

ROBUST OPTIMAL PROPORTIONAL REINSURANCE AND INVESTMENT STRATEGY FOR AN INSURER WITH ORNSTEIN-UHLENBECK PROCESS

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ABSTRACT. This paper analyzes a robust optimal reinsurance and investment strategy for an Ambiguity-Averse Insurer (AAI), who worries about model misspecification and insists on seeking robust optimal strategies. The AAI's surplus process is assumed to follow a jump-diffusion model, and he is allowed to purchase proportional reinsurance or acquire new business, meanwhile invest his surplus in a risk-free asset and a risky-asset, whose price is described by an Ornstein-Uhlenbeck process. Under the criterion for maximizing the expected exponential utility of terminal wealth, robust optimal strategy and value function are derived by applying the stochastic dynamic programming approach. Several numerical examples are given to illustrate the impact of model parameters on the robust optimal strategies and the loss utility function from ignoring the model uncertainty.

1. Introduction

In recent years, there are a bulk of literatures on optimal reinsurance and/or investment problems with various objectives for insurers. To name a few, Browne [7], Yang and Zhang [16] and Bai and Guo [4] focused on utility maximization problems; Bäuerle [5], Zeng et al. [19], Zeng et al. [20] used mean-variance criteria. Schmidli [14], Bai and Guo [4] investigated the optimal reinsurance and investment problems to minimize the ruin probability. For more papers, see reference therein.

All the works mentioned above assumed that the insurer has complete confidence in the specific law of the motion of asset returns, including both the surplus process and the dynamics of the risky assets the insurer invest in, and their beliefs are represented by specific stochastic models under a single measure P . However, in reality, there is no consensus on which model should be used when studying optimal dynamic portfolio strategies. Rather than make

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ad-hoc decisions about how much error is contained in the estimates for the parameters of risky assets and surplus process of the insurance company, investors may consider alternative models that are close to the estimated model. This leads to the study of robust optimal control problems, where one seeks an optimal strategy among a family of possible situations, and the model uncertainty is characterized by a family of probability measures. Such method has been used successfully in quantitative finance. For example, Anderson et al. [1] introduced the concept of ambiguity-aversion and formulated a robust control problem for investors who worried about model uncertainty. Uppal and Wang [15] extended Anderson et al. [1] by allowing different degrees of ambiguity attitude toward different assets. Maenhout [10] innovated a ‘homothetic robustness’ framework who insisted that the level of ambiguity should be weighted by a state-dependent preference parameter. Maenhout [11] considered a dynamic portfolio and consumption problem consisting model uncertainty in the presence of a mean-reverting risk premium. Liu [9] extended Maenhout’s model [11] to recursive preferences.

For an ambiguity-averse insurer (we call him an AAI thereafter), he would also consider robust optimal control problems, and there has been a few papers on such topics. For example, Yi et al. [17] and Yi et al. [18] studied the problem of robust optimal reinsurance-investment for an AAI under the Heston’s stochastic volatility (HSV) model and Geometric Brownian Motion (GBM) model respectively. Zheng et al. [21] obtained the robust optimal reinsurance-investment strategy, where the stock price was modeled by a Constant Elasticity of Variance (CEV) model. In Zeng et al. [21], he supposed that the stock was described by a jump-diffusion (JD) model, and solved an robust equilibrium reinsurance-investment problem.

Apart from GBM, HSV, CEV and JD models, stochastic premium models are also good tools to describe the stock price (See Liang et al. [8] for more extensive review). In our paper, we assume that the instantaneous rate of the stock follows an Ornstein-Uhlenbeck process, which has been used in Rishel [13], Bai and Guo [3] and Liang et al. [8] for other control problems. As the growth rate of the stock is described by an Ornstein-Uhlenbeck process, which has mean-reverting property, this model can have features of bull and bear markets. The AAI is allowed to buy proportional reinsurance or acquire new business, meanwhile invest in the financial market. His objective is to maximize the exponential utility of the terminal wealth. Using the dynamic programming theory, we derive the explicit expressions for the robust control strategies and the corresponding value function.

The rest of the paper is organized as follows. In Section 2, we introduce the dynamics of the financial market and the surplus process of the insurance company, and formulate the robust optimal reinsurance and investment problem. In Section 3, using dynamic programming approach, we obtain closed-form expressions for the robust optimal strategies and value functions. Section 4 presents some numerical examples to illustrate our results. Section 5 concludes.

2. Model formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ be a filtered probability space, where $T > 0$ is a finite constant representing the investment horizon time. The filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ is the reference filtration generated by the following three stochastic processes: a compound Poisson process $C(t)$, two one-dimension standard Brownian Motion $W_1(t)$, $W_2(t)$. P is a reference measure.

2.1. Dynamics of surplus process

If both reinsurance and investment are absent, the insurer's surplus process $\{R(t)\}_{t \geq 0}$ is assumed to follow the *Crámer-Lundberg* model:

$$dR(t) = cdt - dC(t) = cdt - \int_0^\infty yN(dt, dy),$$

where $c > 0$ is the premium rate, $C(t)$ represents the cumulative claims up to time t . We assume that $C(t) := \sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process, where $\{N(t)\}$ is a homogeneous Poisson process with intensity $\lambda > 0$, and claim sizes Y_1, Y_2, \dots , independent of $\{N(t)\}$, are i.i.d positive random variables having common distribution with a generic random variable Y . Further we suppose that Y has finite first moment μ . We denote the moment generating function of Y as $M_Y(u) = Ee^{uY}$. $N(dt, dy)$ is the Poisson random measure corresponding to the compound Poisson process $C(t)$.

