) + T(r,b(z+c)) = o(T(r,e^{\alpha}) + T(r,e^{\beta})). \tag{11}$$

Next, we consider three cases.

Case 1. $\deg\alpha(z) > \deg\beta(z)$. Then (9) can be rewritten as

$$H_2(z)e^{2\alpha(z)} + H_1(z)e^{\alpha(z)} + H_0(z) \equiv 0, \tag{12}$$

where

$$\begin{aligned}H_2(z) &= [a(z)-b(z)]e^{\Delta_c\alpha(z)+\beta(z)} + [a(z)-a(z+c)+b(z)]e^{\Delta_c\alpha(z)},\\ H_1(z) &= -[a(z)-b(z)]e^{\beta(z)+\beta(z+c)} - [a(z)+b(z)-b(z+c)]e^{\beta(z+c)}\\ &\quad + [a(z+c)-b(z+c)] - [a(z)-a(z+c)+b(z)]e^{\Delta_c\alpha(z)+\beta(z)}\\ &\quad - [a(z)-b(z)]e^{\Delta_c\alpha(z)},\\ H_0(z) &= [a(z)+b(z)-b(z+c)]e^{\beta(z)+\beta(z+c)} - [a(z+c)-b(z+c)]e^{\beta(z)}\\ &\quad + [a(z)-b(z)]e^{\beta(z+c)}.\end{aligned} \tag{13}$$

It follows from (10), (11) and $\deg\alpha(z) > \deg\beta(z)$ that $T(H_i(z)) = S(r, e^{\alpha(z)})$ $(i = 0, 1, 2)$. If $H_2(z) \not\equiv 0$, then by Lemma 2.5, we have

$$\begin{aligned} 2T(r, e^{\alpha}) + S(r, e^{\alpha}) &= T(r, H_2 e^{2\alpha}) + S(r, e^{\alpha}) \\ &= T(r, -H_1 e^{\alpha} - H_0) + S(r, e^{\alpha}) \\ &\le T(r, e^{\alpha}) + S(r, e^{\alpha}). \end{aligned}$$

Hence, we get $T(r, e^{\alpha}) = S(r, e^{\alpha})$. This implies e^{α} is a constant, but it contradicts $\deg\alpha(z) > \deg\beta(z) \ge 0$.

By the same way, we deduce that $H_0(z) \equiv H_1(z) \equiv H_2(z) \equiv 0$.

Hence, by (13) and $H_2(z) \equiv H_0(z) \equiv 0$, we obtain

$$[a(z) - b(z)]e^{\beta(z)} + [a(z) - a(z + c) + b(z)] \equiv 0, \tag{14}$$

$$\begin{aligned} &[a(z) + b(z) - b(z + c)]e^{\beta(z+c)} - [a(z + c) - b(z + c)] \\ &+ [a(z) - b(z)]e^{\beta(z+c)-\beta(z)} \equiv 0. \end{aligned} \tag{15}$$

It follows from $H_1(z) \equiv 0$ and (14)-(15) that

$$e^{\beta(z)+\beta(z+c)} - e^{\beta(z+c)-\beta(z)} - [e^{2\beta(z)} - 1]e^{\Delta_c\alpha(z)} \equiv 0. \tag{16}$$

Next, we consider two subcases.

Case 1.1. $\deg\Delta_c\alpha(z) > \deg\beta(z)$. Then it is easy to deduce that $e^{2\beta(z)} = 1$, which implies that $\beta(z)$ is a constant satisfying $e^{\beta} = 1$, or $e^{\beta} = -1$.

If $e^{\beta} = 1$, then by the second equation of (3), we obtain $f(z) \equiv \Delta_c f(z)$.

If $e^{\beta} = -1$, then it follows from (14)-(15) that

$$\begin{aligned} 2b(z) - a(z + c) &\equiv 0, \\ 2b(z) - b(z + c) &\equiv 0. \end{aligned}$$

Hence, we get $a(z) \equiv b(z)$, a contradiction.

Case 1.2. $\deg\Delta_c\alpha(z) = \deg\beta(z) \ge 1$.

