논리연구 22-3(2019) pp. 397-415

Involutive Idempotent Uninorm Logics and Pretabularity*

Eunsuk Yang

【Abstract】This paper deals with the pretabular property of some fuzzy logics. For this, we first introduce the involutive idempotent uninorm logics **IdIUL** and **IUML** and examine the relationship between **IdIUL** and the another well-known system $\mathbf{RM}^{\mathbf{T}}$. Next, we show that **IUML** is pretabular, whereas **IdIUL** is not.

【Key Words】Pretabularity; Involutive idempotent uninorm logics, **IUML,** Algebraic semantics; Fuzzy logic; Finite model property.

Received: Jul. 4. 2019. Revised: Oct. 15. 2019. Accepted: Oct. 19. 2019.

* This research was supported by "Research Base Construction Fund Support Program" funded by Jeonbuk National University in 2019. I must thank the referees for their helpful comments.

1. Introduction

The purpose of this paper is to introduce a pretabular fuzzy logic system. In general, a logic L is said to be pretabular if it does not itself have a finite characteristic matrix (algebra, or frame), but every proper extension of it does (see Dunn & Hardegree (2001)). In the early 1970s, Dunn investigated the pretabular properties of the semi-relevance logic **RM** (**R** with mingle[1]) in Dunn (1970) and he and Meyer studied such properties of the Dummett-Gödel logic **G** in Dunn & Meyer (1971).

It is interesting that these two systems can be regarded as *fuzzy* logic systems.[2] However, unfortunately, since then, no further pretabular fuzzy logics have been introduced. This situation is understandable because most basic fuzzy logics such as **UL** (Uninorm logic) are not pretabular. Here we show that some other fuzzy logic systems still can have pretabular properties. To verify this, we consider the fuzzy logic **IUML** (Involutive uninorm mingle logic) introduced in Metcalfe & Montagna (2007) as a pretabular logic.

The paper is organized as follows: In Section 2, we introduce two fuzzy systems **IdIUL** (Involutive idempotent uninorm logic) and **IUML** and discuss their algebraic completeness. We in

---

1) This system can be more exactly denoted by $\mathbf{RM^0}$ (see below for $\mathbf{RM^0}$).

2) According to Cintula (and Běhounek) (2006; 2006), a (weakly implicative) logic L is said to be *fuzzy* if it is complete with respect to (w.r.t.) linearly ordered matrices (or algebras) and *core fuzzy* if it is complete w.r.t. *standard* algebras (i.e., algebras on the real unit interval [0, 1]).

particular examine the relationship between **IdIUL** and $\mathbf{RM}^{\mathbf{T}}$, a version of **RM**.[3] In Section 3, we show that **IUML** is pretabular, whereas **IdIUL** is not. This implies one interesting and surprising result that $\mathbf{RM}^{\mathbf{0}}$ is pretabular, whereas $\mathbf{RM}^{\mathbf{T}}$ is not.

For convenience, we adopt notations and terminology similar to those in Dunn (1970), Dunn & Hardegree (2001), and Dunn & Meyer (1971), and we assume reader familiarity with them (along with the results therein).

2. Involutive idempotent uninorm logics

We base involutive idempotent uninorm logics on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR* and connectives →, ∧, ∨, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\sim\phi := \phi \rightarrow \mathbf{f}$
df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$
df3. $\phi \,\&\, \psi := \sim(\phi \rightarrow \sim\psi)$.

We moreover define $\phi_{\mathbf{t}} := \phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notations and terminology. We use the axiom systems to provide a consequence relation.

---

3) Note that there are at least three versions of **R** of Relevance and thus three versions of **RM** (see Yang (2013)). By $\mathbf{R}^{\mathbf{0}}$, $\mathbf{R}^{\mathbf{t}}$, and $\mathbf{R}^{\mathbf{T}}$, Yang denoted the **R** without constants, the **R** with constants **t**, **f**, and the **R** with constants **t**, **f**, **F**, **T**, respectively. Similarly, he introduced $\mathbf{RM}^{\mathbf{0}}$, $\mathbf{RM}^{\mathbf{t}}$, and $\mathbf{RM}^{\mathbf{T}}$ as their corresponding extensions of **R** with mingle. Here, we follow his notations.

