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Stability of superconductor by integration formula

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Abstract

The superconductor stability theories are consistently described by the integral formula. If the defined stability function is a simple decreasing function, it becomes a cryogenic stability condition. If the stability function has a maximum value and a minimum value, and the maximum value is less than 0, then it is a cold-end recovery condition. If the maximum value is more than 0, it can be shown that the unstable equilibrium temperature, that is, the MPZ (minimum propagation zone) temperature distribution can exist. The MPZ region is divided into two regions according to the current ratio. At the low current ratio, the maximum dimensionless temperature is greater than 1, and at the relatively high current ratio, the maximum dimensionless temperature is less than 1. In order to predict the minimum quench energy, the dimensionless energy was obtained for the MPZ temperature distribution. In particular, it was shown that the dimensionless energy can be obtained even when the MPZ maximum temperature is 1 or more.

Keywords: superconductor, Stekly parameter, cryogenic stability, cold-end recovery, MPZ (minimum propagation zone), stability function, critical energy

Nomenclature

α : Stekly parameter
ε : dimensionless energy
ρ_c : electric resistivity of copper
θ : dimensionless temperature
A : cross-sectional area of conductor, m^2
C : volumetric heat capacity, J/K-m^3
E : energy, J
F : stability function
f : volume fraction of copper, dimensionless
G : integration of current sharing function
g : current sharing function
h : heat transfer coefficient, W/m^2-K
I : current, A
i : current ratio, I/I_c
J : current density, A/m^2
k : thermal conductivity, W/m-K
P : perimeter, m
Q : Joule heat generation per unit area, W/m^2
q : convection heat flux, W/m^2
s : dimensionless temperature gradient
T : temperature, K
t : time, s
x : dimensionless coordinate
z : coordinate in conductor direction, m

Subscript :

c : critical
b : bath of coolant
m : maximum temperature
min : position of minimum value of stability function
max : position of maximum value of stability function
mpz : minimum propagation zone
n : normal metal
s : cryogenic stability
1 : condition for $\theta_m = 1$

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1. INTRODUCTION

The classical interpretation of superconductor stability has a sophisticated theoretical system leading to cryogenic stability, cold-end recovery, equal area theorem and MPZ (minimum propagation zone). In this paper, based on the integral formula, we intend to provide a simple and intuitive framework for understanding these theories.

The cryogenic stability (or unconditional stability) criterion [1, 2] suggests that conductors can recover superconductivity if the Joule heat generation across the conductor is less than the heat release to the coolant. This theory is most stable but can only deliver an operating current density that is too low for the critical current density. In cryogenic stability criterion, all parts of the conductor driven normal by a thermal perturbation (called normal zone) recover simultaneously.

Maddock [3] showed that unconditional stability can preserved at somewhat higher current densities. The so called cold-end recovery condition assumes that both ends are immersed in the coolant even if the temperature at which the disturbance energy is applied rises. The generated Joule heat can be released by the convection heat transfer to the coolant as well as by conduction heat transfer in the conductor length direction to the cold end. This theory is summarized by the famous equal-area theorem, which suggests conditions that can not constitute an unstable equilibrium temperature distribution.

Large superconducting magnets operate under unconditional stability conditions. Even when operating under cold-end recovery conditions, the operating current density is still too low. The MPZ theory [4, 5] is developed to raise the current density above the Maddock limit, assuming that the perturbation energy is small. The MPZ theory finds the unstable equilibrium temperature distribution (MPZ temperature distribution) and calculates the energy of the temperature distribution to estimate the minimum quench energy.

The above theories are analyzed by using the one-dimensional heat conduction equation, but the analysis considering the two-dimensional heat transfer is also performed by extending it [6].

This paper considers the classical theories on the stability of superconductors in terms of the stability function proposed in this paper and presents the numerical results. It is shown that the current ratio (ratio of operating current to critical current) in a given condition (Stekly parameter) is sequentially categorized into cryogenic stability, cold-end recovery, and MPZ criterions. In addition, the MPZ region is divided into two regions having a dimensionless maximum temperature greater than 1 or less than 1. The MPZ temperature distribution was obtained and the MPZ formation energy was calculated by integration.

In cryogenic stability and cold-end recovery criterions, boiling heat transfer is assumed, which has several modes such as nucleate boiling, transition boiling, and film boiling. In this paper, it is assumed that all heat release to the coolant have a constant boiling heat transfer coefficient for the purpose of consistently examining the stability theory.

2. ANALYSIS

The following one-dimensional heat conduction equation is used for the stability analysis of a superconductor [4].

$$C\frac{\partial T}{\partial t}=\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\frac{QP}{A}-\frac{qP}{A} \tag{1}$$

The coordinate z is the conductor length direction, and at $z=0$, the conductor has the highest temperature, and the temperature approaches the coolant temperature as z increases in the positive and negative directions. Because of the symmetry, only the $z>0$ region is considered.