Case 1.2.1. $\deg[2\beta(z) - \Delta_c\alpha(z)] = \deg\beta(z)$ and $\deg[2\beta(z) + \Delta_c\alpha(z)] = \deg\beta(z)$. Then (16) can be rewritten as follows.

$$\sum_{i=1}^{4} L_i(z)e^{g_i(z)} \equiv 0, \tag{17}$$

where

$$\begin{aligned} L_1(z) &= e^{\Delta_c\beta(z)}, & g_1(z) &= 2\beta(z), \\ L_2(z) &= -1, & g_2(z) &= \Delta_c\beta(z), \\ L_3(z) &= -1, & g_3(z) &= 2\beta(z) + \Delta_c\alpha(z), \\ L_4(z) &= 1, & g_4(z) &= \Delta_c\alpha(z). \end{aligned}$$

Obviously, for any $1 \le i < j \le 4$, $n = 1, 2, 3, 4$, we have

$$T(r, L_n) = o\left\{T(r, e^{g_i - g_j})\right\}.$$

Hence, it follows from (17) and Lemma 2.4 that $L_1(z) \equiv L_2(z) \equiv L_3(z) \equiv L_4(z) \equiv 0$. But $L_1(z) = e^{\Delta_c \beta(z)}(\neq 0)$, $L_2(z) = -1(\neq 0)$, $L_3(z) = -1(\neq 0)$, $L_4(z) = 1(\neq 0)$, we get a contradiction.

Case 1.2.2. $\deg[2\beta(z) - \Delta_c\alpha(z)] < \deg \beta(z)$. Let $2\beta(z) - \Delta_c\alpha(z) = -p_1(z)$, then $\Delta_c\alpha(z) = 2\beta(z) + p_1(z)$. So (16) can be rewritten as follows.

$$\sum_{i=1}^{3} M_i(z) e^{g_i(z)} \equiv 0, \tag{18}$$

where

$$\begin{aligned} M_1(z) &= e^{\Delta_c\beta(z)} + e^{p_1(z)}, & g_1(z) &= 2\beta(z), \\ M_2(z) &= -1, & g_2(z) &= \Delta_c\beta(z), \\ M_3(z) &= -e^{p_1(z)}, & g_3(z) &= 4\beta(z). \end{aligned}$$

Obviously, for any $1 \leq i < j \leq 3$, $n = 1, 2, 3$, we have

$$T(r, M_n) = o\left\{T(r, e^{g_i - g_j})\right\}.$$

Hence, it follows from (18) and Lemma 2.4 that $M_1(z) \equiv M_2(z) \equiv M_3(z) \equiv 0$. But $M_2(z) = -1(\neq 0)$, $M_3(z) = -e^{p_1(z)}(\neq 0)$, we get a contradiction.

Case 1.2.3. $\deg[2\beta(z) + \Delta_c\alpha(z)] < \deg \beta(z)$. Let $2\beta(z) + \Delta_c\alpha(z) = p_2(z)$, then $\Delta_c\alpha(z) = -2\beta(z) + p_2(z)$. So (16) can be rewritten as follows.

$$\sum_{i=1}^{3} M_i(z) e^{g_i(z)} \equiv 0, \tag{19}$$

where

$$\begin{aligned} M_1(z) &= e^{\Delta_c\beta(z)}, & g_1(z) &= 2\beta(z), \\ M_2(z) &= -e^{\Delta_c\beta(z)} - e^{p_2(z)}, & g_2(z) &= 0, \\ M_3(z) &= e^{p_2(z)}, & g_3(z) &= -2\beta(z). \end{aligned}$$

Obviously, for any $1 \leq i < j \leq 3$, $n = 1, 2, 3$, we have

$$T(r, M_n) = o\left\{T(r, e^{g_i - g_j})\right\}.$$

Hence, it follows from (19) and Lemma 2.4 that $M_1(z) \equiv M_2(z) \equiv M_3(z) \equiv 0$. But $M_1(z) = e^{\Delta_c\beta(z)}(\neq 0)$, we get a contradiction.

Case 1.3. $\deg \Delta_c\alpha(z) = \deg \beta(z) = 0$. Then $\beta(z)$ and $\Delta_c\alpha(z)$ are two constants, and $\alpha(z) = Az + B, (A \neq 0)$. So, by (14)-(15), we obtain

$$[2b(z) - b(z + c)](e^{\beta} - 1) \equiv 0.$$

This implies that $e^{\beta} = 1$, or $2b(z) - b(z + c) \equiv 0$.