Definition 2.1 (i) **IdIUL** consists of the following axiom schemes and rules:

A1. $\phi \to \phi$ (self-implication, SI)

A2. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$ ($\wedge$-elimination, $\wedge$-E)

A3. $((\phi\to\psi)\wedge(\phi\to\chi)) \to (\phi\to(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\phi\to\chi)\wedge(\psi\to\chi)) \to ((\phi\vee\psi)\to\chi)$ ($\vee$-elimination, $\vee$-E)

A6. $\phi \to \mathbf{T}$ (verum ex quolibet, VE)

A7. $\mathbf{F} \to \phi$ (ex falso quadlibet, EF)

A8. $(\phi \,\&\, \psi) \to (\psi \,\&\, \phi)$ (&-commutativity, &-C)

A9. $(\phi \,\&\, \mathbf{t}) \leftrightarrow \phi$ (push and pop, PP)

A10. $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$ (suffixing, SF)

A11. $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \,\&\, \psi) \to \chi)$ (residuation, RE)

A12. $(\phi \to \psi)_{\mathbf{t}} \vee (\psi \to \phi)_{\mathbf{t}}$ (**t**-prelinearity, $PL_{\mathbf{t}}$)

A13. $\sim\sim\phi \to \phi$ (double negation elimination, DNE)

A14. $(\phi \,\&\, \phi) \leftrightarrow \phi$ (idempotence, ID)

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj).

(ii) (Metcalfe & Montagna (2007)) Involutive uninorm mingle logic **IUML** is **IdIUL** plus $\mathbf{t} \leftrightarrow \mathbf{f}$ (fixed-point, FP).

A12 is the axiom scheme for linearity, and logics being complete w.r.t. linearly ordered (corresponding) algebras are said to be fuzzy logics (see e.g. Cintula (2006)).[4] Note that $\phi \to \psi$ can be instead defined as $\sim(\phi \,\&\, \sim\psi)$ (df4).

[4] Note that while $\wedge$ is the extensional conjunction connective, & is the intensional conjunction one.

Proposition 2.2 **IdIUL** proves:

(1) $(\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi)$ (contraction, CTR)

(2) $(\phi \wedge (\psi \vee \chi)) \rightarrow ((\phi \wedge \psi) \vee (\phi \wedge \chi))$ (distributivity, D)

(3) $(\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\psi \rightarrow (\phi \rightarrow \chi))$ (permutation, PM)

(4) $(\phi \rightarrow {\sim}\phi) \rightarrow {\sim}\phi$ (reductio, RD)

(5) $(\phi \rightarrow \psi) \rightarrow ({\sim}\psi \rightarrow {\sim}\phi)$ (contraction, CTR)

(6) $\mathbf{t}$

(7) $\phi \leftrightarrow (\mathbf{t} \rightarrow \phi)$

(8) $\phi \rightarrow (\phi \rightarrow \phi)$ (mingle, M).

Proof: We prove (1) as an example. Using A10, A14, and mp, we have $((\phi \ \& \ \phi) \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$. Thus, we obtain the claim further using A11.

The proof for the other cases is left to the interested reader. □

A *theory* over L ($\in$ {**IdIUL**, **IUML**}) is a set T of formulas. A *proof* in a theory T over L is a sequence of formulas each of whose members is either an axiom of L or a member of T or follows from some preceding members of the sequence using the two rules in Definition 2.1. $T \vdash \phi$, more exactly $T \vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. The relevant deduction theorem (RDT_t) for L is as follows:

Proposition 2.3 (Meyer, Dunn, & Leblanc (1976)) Let T be a theory, and ϕ, ψ formulas.

(RDT_t) $T \cup \{\phi\} \vdash \psi$ iff $T \vdash \phi_t \rightarrow \psi$.

A theory T is *inconsistent* if $T \vdash$ **F**; otherwise it is *consistent*.

For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\rightarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meaning.

The algebraic counterpart of L ($\in$ {**IdIUL**, **IUML**}) is the class of the so-called *L-algebras*. Let $x_t := x \wedge t$. They are defined as follows.

Definition 2.4 (i) (IdIUL-algebra) An *IdIUL-algebra* is a structure **A** = $(A, \top, \bot, t, f, \wedge, \vee, *, \rightarrow)$ such that:

(I) $(A, \top, \bot, \wedge, \vee)$ is a bounded lattice with top element $\top$ and bottom element $\bot$.

(II) $(A, *, t)$ is a commutative monoid.

(III) $y \leq x \rightarrow z$ iff $x * y \leq z$ (residuation).