Suppose the insurer can control his insurance risk by purchasing proportional reinsurance or acquiring new business (by acting as a reinsurer of other insurers. See Bäuerle [5] with a retention level $q(t) \in [0, \infty)$ at time t .

- When $q(t) \in [0, 1]$, it corresponds to a proportional reinsurance cover, which means that at time t , if there was a claim Y_i happen, the insurer pays $q(t)Y_i$, and the remainder is paid by the reinsurer. Let $\delta(q(t))$ be the premium rate for the reinsurance. We assume that the reinsurance premium is calculated according to the expected value principle: $\delta(q(t)) = (1 + \eta)(1 - q(t))\lambda\mu$, where $\eta > 0$ is the relative safety loading of the reinsurer.
- When $q(t) \in [1, +\infty)$, it corresponds to acquiring a new business, e.g. acting as a reinsurer for other insurers, whose risks are independent and identically distributed to the original insurance business. Hence, the insurer's safety loading on the new business (i.e., the proportion of the risk exposure $q(t)$ over 1) is assumed to be η .

For the sake of simplicity, we call $\{q(t)\}_{t \in [0, T]}$ a reinsurance strategy hereafter. Then the surplus process under such a reinsurance strategy can be described as:

$$(2.1) \quad dU(t) = [c - \delta(q(t))]dt - \int_0^\infty q(t)yN(dt, dy).$$

2.2. Dynamics of financial securities

We consider a financial market consisting of one risk-free asset (e.g. a bond) and one risky asset (e.g. a stock). We further make the standard assumption that the market is frictionless. All assets can be traded continuously over a finite time interval $[0, T]$, and no traction costs or taxes are involved in trading.

The price process of the risk-free asset is given by:

$$(2.2) \quad dS^0(t) = rS^0(t)dt,$$

where $r > 0$ is the risk-free interest rate. The price process of the risky asset is described by:

$$(2.3) \quad \begin{cases} dS(t) = S(t)[(\bar{r} + m(t))dt + \sigma d\bar{W}_1(t)], \\ dm(t) = am(t)dt + b d\bar{W}_2(t), \end{cases}$$

where \bar{r} , σ , a , b are known constants, and are all positive except a and b . $\bar{W}_1(t)$ and $\bar{W}_2(t)$ are two Brownian Motions with correlation coefficient ρ , i.e., $E[\bar{W}_1(t)\bar{W}_2(t)] = \rho t$.

To obtain independent Brownian Motions, we use Cholesky decomposition here:

$$(2.4) \quad \begin{cases} \bar{W}_1(t) := W_1(t), \\ \bar{W}_2(t) := \rho W_1(t) + \sqrt{1 - \rho^2} W_2(t), \end{cases}$$

where $W_1(t)$ and $W_2(t)$ are two independent standard Brownian Motions. Then the dynamics of the risky asset can be rewrite as:

$$(2.5) \quad \begin{cases} dS(t) = S(t)[(\bar{r} + m(t))dt + \sigma dW_1(t)], \\ dm(t) = am(t)dt + b\rho dW_1(t) + b\sqrt{1 - \rho^2} dW_2(t). \end{cases}$$

This model has been used in Rishel [13], Bai and Guo [3] and Liang et al. [8] for other control problems. As the growth rate of the stock is described by an Ornstein-Uhlenbeck, which has mean-reverting property, this model can have features of bull and bear markets. If there is a period for which $m(t)$ is substantially larger than 0, then this could be considered as a bull market. Conversely, when $m(t)$ is substantially less than 0, this could be considered as a bear market.

2.3. The wealth process

In addition to reinsurance, we assume that the insurer is allowed to invest all his surplus in the financial market defined above. Let $\alpha(t)$ denote the total amount of wealth invested in the risky asset at time t , the remainder amount is invested in the risk-free asset. The insurer's trading strategy is therefore a two-dimensional stochastic process $\pi(t) = (q(t), \alpha(t))$, where $q(t)$ represents the value of risk exposure as described above. The wealth process subjected to

this choice is denoted by $X^\pi(t)$, then its dynamics is given by:

$$\begin{aligned}
 dX^\pi(t) &= \frac{X^\pi(t) - \alpha(t)}{S^0(t)} dS^0(t) + \frac{\alpha(t)}{S(t)} dS(t) + (c - \delta(q(t)))dt \\
 &\quad - \int_0^\infty q(t)yN(dt, dy), \\
 (2.6) \quad &= \{rX^\pi(t) + \alpha(t)[\bar{r} - r + m(t)] + c - \delta(q(t))\}dt \\
 &\quad + \sigma\alpha(t)dW_1(t) - \int_0^\infty q(t)yN(dt, dy).
 \end{aligned}$$

2.4. Robust control problem for an AAI

In the following, we suppose that the insurer has a CARA utility function $U(x) = -\frac{1}{v} \exp\{-vx\}$, and aims to maximize the expected utility of terminal wealth at time T , i.e., $\max_{(q, \alpha) \in \Pi} E^P[U(X^\pi(T))]$. The insurer is always assumed to be an ambiguity-neutral investor (ANI). However, large number of insurers are ambiguity-averse investors (AAI) in reality, which means that although the AAI takes the model under measure P as his reference model, he recognizes that it is only an approximation of the true model, and he is willing to consider other alternative models, which can be represented by another probability measure Q which is equivalent to the original measure P . In other words, he considers all Q in the set of probability measures \mathcal{Q} defined by:

$$(2.7) \quad \mathcal{Q} = \{Q \mid Q \sim P\}.$$

Following Zheng et al. [21], we construct the set of alternative measures in the following way. Suppose $\theta(t) := (\theta_1(t), \theta_2(t), \theta_3(t))$ is a $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted process such that:

- (i) $\theta_1(t), \theta_2(t), \theta_3(t)$ are \mathcal{F} -progressively measurable, for each fixed $t \in [0, T]$;
- (ii) $\theta_1(t), \theta_2(t), \theta_3(t) > 0$ for a.e. $(t, \omega) \in [0, T] \times \Omega$;
- (iii) $E\{\frac{1}{2} \int_0^T \theta_1^2(t)dt + \frac{1}{2} \int_0^T \theta_2^2(t)dt + \lambda \int_0^T [\theta_3(t) \ln \theta_3(t) - \theta_3(t)]dt\} < \infty$.

We denote Θ for the space of all such processes θ . Define:

$$\begin{aligned}
 \Lambda^\theta(t) &= \exp\left\{-\int_0^t \theta_1(u)dW_1(u) - \frac{1}{2} \int_0^t \theta_1^2(u)du\right\} \\
 &\quad \cdot \exp\left\{-\int_0^t \theta_2(u)dW_2(u) - \frac{1}{2} \int_0^t \theta_2^2(u)du\right\} \\
 &\quad \cdot \exp\left\{\int_0^t \int_0^\infty \ln \theta_3(u)N(du, dy) + \lambda \int_0^t (1 - \theta_3(u))du\right\},
 \end{aligned}$$

and let Q be the probability measure defined by

$$(2.8) \quad \frac{dQ}{dP} \Big|_{\mathcal{F}_T} = \Lambda^\theta(T).$$

Then according to Girsanov's Theorem, the processes

$$\{W_1^Q(t)\}_{t \in [0, T]}, \quad \{W_2^Q(t)\}_{t \in [0, T]}$$

defined by

$$(2.9) \quad W_1^Q(t) = W_1(t) + \int_0^t \theta_1(u) du,$$

and

$$(2.10) \quad W_2^Q(t) = W_2(t) + \int_0^t \theta_2(u) du,$$

are two independent standard Brownian Motions under the measure Q , and $N(t)$ is a Poisson process with new intensity $\lambda^Q(t) = \lambda\theta_3(t)$.

For tractability and ease of interpretation, the distribution of jump size Y_i are assumed to be known and are restricted to be identical under P and Q . Then we rewrite the risky asset prices and the wealth process in terms of these new processes, to find: the dynamics of the risky asset can be described as:

$$(2.11) \quad \begin{cases} dS(t) = S(t)\{\bar{r} + m(t) - \sigma\theta_1(t)\}dt + \sigma dW_1^Q(t), \\ dm(t) = [am(t) - b\rho\theta_1(t) - b\sqrt{1-\rho^2}\theta_2(t)]dt + b\rho dW_1^Q(t) \\ \quad + b\sqrt{1-\rho^2}dW_2^Q(t). \end{cases}$$

Then for a given admissible strategy $\pi(t) = (q(t), \alpha(t))$, the wealth process will satisfy the following dynamics:

$$\begin{aligned} dX^\pi(t) = & \{rX^\pi(t) + \alpha(t)[\bar{r} - r + m(t) - \sigma\theta_1(t)] + c - \delta(q(t))\}dt \\ & - \lambda\theta_3(t)q(t)\mu_1 dt + \sigma\alpha(t)dW_1^Q(t) - \int_0^\infty q(t)y\tilde{N}^Q(dt, dy), \end{aligned}$$

where $\tilde{N}^Q(dt, dy) := N(dt, dy) - \lambda\theta_3(t)dtdF(y)$ is a compensated Poisson random measure. We notice that the wealth process under the alternative model in the class \mathcal{Q} differs only in the drift term as it should.

Assume that the insurer seeks a robust optimal control, which is the best choice under some worst-case model. Following Anderson et al. [2] and Maenhout [10], we shall modify the original utility maximization problem as follows: we first propose the following definition for the set of admissible strategies:

Definition 2.1. For any fixed $t \in [0, T]$, a strategy $\pi(t) = (q(t), \alpha(t))$ is said to be admissible if

- (i) $(q(t), \alpha(t))$ are \mathcal{F} -progressively measurable, and $q(t), \alpha(t) \geq 0$ for a.e. $(t, \omega) \in [0, T] \times \Omega$.
- (ii) $\int_0^T E^{Q^*}[q^2(t) + \alpha^2(t)]dt < \infty$.
- (iii) SDE(2.6) has a unique strong solution,

where Q^* is the chosen model to describe the worst case and will be specified later.

We denote the set of all admissible strategies by Π .

The value function is defined as:

$$(2.12) \quad V(t, x, m) := \sup_{\pi \in \Pi} \inf_{Q \in \mathcal{Q}} E_{t,x,m}^Q [U(X^\pi(T)) + \int_t^T \psi(u, X^\pi(u), \theta(u))],$$

where $E_{t,x,m}^Q[\cdot] = E^Q[\cdot \mid X_t = x, m_t = m]$,

$$(2.13) \quad \psi(t, X^\pi(t), \theta(t)) := \frac{\theta_1^2(t)}{2\phi_1(t, X^\pi(t))} + \frac{\theta_2^2(t)}{2\phi_2(t, X^\pi(t))} + \frac{\lambda[\theta_3(t) \ln \theta_3(t) - \theta_3(t) + 1]}{\phi_3(t, X^\pi(t))}.$$

$\psi(t, X^\pi(t), \theta(t))$ is called the penalization term, measuring the discrepancy between the probability measure P and Q , and the penalty factors $\{\phi_i\}_{i=1,2,3}$ are nonnegative functions of time and wealth, whose representations will be specified later.