If $e^{\beta} = 1$, then it follows from the second equation of (3) that $f(z) \equiv \Delta_c f(z)$.

Next, we consider the case $2b(z) - b(z + c) \equiv 0$. In this case, we divide it into two subcases.

Case 1.3.1. $b(z) \not\equiv 0$. Then by Lemma 2.1, there exists $D_1 > 0$ such that $T(r, b(z)) \geq D_1 r$. On the other hand, by $\alpha(z) = Az + B$, there exists

$D_2 > 0$ such that $T(r, e^{\alpha(z)}) = T(r, e^{Az+B}) \le D_2 r$. Hence, we have $T(r, b(z)) \ge \frac{D_1}{D_2} T(r, e^{\alpha(z)})$, but it contradicts with (10).

Case 1.3.2. $b(z) \equiv 0$. Then it follows from (14) that $(e^{\beta}+1)a(z) - a(z+c) \equiv 0$, obviously, $e^{\beta} + 1 \ne 1$. Since $a(z) \not\equiv b(z)$, we have $a(z) \not\equiv 0$. Using the same argument as case 1.3.1, we get a contradiction.

Case 2. $\deg \alpha(z) < \deg \beta(z)$. Then (9) can be rewritten as

$$K_2(z)e^{2\beta(z)} + K_1(z)e^{\beta(z)} + K_0(z) \equiv 0, \tag{20}$$

where

$$\begin{aligned}
K_2(z) = & -[a(z) - b(z)]e^{\alpha(z)+\Delta_c\beta(z)} + [a(z) + b(z) - b(z+c)]e^{\Delta_c\beta(z)}, \\
K_1(z) = & [a(z) - b(z)]e^{\alpha(z)+\alpha(z+c)} - [a(z) + b(z) - b(z+c)]e^{\alpha(z)+\Delta_c\beta(z)} \\
& - [a(z) - a(z+c) + b(z)]e^{\alpha(z+c)} - [a(z+c) - b(z+c)] \\
& + [a(z) - b(z)]e^{\Delta_c\beta(z)}, \\
K_0(z) = & [a(z) - a(z+c) + b(z)]e^{\alpha(z)+\alpha(z+c)} + [a(z+c) - b(z+c)]e^{\alpha(z)} \\
& - [a(z) - b(z)]e^{\alpha(z+c)}.
\end{aligned} \tag{21}$$

It follows from (10)-(11) and $\deg \alpha(z) < \deg \beta(z)$ that $T(r, K_i(z)) = S(r, e^{\beta(z)})$ $(i = 0, 1, 2)$. Using the same argument as Case 1, we obtain $K_0(z) \equiv K_1(z) \equiv K_2(z) \equiv 0$.

Hence, by (21) and $K_2(z) \equiv K_0(z) \equiv 0$, we obtain

$$[a(z) - b(z)]e^{\alpha(z)} - [a(z) + b(z) - b(z+c)] \equiv 0, \tag{22}$$

$$\begin{aligned}
&[a(z) - a(z+c) + b(z)]e^{\alpha(z+c)} + [a(z+c) - b(z+c)] \\
&\quad - [a(z) - b(z)]e^{\alpha(z+c)-\alpha(z)} \equiv 0.
\end{aligned} \tag{23}$$

It follows from $K_1(z) \equiv 0$ and (22)-(23) that

$$e^{\alpha(z)+\alpha(z+c)} - e^{\alpha(z+c)-\alpha(z)} - [e^{2\alpha(z)} - 1]e^{\Delta_c\beta(z)} \equiv 0. \tag{24}$$

Next, we consider two subcases.

Case 2.1. $\deg \Delta_c\beta(z) > \deg \alpha(z)$. Then it is easy to deduce that $e^{2\alpha(z)} = 1$, which implies that $\alpha(z)$ is a constant satisfying $e^{\alpha} = 1$, or $e^{\alpha} = -1$.

If $e^{\alpha} = 1$, then by the first equation of (3), we obtain $f(z) \equiv \Delta_c f(z)$.