(IV) $t \leq (x \rightarrow y)_t \vee (y \rightarrow x)_t$ (prelinearity, pl_t).

(V) $(x \rightarrow f) \rightarrow f \leq x$ (double negation elimination, dne).

(VI) $x = x * x$ (idempotence, id).

(ii) (Metcalfe & Montagna (2007)) An *IUML*-algebra is an IdIUL-algebra satisfying (fixed-point, fp) $t = f$.

Additional (unary) negation and (binary) equivalence operations are defined as follows: $\sim x := x \rightarrow f$ and $x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x)$.

The class of all L-algebras is a variety which will be denoted by L.

Definition 2.5 (Evaluation) Let $\mathcal{A}$ be an L-algebra. An $\mathcal{A}$*-evaluation* is a function v : *Fm* $\rightarrow$ $\mathcal{A}$ satisfying: $v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \& \psi) = v(\phi) * v(\psi)$, $v(\mathbf{f}) = f$, $v(\mathbf{t}) = t$, $v(\mathbf{F}) = \perp$, $v(\mathbf{T}) = \top$, and hence $v(\sim\phi) = \sim v(\phi)$.

Definition 2.6 Let $\mathcal{A}$ be an L-algebra, T a theory, ϕ a formula, and K a class of L-algebras.

(i) (Tautology) ϕ is a *t-tautology* in $\mathcal{A}$, briefly an $\mathcal{A}$*-tautology* (or $\mathcal{A}$*-valid*), if $v(\phi) \geq t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an $\mathcal{A}$*-model* of T if $v(\phi) \geq t$ for each $\phi \in T$. By *Mod(T,* $\mathcal{A}$*)*, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_K \phi$, if Mod(T, $\mathcal{A}$) = Mod(T $\cup$ $\{\phi\}$, $\mathcal{A}$) for each $\mathcal{A} \in$ K.

Definition 2.7 (**L**-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 2.6. $\mathcal{A}$ is an ***L****-algebra* iff whenever ϕ is L-provable in T (i.e. $T \vdash_L \phi$), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. $T \vDash_{\{A\}} \phi$). By *MOD(**L**)*, we denote the class of **L**-algebras. Finally, we write $T \vDash_L \phi$ in place of $T \vDash_{MOD(\mathbf{L})} \phi$.[5)]

---

5) Note that the boldface L-algebras, **L**-algebras, are different from L-algebras in the sense that the former algebras are L-algebras satisfying soundness.

Note that since each condition for the L-algebra has a form of equation or can be defined in equation (exercise), it can be ensured that the class of all L-algebras is a variety.[6)]

We first show that classes of provably equivalent formulas form an L-algebra. Let T be a fixed theory over L. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that $T \vdash_L \phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \rightarrow [\psi]_T = [\phi \rightarrow \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \,\&\, \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $t = [\mathbf{t}]_T$, $f = [\mathbf{f}]_T$, $\top = [\mathbf{T}]_T$, $\bot = [\mathbf{F}]_T$, and thus $\sim[\phi]_T = [\sim\phi]_T$. By $\boldsymbol{A_T}$, we denote this algebra.

Proposition 2.8 For T a theory over L, $\mathbf{A}_T$ is an **L**-algebra.

Proof: In order to show that $\mathbf{A}_T$ (T over L) is an **L**-algebra, we just consider (id) for **IdIUL**. $[\phi]_T * [\phi]_T = [\phi \,\&\, \phi]_T = [\phi]_T$ iff $T \vdash (\phi \,\&\, \phi) \leftrightarrow \phi$. Thus, it is an **L**-algebra. □

Theorem 2.9 (Strong completeness) Let T be a theory, and ϕ a formula. $T \vdash_L \phi$ iff $T \vDash_L \phi$.