In fact, this form of penalization term depends on the relative entropy of two different probability measures P and Q , which is defined as:

$$(2.14) \quad KL(Q, P) = E^Q[\ln \frac{dQ}{dP}],$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative corresponding to this measure change.

Similar as Branger and Larsen [6], we can show that the increase in relative entropy from t to $t + dt$ equals:

$$(2.15) \quad \frac{1}{2}\theta_1^2(t)dt + \frac{1}{2}\theta_2^2(t)dt + \lambda[\theta_3(t) \ln \theta_3(t) - \theta_3(t) + 1]dt.$$

In the penalty term (2.13), three terms in (2.15) are scaled by ϕ_1 , ϕ_2 and ϕ_3 , which stand for preference for ambiguity aversion with respect to diffusion risk and jump risk respectively.

Let $C^{1,2,2}([0, T] \times R \times R) = \{\varphi(t, x, m) \mid \varphi(t, \cdot, \cdot)$ is continuously differentiable on $[0, T]$, and $\varphi(\cdot, x, \cdot)$, $\varphi(\cdot, \cdot, m)$ are twice continuously differentiable on $R\}$.

Let $\mathcal{A}^{\pi, \theta}$ be the infinitesimal generator applied to the value function V and is defined by:

$$(2.16) \quad \begin{aligned} \mathcal{A}^{\pi, \theta} V(t, x, m) = & V_t(t, x, m) + \{rx + \alpha[\bar{r} - r + m - \sigma\theta_1] + c - \delta(q)\}V_x(t, x, m) \\ & + \frac{1}{2}\sigma^2\alpha^2V_{xx}(t, x, m) + (am - b\rho\theta_1 - b\sqrt{1 - \rho^2}\theta_2)V_m(t, x, m) \\ & + \frac{1}{2}b^2V_{mm}(t, x, m) + b\rho\sigma\alpha V_{xm}(t, x, m) \\ & + \lambda\theta_3 E[V(t, x - qY, m) - V(t, x, m)]. \end{aligned}$$

From standard arguments of dynamic programming approach, we see that if $V(t, x, m) \in C^{1,2,2}([0, T] \times R \times R)$, then $V(t, x, m)$ satisfies the following HJBI

equations:

$$(2.17) \quad \sup_{\pi \in \Pi} \inf_{Q \in \mathcal{Q}} \{ \mathcal{A}^{\pi, \theta} V(t, X^\pi(t), m) + \psi(t, X^\pi(t), \theta(t)) \} = 0, \\ V(T, x, m) = U(x).$$

The verification theorem is presented as follows:

Theorem 2.1 (Verification Theorem). *Suppose there exist a function $\varphi(t, x, m) \in C^{1,2,2}([0, T] \times R \times R)$, and a Markov control $(\theta^*, \pi^*) \in \Theta \times \Pi$, such that:*

- (i) $\mathcal{A}^{\pi, \theta^*} \varphi(t, X^\pi(t), m(t)) + \psi(t, X^\pi(t), \theta^*(t)) \leq 0$ for any $\pi \in \Pi$.
- (ii) $\mathcal{A}^{\pi^*, \theta} \varphi(t, X^{\pi^*}(t), m(t)) + \psi(t, X^{\pi^*}(t), \theta(t)) \geq 0$ for any $\theta \in \Theta$.
- (iii) $\mathcal{A}^{\pi^*, \theta^*} \varphi(t, X^{\pi^*}(t), m(t)) + \psi(t, X^{\pi^*}(t), \theta^*(t)) = 0$.
- (iv) $\{\varphi(\tau, X^{\pi^*}(\tau), m(\tau))\}_{\tau \in \mathcal{F}}$, and $\{\psi(\tau, X^{\pi^*}(\tau), \theta^*(\tau))\}_{\tau \in \mathcal{F}}$ are uniformly integrable, where τ denotes the set of stopping times $\tau \leq T$.
Then, $\varphi(t, x, m) = V(t, x, m)$, and (θ^*, π^*) is an optimal Markov control.

The proof of the verification theorem can be adapted from Theorem 3.2 in Mataramvura and Øksendal [12], so we omit it here.

3. Solution to the model

In this section, we try to solve HJBI equation (2.17).

For analytical tractability, we follow Maenhout [10] and choose:

$$(3.18) \quad \phi_k = \frac{-\beta_k}{vV(t, x, m)}, \quad k = 1, 2, 3,$$

where β_k are nonnegative parameters ($\beta_k = 0$ correspond to the expected utility maximization) reflecting the insurer's ambiguity aversion with respect to diffusion and jump risks.

First, we conjecture that the solution has the following form:

$$(3.19) \quad V(t, x, m) = -\frac{1}{v} \exp[-vxe^{r(T-t)} + G(t, m)],$$

where $G(t, m)$ is a function satisfying the terminal condition $G(T, m) = 0$.