If $e^{\alpha} = -1$, then it follows from (22)-(23) that

$$\begin{aligned}
2a(z) - b(z+c) &\equiv 0, \\
2a(z) - a(z+c) &\equiv 0.
\end{aligned}$$

Hence, we get $a(z) \equiv b(z)$, a contradiction.

Case 2.2. $\deg \Delta_c\beta(z) = \deg \alpha(z) \ge 1$.

Case 2.2.1. $\deg[2\alpha(z) - \Delta_c\beta(z)] = \deg\alpha(z)$ and $\deg[2\alpha(z) + \Delta_c\beta(z)] = \deg\alpha(z)$. Then (24) can be rewritten as follows.

$$\sum_{i=1}^{4} J_i(z)e^{g_i(z)} \equiv 0, \tag{25}$$

where

$$\begin{aligned} J_1(z) &= e^{\Delta_c\alpha(z)}, & g_1(z) &= 2\alpha(z), \\ J_2(z) &= -1, & g_2(z) &= \Delta_c\alpha(z), \\ J_3(z) &= -1, & g_3(z) &= 2\alpha(z) + \Delta_c\beta(z), \\ J_4(z) &= 1, & g_4(z) &= \Delta_c\beta(z). \end{aligned}$$

Obviously, for any $1 \le i < j \le 4$, $n = 1, 2, 3, 4$, we have

$$T(r, J_n) = o\left\{T(r, e^{g_i - g_j})\right\}.$$

Hence, it follows from (25) and Lemma 2.4 that $J_1(z) \equiv J_2(z) \equiv J_3(z) \equiv J_4(z) \equiv 0$. But $J_1(z) = e^{\Delta_c\alpha(z)}$, $J_2(z) = -1$, $J_3(z) = -1$, $J_4(z) = 1$, we get a contradiction.

Case 2.2.2. $\deg[2\alpha(z) - \Delta_c\beta(z)] < \deg\alpha(z)$. Let $2\alpha(z) - \Delta_c\beta(z) = -p_3(z)$, then $\Delta_c\beta(z) = 2\alpha(z) + p_3(z)$. So (24) can be rewritten as follows.

$$\sum_{i=1}^{3} N_i(z)e^{g_i(z)} \equiv 0, \tag{26}$$

where

$$\begin{aligned} N_1(z) &= e^{\Delta_c\alpha(z)} + e^{p_3(z)}, & g_1(z) &= 2\alpha(z), \\ N_2(z) &= -1, & g_2(z) &= \Delta_c\alpha(z), \\ N_3(z) &= -e^{p_3(z)}, & g_3(z) &= 4\alpha(z). \end{aligned}$$

Obviously, for any $1 \le i < j \le 3$, $n = 1, 2, 3$, we have

$$T(r, N_n) = o\left\{T(r, e^{g_i - g_j})\right\}.$$

Hence, it follows from (26) and Lemma 2.4 that $N_1(z) \equiv N_2(z) \equiv N_3(z) \equiv 0$. But $N_1(z) = e^{\Delta_c\alpha(z)} + e^{p_3(z)}$, $N_2(z) = -1$, $N_3(z) = -e^{p_3(z)}$, we get a contradiction.

Case 2.2.3. $\deg[2\alpha(z) + \Delta_c\beta(z)] < \deg\alpha(z)$. Let $2\alpha(z) + \Delta_c\beta(z) = p_4(z)$, then $\Delta_c\beta(z) = -2\alpha(z) + p_4(z)$. So (24) can be rewritten as follows.

$$\sum_{i=1}^{3} N_i(z)e^{g_i(z)} \equiv 0, \tag{27}$$

where

$$\begin{aligned}
N_1(z) &= e^{\Delta_c \alpha(z)}, & g_1(z) &= 2\alpha(z),\\
N_2(z) &= -e^{\Delta_c \alpha(z)} - e^{p_4(z)}, & g_2(z) &= 0,\\
N_3(z) &= e^{p_4(z)}, & g_3(z) &= -2\alpha(z).
\end{aligned}$$

Obviously, for any $1 \le i < j \le 3$, $n = 1, 2, 3$, we have

$$T(r, N_n) = o\left\{T(r, e^{g_i - g_j})\right\}.$$

Hence, it follows from (27) and Lemma 2.4 that $N_1(z) \equiv N_2(z) \equiv N_3(z) \equiv 0$. But $N_1(z) = e^{\Delta_c \alpha(z)}$, we get a contradiction.