Proof: The left-to-right direction follows from definition. The right-to-left direction is as follows: from Proposition 2.8, we obtain $\mathbf{A}_T \in \mathrm{MOD}(L)$, and for $\mathbf{A}_T$-evaluation v defined as $v(\psi) = [\psi]_T$, it holds that $v \in \mathrm{Mod}(T, \mathbf{A}_T)$. Thus, since from $T \vDash_L \phi$

6) Variety is the class of algebras closed under homomorphic images, subalgebras, and direct products (see Dunn & Hardegree (2001)).

we obtain that $[\phi]_T = v(\phi) \geq t$, $T \vdash_L \mathbf{t} \to \phi$. Then, since $T \vdash_L \mathbf{t}$, by (mp) $T \vdash_L \phi$, as required. □

We finally examine the relationship between **IdIUL** and $\mathbf{RM}^{\mathbf{T}}$, which is $\mathbf{R}^{\mathbf{T}}$ (the **R** with constants **t**, **f**, **F**, and **T**) with mingle. First note that $\mathbf{RM}^{\mathbf{T}}$ can be axiomatized as the system having the axioms and rules A1 to A7, A10, A13, (mp), (adj), and Proposition 2.2 (1) to (8) (see Yang (2013)).

Theorem 2.10 The system **IdIUL** is proof-theoretically equivalent to $\mathbf{RM}^{\mathbf{T}}$.

Proof: Definition 2.1 (i) and Proposition 2.2 ensure the axioms and rules for $\mathbf{RM}^{\mathbf{T}}$ are provable in **IdIUL**. Here, we consider the converse direction. We have to show that A8, A9, A11, A12, and A14 are provable in $\mathbf{RM}^{\mathbf{T}}$. We prove A8 as an example. For this, first note that we have $(\psi \to {\sim}\phi) \to (\phi \to {\sim}\psi)$ using (df1) and PM. Thus, we further have ${\sim}(\phi \to {\sim}\psi) \to {\sim}(\psi \to {\sim}\phi)$ using (df1) and A10. Therefore, we obtain $(\phi \,\&\, \psi) \to (\psi \,\&\, \phi)$ by (df3).

The proof for the other cases is left to the interested reader. □

3. Pretabularity

Here we show that **IUML** is pretabular, whereas **IdIUL** is not. This will imply that $\mathbf{RM}^{\mathbf{0}}$ is pretabular but $\mathbf{RM}^{\mathbf{T}}$ is not.

By *e*, *o*, *1*, and *0*, we express t, f, ⊤ and ⊥, respectively, on the real unit interval [0,1] or on a subset of it with top and bottom elements 1, 0.[7] We refer to L-algebras on such a carrier set as *S^L-algebras*. S^L-algebras are defined as follows:

Proposition 3.1 The operations for an S^L-algebra are defined as follows.

(1) (Metcalfe & Montagna (2007)) Let the carrier set S be [0,1]. An *S^{IUML}-algebra* is an algebra satisfying:

T1. $x \wedge y = \min(x, y)$

T2. $x \vee y = \max(x, y)$

T3. $x \to y = \max(1 - x, y)$ if $x \leq y$, and otherwise $x \to y = \min(1 - x, y)$

T4. $\sim x = 1 - x$.[8]

(2) Let the carrier set S be a subset of [0,1] with top and bottom elements 1, 0. An *S^{IdIUL}-algebra* is an S^{IUML}-algebra whose carrier set S does not necessarily have a fixed-point.

By *$S^L{}_{[0,1]}$-algebra*, we henceforth denote the S^L-algebra on [0,1]; by *$S^L{}_{[0,1]-}$-algebra*, the S^L-algebra on $[0,1] \setminus \{1/2\}$; by *$S^L{}_n$-algebra*, the S^L-algebra whose elements are in {0, 1/n – 1, ⋯, n – 2/n – 1, 1}. Generalizing, *S-algebra* refers to any algebra whose elements form a chain with the greatest and least elements,

7) Note that *e*, *o*, *1*, and *0* correspond to identity, its negation, top, and bottom elements, respectively.

8) In general, the *involutive* negation is defined as the negation n satisfying $n(n(x)) = x$ for all $x \in [0, 1]$. Since any involutive negation [0, 1] can be isomorphic to $1 - x$, for convenience, we take this definition.

and whose operations are defined in an analogous way.

Note that S-algebras having 1/2 as an element x such that x = ~x are said to be *fixed-pointed* and otherwise *non-fixed-pointed*. A logic L is said to be *fixed-pointed* if L is characterized by an S-algebra having a fixed-point, and otherwise is *non-fixed-pointed*. An extension of L is said to be *proper* if it does not have exactly the same theorems as L.

Definition 3.2

(i) (Tabularity) A logic L is *tabular* if L has some finite characteristic algebra.

(ii) (Pretabularity) A logic L is *pretabular* if (a) L is not tabular and (b) every proper extension of L has some finite characteristic algebra.