From (3.19), we get:

$$(3.20) \quad \begin{cases} V_t(t, x, m) = V(t, x, m)[rxve^{r(T-t)} + G_t], \\ V_x(t, x, m) = V(t, x, m)[-ve^{r(T-t)}], \\ V_{xx}(t, x, m) = V(t, x, m)[v^2e^{2r(T-t)}], \\ V_m(t, x, m) = V(t, x, m)G_m, \\ V_{mm}(t, x, m) = V(t, x, m)(G_m^2 + G_{mm}), \\ V_{xm}(t, x, m) = V(t, x, m)G_m(-ve^{r(T-t)}), \\ E[V(t, x - qY, m) - V(t, x, m)] = V(t, x, m)[M_Y(qve^{r(T-t)}) - 1]. \end{cases}$$

For fixed strategy $\pi = (q, \alpha)$, substituting (3.18), (3.19) and (3.20) into (2.17), and applying the first-order conditions with respect to θ yields:

$$(3.21) \quad \begin{cases} \theta_1^* = \beta_1 \alpha \sigma e^{r(T-t)} - \frac{\beta_1 b \rho}{v} G_m, \\ \theta_2^* = -\frac{\beta_2 b \sqrt{1-\rho^2}}{v} G_m, \\ \theta_3^* = \exp\left\{\frac{\beta_3}{v} [M_Y(qv e^{r(T-t)}) - 1]\right\} := \exp\left\{\frac{\beta_3}{v} H(q, t)\right\}. \end{cases}$$

Then putting back $\{\theta_i^*\}_{i=1,2,3}$ yields:

$$(3.22) \quad \begin{aligned} 0 = & G_t + amG_m + \frac{\beta_1 b^2 \rho^2}{2v} G_m^2 + \frac{\beta_2 b^2 (1-\rho^2)}{2v} G_m \\ & + \frac{b^2}{2} G_m^2 + \frac{b^2}{2} G_{mm} - \frac{\lambda v}{\beta_3} - cve^{r(T-t)} \\ & + \inf_q \{f_1(q, t)\} + \inf_\alpha \{f_2(\alpha, t)\}, \end{aligned}$$

where

$$(3.23) \quad \begin{cases} f_1(q, t) = \delta(q)ve^{r(T-t)} + \frac{\lambda v}{\beta_3} \exp\left[\frac{\beta_3}{v} H(q, t)\right], \\ f_2(\alpha, t) = \frac{1}{2} \alpha^2 \sigma^2 (\beta_1 + v) e^{r(T-t)} - \alpha[\bar{r} - r + m + \frac{\sigma b \rho (\beta_1 + v)}{v} G_m]. \end{cases}$$

For the existence and uniqueness of the minimizer of $f_1(q, t)$, we have the following lemma:

Lemma 3.1. *For any $t \in [0, T]$, $f_1(q, t)$ has a unique minimizer $\bar{q}(t) \in (0, \infty)$.*

Proof. Putting the expressions of $\delta(q)$ and $H(q, t)$ into (3.23), direct calculation yields:

$$(3.24) \quad \begin{aligned} \frac{\partial f_1(q, t)}{\partial q} &= \lambda v e^{r(T-t)} \left\{ \exp\left[\frac{\beta_3}{v} H(q, t)\right] \cdot E[Y \exp(qv Y e^{r(T-t)})] - \mu(1 + \eta) \right\}, \\ \frac{\partial^2 f_1(q, t)}{\partial q^2} &= \lambda \exp\left[\frac{\beta_3}{v} H(q, t)\right] \left\{ \frac{\beta_3}{v} \cdot \left[\frac{\partial H(q, t)}{\partial q} \right]^2 \right. \\ &\quad \left. + v^2 e^{2r(T-t)} \cdot E[Y^2 \exp(qv Y e^{r(T-t)})] \right\} > 0. \end{aligned}$$

It is easy to know that $\lim_{q \rightarrow \infty} H(q, t) = \infty$ and $\lim_{q \rightarrow 0} H(q, t) = 0$. So we have $\lim_{q \rightarrow \infty} \frac{\partial f_1(q, t)}{\partial q} > 0$ and $\lim_{q \rightarrow 0} \frac{\partial f_1(q, t)}{\partial q} = -\lambda v e^{r(T-t)} \mu \eta < 0$.

Based on (3.24), we know that $\frac{\partial f_1(q, t)}{\partial q}$ is strictly increasing in q , therefore there exists a unique $\bar{q}(t) \in (0, \infty)$ such that $\frac{\partial f_1(q, t)}{\partial q} = 0$. \square

From Lemma 3.1, we obtain

$$(3.25) \quad q^*(t) = \bar{q}(t).$$

Applying the first-order conditions with respect to α yields:

$$(3.26) \quad \alpha^*(t) = \frac{\bar{r} - r + m + \frac{(\beta_1 + v)}{v} \sigma b \rho G_m}{\sigma^2 e^{r(T-t)} (\beta_1 + v)}.$$

Plugging $(q^*(t), \alpha^*(t))$ back to (3.5), we obtain:

$$(3.27) \quad \begin{aligned} 0 = & G_t + a m G_m + \frac{\beta_1 b^2 \rho^2}{2v} G_m^2 + \frac{\beta_2 b^2 (1 - \rho^2)}{2v} G_m \\ & + \frac{b^2}{2} G_m^2 + \frac{b^2}{2} G_{mm} - \frac{\lambda v}{\beta_3} - c v e^{r(T-t)} \\ & + \delta(q^*) v e^{r(T-t)} + \frac{\lambda v}{\beta_3} \exp\left\{\frac{\beta_3}{v} H(q^*, t)\right\} \\ & - \frac{v[\bar{r} - r + m + \frac{(\beta_1 + v)}{v} \sigma b \rho G_m]^2}{2\sigma^2(\beta_1 + v)}. \end{aligned}$$