Case 2.3. $\deg \Delta_c \beta(z) = \deg \alpha(z) = 0$. Then $\alpha(z)$ and $\Delta_c \beta(z)$ are two constants, and $\beta(z) = A_2 z + B_2, (A_2 \neq 0)$. So, by (22)-(23), we obtain

$$[2a(z) - a(z+c)](e^{\alpha} - 1) \equiv 0.$$

This implies that $e^{\alpha} = 1$, or $2a(z) - a(z+c) \equiv 0$.

If $e^{\alpha} = 1$, then it follows form the first equation of (3) that $f(z) \equiv \Delta_c f(z)$.

Next, we consider the case $2a(z) - a(z+c) \equiv 0$. In this case, we divide it into two subcases.

Case 2.3.1. $a(z) \not\equiv 0$. Then by Lemma 2.1, there exists $D_1 > 0$ such that $T(r, a(z)) \ge D_1 r$. On the other hand, by $\beta(z) = Az + B$, there exists $D_2 > 0$ such that $T(r, e^{\beta(z)}) = T(r, e^{Az+B}) \le D_2 r$. Hence, we have $T(r, a(z)) \ge \frac{D_1}{D_2} T(r, e^{\beta(z)})$, but it contradicts with (10).

Case 2.3.2. $a(z) \equiv 0$. Then it follows from (22) that $(e^{\alpha}+1)b(z) - b(z+c) \equiv 0$, obviously, $e^{\alpha} + 1 \neq 1$. Since $a(z) \not\equiv b(z)$, we have $b(z) \not\equiv 0$. Using the same argument as case 2.3.1, we get a contradiction.

Case 3. $\deg \alpha(z) = \deg \beta(z)$. Then (9) can be rewritten as follows.

$$\sum_{i=1}^{7} W_i(z) e^{g_i(z)} \equiv 0, \tag{28}$$

where

$$\begin{aligned}
W_1(z) &= [a(z) - b(z)]e^{\Delta_c \alpha(z)}, & g_1(z) &= 2\alpha(z) + \beta(z),\\
W_2(z) &= [a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z)}, & g_2(z) &= 2\alpha(z),\\
W_3(z) &= -[a(z) - b(z)]e^{\Delta_c \beta(z)}, & g_3(z) &= \alpha(z) + 2\beta(z),\\
W_4(z) &= [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)}, & g_4(z) &= 2\beta(z),\\
W_5(z) &= -[a(z) - a(z+c) + b(z)]e^{\Delta_c \alpha(z))} & g_5(z) &= \alpha(z) + \beta(z),\\
&\quad - [a(z) + b(z) - b(z+c)]e^{\Delta_c \beta(z)}, & &\\
W_6(z) &= [a(z+c) - b(z+c)] - [a(z) - b(z)]e^{\Delta_c \alpha(z)}, & g_6(z) &= \alpha(z),\\
W_7(z) &= [a(z) - b(z)]e^{\Delta_c \beta(z)} - [a(z+c) - b(z+c)], & g_7(z) &= \beta(z).
\end{aligned}$$

If $\deg[\alpha(z)-\beta(z)], \deg[\alpha(z)+\beta(z)], \deg[2\alpha(z)-\beta(z)], \deg[2\beta(z)-\alpha(z)]$ are all equal to $\deg\alpha(z)$. Then, for any $1 \le i < j \le 7$, $n = 1, 2, \ldots, 7$, we have $T(r, W_n) = o\{T(r, e^{g_i - g_j})\}$. Hence, it follows from (28) and Lemma 2.4 that $W_i(z) \equiv 0$ $(i = 1, 2, \ldots, 7)$. By $W_1(z) = (a(z) - b(z))e^{\Delta_c \alpha(z)} \equiv 0$, we get $a(z) \equiv b(z)$, a contradiction.

Next, we only need to discuss the cases that some of $\deg[\alpha(z) - \beta(z)]$, $\deg[\alpha(z) + \beta(z)]$, $\deg[2\alpha(z) - \beta(z)]$, $\deg[2\beta(z) - \alpha(z)]$ are less than $\deg\alpha(z)$.