Now, we show that **IUML** is pretabular, but the systems **IdIUL** is not. We first introduce some known pretabular logics.

Fact 3.3

(1) (Dunn & Meyer (1971)) **G** is pretabular.

(2) (Dunn (1970)) $\mathbf{RM^0}$ is pretabular.

We then divide the work into a number of propositions following the line in Dunn (1970) and Dunn & Meyer (1971).

Proposition 3.4 Let **X** be an extension of **IUML**, **A** be an **X**-algebra, and a $\in$ **A** be such that a < t. Then, there is a homomorphism h of **A** onto an S-algebra which is an **X**-algebra, such that h(a) < *e*.

Proof: The proof is analogous to Theorem 3 in Dunn (1970) and Theorem 11.10.4 in Dunn & Hardegree (2001). □

Proposition 3.5 For the system **IUML**, let $S^{IUML}{}_1$, $S^{IUML}{}_3$, $S^{IUML}{}_5$, $S^{IUML}{}_7$, ⋯, i.e., $S^{IUML}{}_{2n-1}$, $1 \leq n \in N$, be the sequence of S^{IUML}-algebras relabeled in order as $M^{IUML}{}_1$, $M^{IUML}{}_2$, $M^{IUML}{}_3$, ⋯. If a sentence ϕ is valid in $M^{IUML}{}_i$, then ϕ is valid in $M^{IUML}{}_j$, for all j, $j \leq i$.

Proof: Since each $S^{IUML}{}_j$ is (isomorphic to) a subalgebra or a homomorphic image of $S^{IUML}{}_i$, (i) and (ii) are immediate. □

Proposition 3.6 In S^{IdIUL}-algebras, when i is even ($\geq$ 3), $S^{IdIUL}{}_i$ validates a sentence ϕ that is not valid in any even-valued $S^{IdIUL}{}_j$, $2 \leq j \leq I$.

Proof: The claim can be verified by considering the sentence (FP), which is valid in every odd-valued $S^{IdIUL}{}_i$, but not in $S^{IdIUL}{}_2$ (and thus not in any even-valued $S^{IdIUL}{}_j$, $j \geq 2$). □

Remark 3.7 Proposition 3.6 implies that every valid sentence in $S^{IdIUL}{}_{[0,1]-}$ must be valid in $S^{IUML}{}_{[0,1]}$, but there is a valid sentence in $S^{IUML}{}_{[0,1]}$ that is not in $S^{IdIUL}{}_{[0,1]-}$.

Now, we recall the concept of a Lindenbaum-Tarski algebra. Let L be **IUML** and T be a theory in L. We define [ϕ] = {ψ: T $\vdash_L$ $\phi \leftrightarrow \psi$} and L = {[$\phi$] : $\phi \in Fm$}. The *Lindenbaum-Tarski algebra* $Lind_T$ w.r.t. L and T is L-algebra having the domain L_T, operations $\#^{Lind_T}([\phi_1], \cdots, [\phi_n]) = [\#(\phi_1, \cdots, \phi_n)]$, where $\# \in \{\wedge, \vee, \rightarrow\}$, identity t, any f, and top and bottom elements are [**t**], [**f**], [**T**], and [**F**], respectively.

Where **X** is a propositional system and **V** is a set of atomic sentences, let **X/V** be that propositional system like **X** except that its sentences contain no atomic sentences other than those in **V**. The following is obvious.

Proposition 3.8 Let **X** be an extension of **IUML**. Then, **A**(**X/V**) is an **X**-algebra and is characteristic for **X/V**, since any non-theorem may be falsified under the canonical evaluation v_c, which sends every sentence ϕ to [ϕ], where [ϕ] is the set of all sentences ψ such that $\psi \leftrightarrow \phi$.

Then, using Propositions 3.4 and 3.8, we further have the proposition below.

Proposition 3.9 Let **X** be an extension of **IUML**. Then, if a sentence ϕ is not a theorem of **X**, there is some S^{IUML}-algebra $S^{IUML}{}_n$ such that $S^{IUML}{}_n$ is an **X**-algebra and ϕ is not valid in $S^{IUML}{}_n$.