To solve above equation, we further assume that $G(t, m) = K(t)m^2 + J(t)m + L(t)$, therefore the terminal condition $G(T, m) = 0$ implies $K(T) = 0$, $J(T) = 0$ and $L(T) = 0$, and we have:

$$(3.28) \quad \begin{cases} G_t = K'(t)m^2 + J'(t)m + L'(t), \\ G_m = 2K(t)m + J(t), \\ G_{mm} = 2K(t). \end{cases}$$

Putting G_t, G_m, G_{mm} into (3.22), then after some algebra simplifications, and grouping the coefficients according to the power of m , we obtain:

$$(3.29) \quad \begin{aligned} 0 = & [K'(t) + 2b^2(1 - \rho^2)K^2(t) + 2(a - \frac{b\rho}{\sigma})K(t) - \frac{v}{2\sigma^2(\beta_1 + v)}]m^2 \\ & + [J'(t) + (a - \frac{b\rho}{\sigma})J(t) + 2b^2(1 - \rho^2)K(t)J(t) \\ & + [\frac{\beta_2 b^2 (1 - \rho^2)}{v} - \frac{2b\rho(\bar{r} - r)}{\sigma}]K(t) - \frac{v(\bar{r} - r)}{\sigma^2(\beta_1 + v)}]m \\ & + L'(t) + \frac{1}{2}b^2(1 - \rho^2)J^2(t) + [\frac{\beta_2 b^2 (1 - \rho^2)}{2v} - \frac{b\rho(\bar{r} - r)}{\sigma}]J(t) \\ & + b^2 K(t) + M(t), \end{aligned}$$

where

$$(3.30) \quad M(t) = -\frac{\lambda v}{\beta_3} - c v e^{r(T-t)} - \frac{v(\bar{r} - r)^2}{2\sigma^2(\beta_1 + v)} + f_1(q^*(t), t).$$

Because this equation must hold for all m , the coefficient of m^2 and m must be zero. Otherwise, changing the value of m would change the value of the right hand side of equation (3.29), and hence it could not always be equal to

zero. This gives us the following three ordinary differential equations:

$$(3.31) \quad \begin{cases} K'(t) + 2b^2(1 - \rho^2)K^2(t) + 2(a - \frac{b\rho}{\sigma})K(t) - \frac{v}{2\sigma^2(\beta_1 + v)} = 0, \\ J'(t) + (a - \frac{b\rho}{\sigma})J(t) + 2b^2(1 - \rho^2)K(t)J(t) \\ + [\frac{\beta_2 b^2(1 - \rho^2)}{v} - \frac{2b\rho(\bar{r} - r)}{\sigma}]K(t) - \frac{v(\bar{r} - r)}{\sigma^2(\beta_1 + v)} = 0, \\ L'(t) + \frac{1}{2}b^2(1 - \rho^2)J^2(t) + [\frac{\beta_2 b^2(1 - \rho^2)}{2v} - \frac{b\rho(\bar{r} - r)}{\sigma}]J(t) \\ + b^2K(t) + M(t) = 0, \end{cases}$$

with terminal conditions $K(T) = 0$, $J(T) = 0$ and $L(T) = 0$.

For $K(t)$, the related equation is a kind of Riccati equation, having the following standard form:

$$(3.32) \quad K'(t) + AK^2(t) + BK(t) + C = 0.$$

with:

$$(3.33) \quad A = 2b^2(1 - \rho^2), \quad B = 2(a - \frac{b\rho}{\sigma}), \quad C = -\frac{v}{2\sigma^2(\beta_1 + v)}.$$

As obviously $B^2 - 4AC > 0$, the solution of the Riccati equation (3.15) fits the form:

$$(3.34) \quad K(t) = \frac{\sqrt{B^2 - 4AC}}{2A} \tanh[\frac{1}{2}\sqrt{B^2 - 4AC}(t - T) + \operatorname{arctanh}(\frac{B}{\sqrt{B^2 - 4AC}})] - \frac{B}{2A}.$$

Then for $J(t)$ and $L(t)$, the corresponding equations are first order linear ordinary differential equations, so we can easily deduce their solutions:

$$(3.35) \quad J(t) = e^{\int_t^T p(s)ds} [-\int_t^T h(s)e^{-\int_s^T p(u)du} ds],$$

where

$$(3.36) \quad \begin{aligned} p(t) &= a - \frac{b\rho}{\sigma} + 2b^2(1 - \rho^2)K(t), \\ h(t) &= [\frac{\beta_2 b^2(1 - \rho^2)}{v} - \frac{2b\rho(\bar{r} - r)}{\sigma}]K(t) - \frac{v(\bar{r} - r)}{\sigma^2(\beta_1 + v)}, \end{aligned}$$

and

$$(3.37) \quad L(t) = \int_t^T \{ \frac{1}{2}b^2(1 - \rho^2)J^2(s) + [\frac{\beta_2 b^2(1 - \rho^2)}{2v} - \frac{b\rho(\bar{r} - r)}{\sigma}]J(s) + b^2K(s) + M(s) \} ds,$$

where $M(t)$ is defined in (3.30).