Case 3.1. $\deg[\alpha(z) - \beta(z)] < \deg\alpha(z)$. Let $\alpha(z) - \beta(z) = -p_4(z)$, then $\beta(z) = \alpha(z) + p_4(z)$ and (28) can be rewritten as follows.

$$F_3(z)e^{2\alpha(z)} + F_2(z)e^{\alpha(z)} + F_1(z) \equiv 0, \tag{29}$$

where

$$\begin{aligned}
F_3(z) &= [a(z)-b(z)]e^{\Delta_c\alpha(z)+p_4(z)} - [a(z)-b(z)]e^{\Delta_c\beta(z)+2p_4(z)},\\
F_2(z) &= [a(z)-a(z+c)+b(z)]e^{\Delta_c\alpha(z)} + [a(z)+b(z)-b(z+c)]e^{\Delta_c\beta(z)+2p_4(z)}\\
&\quad - \Big\{[a(z)-a(z+c)+b(z)]e^{\Delta_c\alpha(z)}\\
&\qquad +[a(z)+b(z)-b(z+c)]e^{\Delta_c\beta(z)}\Big\}\, e^{p_4(z)},\\
F_1(z) &= a(z+c)-b(z+c)-[a(z)-b(z)]e^{\Delta_c\alpha(z)}\\
&\quad - \Big\{[a(z+c)-b(z+c)] - [a(z)-b(z)]e^{\Delta_c\beta(z)}\Big\}\, e^{p_4(z)}.
\end{aligned}$$

Obviously, for any $1 \le i < j \le 3$, $n = 1, 2, 3$, we have

$$T(r, F_n) = o\{T(r, e^{\alpha})\}.$$

Hence, it follows from (29) and Lemma 2.4 that $F_i(z) \equiv 0$ $(i = 1, 2, 3)$.

By $F_3 \equiv 0$, we get

$$e^{\Delta_c\alpha(z)} - e^{\Delta_c\beta(z)+p_4(z)} \equiv 0. \tag{30}$$

It follows from (30) and $F_1(z) \equiv 0$ that

$$[a(z+c)-b(z+c)][1-e^{p_4(z)}] \equiv 0.$$

Combing this with $a(z) \not\equiv b(z)$, we obtain that $e^{p_4(z)} = 1$, this implies that $e^{\alpha(z)} \equiv e^{\beta(z)}$, which contradicts with our assumption.

Case 3.2. $\deg[\alpha(z) + \beta(z)] < \deg\alpha(z)$. Let $\alpha(z) + \beta(z) = p_5(z)$, then $\beta(z) = -\alpha(z) + p_5(z)$ and (28) can be rewritten as follows.

$$G_2(z)e^{2\alpha(z)} + G_1(z)e^{\alpha(z)} + G_0(z) + G_{-1}e^{-\alpha(z)} + G_{-2}e^{-2\alpha(z)} \equiv 0, \tag{31}$$

where

$$\begin{aligned}
G_2(z) &= [a(z)-a(z+c)+b(z)]e^{\Delta_c\alpha(z)},\\
G_1(z) &= [a(z)-b(z)]e^{\Delta_c\alpha(z)+p_5(z)} + [a(z+c)-b(z+c)]\\
&\quad - [a(z)-b(z)]e^{\Delta_c\alpha(z)},
\end{aligned}$$

$$\begin{aligned} G_0(z) = & -\Big\{[a(z)-a(z+c)+b(z)]e^{\Delta_c\alpha(z)} \\ & +[a(z)+b(z)-b(z+c)]e^{\Delta_c\beta(z)}\Big\}\, e^{p_5(z)}, \\ G_{-1}(z) = & -[a(z)-b(z)]e^{\Delta_c\beta(z)+2p_5(z)}+\{[a(z)-b(z)]e^{\Delta_c\beta(z)} \\ & -[a(z+c)-b(z+c)]\}e^{p_5(z)}, \\ G_{-2}(z) = & \ [a(z)+b(z)-b(z+c)]e^{\Delta_c\beta(z)+2p_5(z)}. \end{aligned}$$

Obviously, for any $-2 \le i < j \le 2$, $n=-2,-1,\ldots,2$, we have

$$T(r,G_n)=o\{T(r,e^\alpha)\}.$$

Hence, it follows from (31) and Lemma 2.4 that $G_i(z)\equiv 0$ $(i=-2,1,\ldots,2)$. By $G_2\equiv 0$ and $G_{-2}\equiv 0$, we get $a(z)-a(z+c)+b(z)\equiv 0$ and $a(z)+b(z)-b(z+c)\equiv 0$, which implies $a(z)\equiv b(z)$, a contradiction.