Proof: If ϕ is not a theorem of **X**, then, by Proposition 3.8,

ϕ is falsifiable in the **X**-algebra **A**(**X**/**V**), where **V** is the set of sentential variables occurring in ϕ, by the canonical evaluation v_c. However, since $[\phi]$ is undesignated in **A**(**X**/**V**), then, by Proposition 3.4, there is a homomorphism h of **A**(**X**/**V**) onto an S^{IUML}-algebra S^{IUML} such that S^{IUML} is an **X**-algebra and $h([\phi]) < e$ in S^{IUML}. However, the composition of h and v_c, $h \circ v_c(\psi) = h([\psi])$, is an evaluation that falsifies ϕ in S^{IUML}. Note that an S^{IUML}-subalgebra, the image h(**A**(**X**/**V**), is finitely generated since it is the homomorphic image of **A**(**X**/**V**), which is finitely generated by the elements [p] such that $p \in$ **V**. Thus, this algebra is finitely generated by the elements [p] such that $p \in$ **V**. It is obvious that every finitely generated S^{IUML}-subalgebra is finite and isomorphic to some $S^{IUML}{}_n$. Thus, this algebra is isomorphic to some $S^{IUML}{}_n$. □

Now, we turn to a proof of our principal results.

Theorem 3.10

(i) **IUML** is pretabular.

(ii) **IdIUL** is not pretabular.

Proof: For (i), we show that every proper extension of **IUML** has a finite characteristic algebra. Let $M^{IUML}{}_1$, $M^{IUML}{}_2$, $M^{IUML}{}_3$, ⋯ be the sequence of S^{IUML}-algebras defined in Proposition 3.5. Let I be the set of indices of those S^{IUML}-algebras that are **X**-algebras, where **X** is the given proper extension of L.

First, if I contains an infinite number of indices, then I contains every index because of Proposition 3.5. However, since every S^{IUML}-algebra $M^{IUML}{}_{i}$ is an IUML-algebra, it follows from Proposition 3.9 and Theorem 2.9 that **X** is identical with **IUML**, which contradicts the hypothesis that **X** is a proper extension of **IUML**.

Second, if I contains only a finite number of indices, then, by Proposition 3.5, there must be some index i such that I contains exactly those indices less then or equal to i. By construction, $S^{IUML}{}_{i}$ is an **X**-algebra. Let a sentence ϕ not be a theorem of **X**. Then, by Proposition 3.9, ϕ is not valid in some **X**-algebra $M^{IUML}{}_{h}$, and, by our choice of i, $h \leq i$. However, by Proposition 3.5, ϕ is not valid in $M^{IUML}{}_{i}$. Therefore, $M^{IUML}{}_{i}$ is the desired finite characteristic algebra.

IUML itself has no finite characteristic algebra, which can easily be shown by a proof similar to that of Sugihara in Sugihara (1955). Therefore, it can be ensured that **IUML** is pretabular.

(ii) directly follows from (i), Proposition 3.6, and Remark 3.7. (Note that the system **IUML** is a pretabular extension of **IdIUL**.) □

Corollary 3.11 $\mathbf{RM}^{\mathbf{0}}$ is pretabular, whereas $\mathbf{RM}^{\mathbf{T}}$ is not.

Proof: The claim follows from Fact 3.3 (2), Theorem 2.10, and Theorem 2.10. □

This corollary gives us an interesting and surprising result in the following sense: When one hears that $\mathbf{RM}^{\mathbf{0}}$ is pretabular, one expects that $\mathbf{RM}^{\mathbf{T}}$ is also pretabular because they are just two different versions of **RM** and thus one may think that they will have almost the same properties. But the result shows that they have a different property to each other w.r.t. pretabularity.

We finally remark some relationships between the results in Theorem 3.10 and algebraic results introduced in Galatos & Raftery (2012) and Raftery (2007).

Remark 3.12 Recall that **IUML** is pretabular, whereas **IdIUL** is not. This fact can be *algebraically* obtained as a consequence of the full description of the lattice of subvarieties of the variety of bounded odd Sugihara monoids $\mathsf{OSM}^{\perp}$, which is a proper non-finitely generated subvariety of the variety of bounded Sugihara monoids $\mathsf{SM}^{\perp}$ (see Fact 7.6 in Galatos & Raftery (2012) and Theorem 5 in Raftery (2007)). Note that $\mathsf{OSM}^{\perp}$ and $\mathsf{SM}^{\perp}$ are algebraic counterparts for the systems **IUML** and **IdIUL**, respectively.