Now we summarize the above analysis in the following theorem:

Theorem 3.1. *For the robust control problem (2.12), the optimal proportional reinsurance and investment strategies are given by:*

$$(3.38) \quad \begin{cases} q^*(t) = \bar{q}(t), \\ \alpha^*(t) = \frac{\bar{r} - r + m + \frac{(\beta_1 + v)}{v} \sigma b \rho [2K(t)m + J(t)]}{\sigma^2 e^{r(T-t)} (\beta_1 + v)} \end{cases}$$

and the corresponding value function is given by:

$$(3.39) \quad V(t, x, m) = -\frac{1}{v} \exp[-vxe^{r(T-t)} + K(t)m^2 + J(t)m + L(t)].$$

The worst case measure is given by:

$$(3.40) \quad \begin{cases} \theta_1^* = \beta_1 \alpha \sigma e^{r(T-t)} - \frac{\beta_1 b \rho}{v} [2K(t)m + J(t)], \\ \theta_2^* = -\frac{\beta_2 b \sqrt{1 - \rho^2}}{v} [2K(t)m + J(t)], \\ \theta_3^* = \exp\left\{\frac{\beta_3}{v} [M_Y(q^* v e^{r(T-t)}) - 1]\right\}, \end{cases}$$

where $K(t)$, $J(t)$ and $L(t)$ are given by equations (3.34), (3.35) and (3.37), respectively.

Remark 3.2. For an ambiguity-neutral insurer (ANI) in the same financial market, who considers a similar optimization problem, we denote the set of all admissible strategies as $\tilde{\Pi} = \{\tilde{\pi}(t) \mid \tilde{\pi}(t) = (\tilde{q}(t), \tilde{\alpha}(t)), t \in [0, T]\}$. His wealth process under the reference measure P will be described as:

$$\begin{aligned} dX^{\tilde{\pi}}(t) &= \{rX^{\tilde{\pi}}(t) + \alpha(t)[\bar{r} - r + m(t)] + c - \delta(q(t))\}dt \\ &\quad + \sigma \alpha(t) dW_1(t) - \int_0^\infty q(t)yN(dt, dy), \end{aligned}$$

and the objective function is:

$$\begin{aligned} \tilde{V}(t, x, m) &:= \sup_{\tilde{\pi} \in \tilde{\Pi}} E_{t,x,m}[U(X^{\tilde{\pi}}(T))] \\ &= \sup_{\tilde{\pi} \in \tilde{\Pi}} E_{t,x,m}\left\{-\frac{1}{v} \exp[-vX^{\tilde{\pi}}(T)] \mid X_t = x, m_t = m\right\}. \end{aligned}$$

Using the same technique, we obtain the value function:

$$(3.41) \quad \tilde{V}(t, x, m) = -\frac{1}{v} \exp[-vxe^{r(T-t)} + \tilde{K}(t)m^2 + \tilde{J}(t)m + \tilde{L}(t)],$$

where $\tilde{K}(t)$, $\tilde{J}(t)$ and $\tilde{L}(t)$ are given by three ordinary differential equations:

$$\begin{cases} \tilde{K}'(t) + 2b^2(1 - \rho^2)\tilde{K}^2(t) + 2(a - \frac{b\rho}{\sigma})\tilde{K}(t) - \frac{1}{2\sigma^2} = 0, \\ \tilde{J}'(t) + (a - \frac{b\rho}{\sigma})\tilde{J}(t) + 2b^2(1 - \rho^2)\tilde{K}(t)\tilde{J}(t) - \frac{2b\rho(\bar{r} - r)}{\sigma}\tilde{K}(t) - \frac{\bar{r} - r}{\sigma^2} = 0, \\ \tilde{L}'(t) + \frac{1}{2}b^2(1 - \rho^2)\tilde{J}^2(t) - \frac{b\rho(\bar{r} - r)}{\sigma}\tilde{J}(t) + b^2\tilde{K}(t) + \tilde{M}(t) = 0, \end{cases}$$

with terminal conditions $\tilde{K}(T) = 0$, $\tilde{J}(T) = 0$ and $\tilde{L}(T) = 0$.
and

$$\begin{cases} \tilde{M}(t) = -cve^{r(T-t)} - \lambda - \frac{(\bar{r} - r)^2}{2\sigma^2} + \tilde{f}_1(\tilde{q}^*(t), t), \\ \tilde{f}_1(\tilde{q}(t), t) = \delta(\tilde{q})ve^{r(T-t)} + \lambda M_Y(\tilde{q}ve^{r(T-t)}), \end{cases}$$

where $\tilde{q}^*(t)$ is the unique minimizer of $\tilde{f}_1(\tilde{q}(t), t)$ over $(0, +\infty)$.

4. Numerical implications

In this section, we provide several numerical examples to illustrate the effects of model parameters on our robust optimal reinsurance and investment strategies. In the following analysis, unless otherwise stated, the basic parameters are given in Table 1. Further we assume that the claim Y_i follows exponential distribution with parameter $m_1 = 0.05$.

TABLE 1. Values of model parameters in the numerical examples

Parameters	r	\bar{r}	a	b	σ	ρ	η	T	t	v	β_1	β_2	β_3
Value	0.03	0.08	-0.7	0.5	0.3	0.5	1.2	2	0	0.5	0.8	0.8	0.8

4.1. Sensitivity analysis of the robust optimal investment strategy

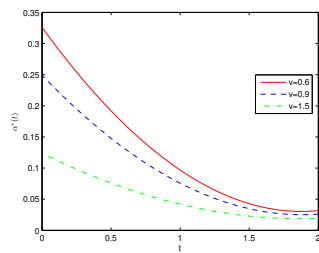


FIGURE 1. The effect of v on $\alpha^*(t)$.

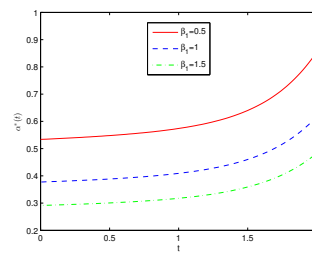


FIGURE 2. The effect of β_1 on $\alpha^*(t)$.