Case 3.3. $\deg[2\alpha(z)-\beta(z)]<\deg\alpha(z)$. Let $2\alpha(z)-\beta(z)=-p_6(z)$, then $\beta(z)=2\alpha(z)+p_6(z)$ and (28) can be rewritten as follows.

$$\sum_{i=1}^{5} D_i(z)e^{i\alpha(z)}\equiv 0, \tag{32}$$

where

$$\begin{aligned} D_1(z) &= [a(z+c)-b(z+c)]-[a(z)-b(z)]e^{\Delta_c\alpha(z)}, \\ D_2(z) &= [a(z)-a(z+c)+b(z)]e^{\Delta_c\alpha(z)} \\ &\quad +\Big\{[a(z)-b(z)]e^{\Delta_c\beta(z)}-[a(z+c)-b(z+c)]\Big\}\, e^{p_6(z)}, \\ D_3(z) &= -\Big\{[a(z)-a(z+c)+b(z)])e^{\Delta_c\alpha(z)} \\ &\quad +[a(z)+b(z)-b(z+c)]e^{\Delta_c\beta(z)}\Big\}\, e^{p_6(z)}, \\ D_4(z) &= [a(z)-b(z)]e^{\Delta_c\alpha(z)+p_6(z)}+[a(z)+b(z)-b(z+c)]e^{\Delta_c\beta(z)+2p_6(z)}, \\ D_5(z) &= -[a(z)-b(z)]e^{\Delta_c\beta(z)+2p_6(z)}. \end{aligned}$$

Obviously, for any $1\le i<j\le 5$, $n=1,2,\ldots,5$, we have

$$T(r,D_n)=o\{T(r,e^\alpha)\}.$$

Hence, it follows from (32) and Lemma 2.4 that $D_i(z)\equiv 0$ $(i=1,2,\ldots,5)$. By $D_5\equiv 0$, we get $a(z)\equiv b(z)$, a contradiction.

Case 3.4. $\deg[2\beta(z)-\alpha(z)]<\deg\alpha(z)$. Let $2\beta(z)-\alpha(z)=-p_7(z)$, then $\alpha(z)=2\beta(z)+p_7(z)$ and (28) can be rewritten as follows.

$$\sum_{i=1}^{5} X_i(z)e^{i\beta(z)}\equiv 0, \tag{33}$$

where

$$X_1(z)=[a(z)-b(z)]e^{\Delta_c\beta(z)}-[a(z+c)-b(z+c)],$$

$$
\begin{aligned}
X_2(z) = {} & [a(z)+b(z)-b(z+c)]e^{\Delta_c\beta(z)} \\
& + \left\{[a(z+c)-b(z+c)]-[a(z)-b(z)]e^{\Delta_c\alpha(z)}\right\} e^{p_7(z)}, \\
X_3(z) = {} & -\left\{[a(z)-a(z+c)+b(z)]e^{\Delta_c\alpha(z)}\right. \\
& \left. +[a(z)+b(z)-b(z+c)]e^{\Delta_c\beta(z)}\right\} e^{p_7(z)}, \\
X_4(z) = {} & [a(z)-a(z+c)+b(z)]e^{\Delta_c\alpha(z)+2p_7(z)} - [a(z)-b(z)]e^{\Delta_c\beta(z)+p_7(z)}, \\
X_5(z) = {} & [a(z)-b(z)]e^{\Delta_c\alpha(z)+2p_7(z)}.
\end{aligned}
$$

Obviously, for any $1 \le i < j \le 5$, $n = 1, 2, \ldots, 5$, we have

$$T(r, X_n) = o\left\{T(r, e^{\beta})\right\}.$$

Hence, it follows from (33) and Lemma 2.4 that $X_i(z) \equiv 0$ $(i = 1, 2, \ldots, 5)$. By $X_5 \equiv 0$, we get $a(z) \equiv b(z)$, a contradiction.

Thus, Theorem 1.6 is proved. □

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