Remark 3.13 Pretabularity is a property related to logics whose associated varieties of algebras are locally finite. A variety of algebras is said to be *locally finite* if each of its finitely generated members is a finite algebra. We first note the following fact:

Fact 3.14 The variety of Sugihara monoids SM is locally finite (see Raftery (2007)) and thus so is $\mathsf{SM}^{\perp}$. Hence, since the variety $\mathsf{OSM}^{\perp}$ is a subvariety of $\mathsf{SM}^{\perp}$, $\mathsf{OSM}^{\perp}$ is locally finite.

The result in Fact 3.14 shows that the varieties for **IdIUL** (= $\mathbf{RM}^{\mathbf{T}}$) and **IUML** are locally finite.

4. Concluding remark

We investigated the pretabular property of the system **IUML**. More precisely, we showed that **IUML** is pretabular, whereas **IdIUL** is not. We also examined that **IdIUL** and $\mathbf{RM}^{\mathbf{T}}$ are equivalent. However, we have not yet investigated pretabular properties of other fuzzy systems. This is a problem left in this paper.

References

Behounek, L. and Cintula, P. (2006), "Fuzzy logics as the logics of chains", *Fuzzy Sets and Systems* 157, pp. 604-610.

Cintula, P. (2006), "Weakly Implicative (Fuzzy) Logics I: Basic properties", *Archive for Mathematical Logic* 45, pp. 673-704.

Dunn, J. M. (1970), "Algebraic completeness for R-mingle and its extensions", *The Journal of Symbolic Logic* 35, pp. 1-13.

Dunn, J. M. and Hardegree, G. (2001), *Algebraic Methods in Philosophical Logic*, Oxford, Oxford Univ Press.

Dunn, J. M. and Meyer, R. K. (1971), "Algebraic completeness results for Dummett's LC and its extensions", *Mathematical Logic Quarterly* 17, pp. 225-230.

Gabbay, D. and Shetman, V. B. (1993), "Undecidability of modal and intermediate first-order logics with two individual variables", *The Journal of Symbolic Logic* 58, pp. 800-823.

Galatos, N. and Raftery, J. G. (2012), "A category equivalence for odd Sugihara monoids and its applications", *Journal of Pure and Applied Algebra* 216, pp. 2177-2192.

Galminas, L. and Mersch, J. G. (2012}, "A pretabular classical relevance logic", *Studia Logica* 100, pp. 1211-1221.

Meyer, R. K., Dunn, J. M., and Leblanc, H. (1976), "Completeness of relevant quantification theories", *Notre Dame Journal of Formal Logic* 15, pp. 97-121.

Metcalfe, G., and Montagna, F. (2007), "Substructural Fuzzy Logics", *Journal of Symbolic Logic*, 72, pp. 834-864.

Raftery, J. G. (2007), “Representable idempotent commutative residuated lattices”, *Transactions of the American Mathematical Society* 359, pp. 4405-4427.
Rybakov, V., Kiyatkin, V., and Terziler, M. (1999), “Independent bases for rules admissible in pretabular logics”, *Logic Journal of the Interest Group in Pure and Applied Logics* 7, pp. 253-266.
Sugihara, T. (1955), “Strict implication free from implicational paradoxes”, *Memoirs of the Faculty of Liberal Arts*, Fukui University, Series 1, pp. 55-59.
Swirydowicz, K. (2008), “There exists an uncountable set of pretabular extensions of the relevant logic **R** and each logic of this set is generated by a variety of finite height”, *The Journal of Symbolic Logic* 73, pp. 1249-1270.
Yang, E. (2013), “**R** and Relevance principle revisited”, *Journal of Philosophical Logic* 42, pp. 767-782.

전북대학교 철학과, 비판적사고와논술연구소
Department of Philosophy & Institute of Critical Thinking and Writing, Jeonbuk National University
eunsyang@jbnu.ac.kr

누승적 멱등 유니놈 논리와 선표성

양 은 석

이 글에서 우리는 퍼지 논리의 선표성 성질을 다룬다. 이를 위하여 먼저 누승적 멱등 유니놈 논리 **IdIUL**과 **IUML** 체계를 소개하고 **IdIUL** 체계와 우리에게 이미 알려진 $\mathbf{RM^T}$ 체계의 관계를 다룬다. 다음으로 **IUML**은 선표성을 만족하지만 **IdIUL**은 그렇지 않다는 것을 보인다.

주요어: 선표성, 누승적 멱등 유니놈 논리, **IUML**, 대수적 의미론, 퍼지 논리, 유한 모형 성질