Fig. 1 reflects that the more risk averse the AAI is, the less amount of his wealth will be invested in the stock. As is shown in Fig. 2, $\alpha^*(t)$ decreases with respect to β_1 . Because the higher β_1 is, the less confidence the AAI has in the reference model, so he will reduce the investment on the stock.

4.2. Sensitivity analysis of the robust optimal reinsurance strategy

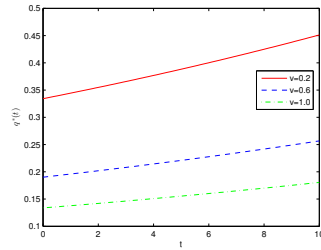


FIGURE 3. The effect of v on $q^*(t)$.

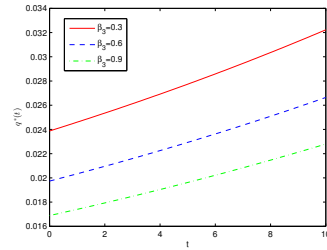


FIGURE 4. The effect of β_3 on $q^*(t)$.

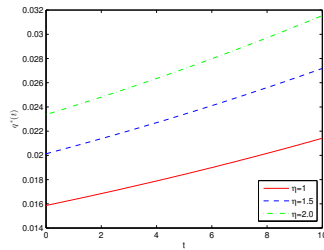


FIGURE 5. The effect of η on $q^*(t)$.

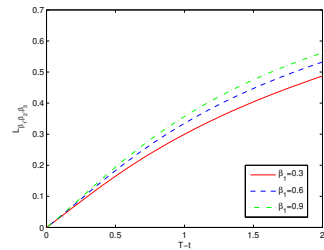


FIGURE 6. The effect of β_1 on $L_{\beta_1, \beta_2, \beta_3}$.

In this subsection, in order to show the relationships more clearly, we prolong the time horizon T to 10. Fig. 3 shows the effect of the risk aversion coefficient v on the robust optimal reinsurance strategy $q^*(t)$. Clearly $q^*(t)$ is a decreasing function of v , which means that the more risk averse the AAI is, the lower risk retention level he will choose, which means that the less risk he would like to undertake by himself. This is also in accordance with people's intuition. From Fig. 4, we find that the AAI with higher ambiguity aversion level β_3 is prone to purchasing more reinsurance. As β_3 reflects the AAI's attitude towards the jump risk, the higher β_3 is, he would like to cede more risks to the reinsurer. Fig. 5 indicates how the reinsurance safety loading parameter η influent the AAI's decision. As the larger η is, the more reinsurance premium he would pay to the reinsurer, then he prefers to raise the retention level.

4.3. Utility loss analysis

Compared with the AAI with ambiguity-aversion coefficients β_1 , β_2 and β_3 , the utility loss function from ignoring the model uncertainty can be defined as:

$$(4.42) \quad L_{\beta_1, \beta_2, \beta_3} = 1 - \frac{\tilde{V}(t, x, m)}{V(t, x, m)},$$

where \tilde{V} is given by (3.41) and V is given by Theorem 3.1.

We show the effects of three ambiguity aversion coefficients $\beta_1, \beta_2, \beta_3$ on the loss utility function $L_{\beta_1, \beta_2, \beta_3}$ in Figs. 6-8. As is shown in Fig. 7, the effect of β_2 on $L_{\beta_1, \beta_2, \beta_3}$ is not so obvious as β_1 and β_3 , implying that $L_{\beta_1, \beta_2, \beta_3}$ is more sensitive to the diffusion risk of the risky asset's price and the jump risk of the surplus process. On the other hand, $L_{\beta_1, \beta_2, \beta_3}$ increases when $\beta_1, \beta_2, \beta_3$ increases respectively, which means that the more ambiguity aversion the AAI is, the more conservative strategies he will choose, and then the more utility loss the AAI will suffer. From Figs. 6-8, we can also see that the utility loss is a increasing function of the remaining time span $T - t$, indicating that in the beginning, the differences between the AAI and the ANI is large. As time elapse, the differences diminish gradually.

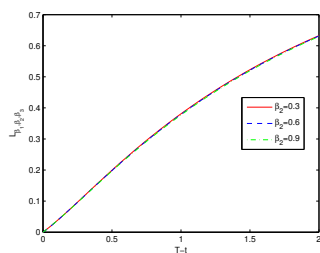


FIGURE 7. The effect of β_2 on $L_{\beta_1, \beta_2, \beta_3}$.

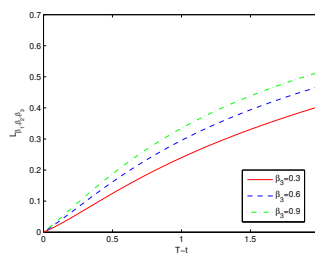


FIGURE 8. The effect of β_3 on $L_{\beta_1, \beta_2, \beta_3}$.

5. Conclusion

In this paper, we consider a robust optimal reinsurance and investment problem for an AAI, who worries about model uncertainty, and aims to find robust optimal strategies. His surplus is described by the classical *Crámer-Lundberg* model, and the dynamics of the risky-asset he invests in is assumed to follow an Ornstein-Uhlenbeck process, which effectively describes the features of bull and bear markets. He aims to maximize the CARA utility of the terminal wealth. By applying stochastic dynamic programming theory, explicit expressions for the robust optimal strategies and value functions are derived. Some numerical examples are given to illustrate the effects of model parameters on the robust optimal strategies.

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