The Joule heat generation Q changes depending on the conductor temperature. When T is greater than critical temperature T_c, all current flow in normal metal and the Joule heat generation becomes maximum value, $Q=Q_n=(\rho_{cu}J^2A)/fP$. When $T_b<T<T_{cs}$, all current flow in superconductor and $Q=0$, where T_b is the bath temperature, and T_{cs} is the current sharing temperature.
Fig. 1 shows the relation between the temperature and critical current of a superconductor. The critical current I_c is the maximum current in superconductor without resistance at T_b. The critical current decreases as temperature increases, and becomes zero at the critical temperature T_c. The current sharing temperature is defined as the temperature at which the operating current is equal to the critical current.

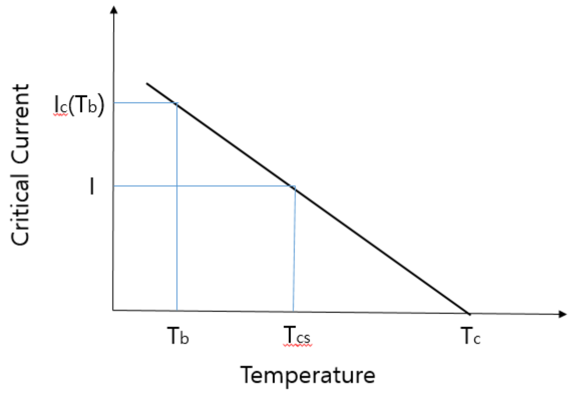

Fig. 1. Critical current vs. temperature.

If the temperature of the section is between the current sharing temperature and the critical temperature, $T_{cs}<T<T_c$, $Q=Q_n(T-T_{cs})/(T_c-T_{cs})$. We can find that $(T_c-T_{cs})/(T_c-T_b)=I/I_c=i$.

The current density is $J=I/A$, and $f=A_{cu}/A$ is the area ratio of copper in the total cross-sectional area A, and ρ_{cu} is the electrical resistivity of copper. The convection heat flux into coolant is $q=h(T-T_b)$, h is the film boiling heat transfer coefficient between the superconductor and the coolant, and P is the perimeter of the conductor.

For $Q<q$, the superconductor is unconditionally stable because the Joule heat generation is smaller than cooling. This condition is called cryogenic stability criterion. Stekly parameter is defined as

$$\alpha=\frac{\rho_{cu}\left((1-f)J_c\right)^2}{fPh(T_c-T_b)} \tag{2}$$

Then the cryogenic stability criterion can be written as $\alpha\, i^2<1$. The coil can regain its superconductivity if cryogenic stability criterion is satisfied even if the entire coil becomes normal state and generates Joule heat.

However, the superconductor is in the coolant and heat transfer occurs in the direction of the coil length. In this case, the temperature distribution of the unstable equilibrium temperature should be obtained for stability analysis. Equation (1) except the transient term, and using $s=d\theta/dx$, $x=z/\sqrt{kA/hP}$, it becomes

$$s\frac{ds}{d\theta}+\alpha i^2 g(\theta)-\theta=0 \tag{3}$$

$$g(\theta)=\begin{cases}0 & \theta<1-i\\ (\theta+i-1)/i & 1-i<\theta<1\\ 1 & 1<\theta\end{cases} \tag{4}$$

$$\theta=\frac{T-T_b}{T_c-T_b} \tag{5}$$

$$i=\frac{I}{I_c}=\frac{JA}{J_cA_s}=\frac{J}{(1-f)J_c} \tag{6}$$

The above equations are cited by several researchers for superconductor stability analysis [4, 5].

If the thermal conduction is neglected in Equation (3), the boundary of the cryogenic stability condition $\alpha i^2 = 1$ is obtained under the condition that the maximum Joule heat generation is equal to the heat release amount.

The temperature distribution has a maximum temperature ($\theta = \theta_m$) at $x = 0$ and $\theta = 0$ at $x = \infty$. Also, the temperature gradient at $x = 0$ and $x = \infty$ should be zero.

By separation of variables and integration, Equation (3) becomes

$$s^2 = 2\,\alpha i^2[G(\theta_m) - G(\theta)] - (\theta_m^2 - \theta^2) \tag{7}$$

$$G(\theta) = \int_0^\theta g(\theta)d\theta \tag{8}$$

$$G(\theta) = \begin{cases} 0 & \theta < 1 - i \\ \dfrac{(\theta + i - 1)^2}{2i} & 1 - i < \theta < 1 \\ \dfrac{i}{2} + \theta - 1 & 1 < \theta \end{cases} \tag{9}$$

Because $s = 0$ at $\theta = 0$ in Equation (7), the following equation can be derived.

$$F(\theta) = 2\,\alpha i^2 G(\theta) - \theta^2 \tag{10}$$

Equation (10) is called a stability function in this paper. To find the maximum and minimum values of stability function, set the derivative values zero. $F'(\theta) = 2[\alpha i^2 g(\theta) - \theta]$. The minimum value of $F(\theta)$ is between $1 - i < \theta < 1$, and its position and value are as follows.

$$\theta_{min} = \frac{\alpha i(1 - i)}{\alpha i - 1} \tag{11}$$

$$F_{min} = -\frac{\alpha i(1 - i)^2}{\alpha i - 1} \tag{12}$$

F_{min} is always negative because $\alpha i^2 > 1$, $0 < i < 1$.

The maximum value of the stability function can be obtained from $1 < \theta$, and its position and maximum value are as follows.

$$\theta_{max} = \alpha i^2 \tag{13}$$

$$F_{max} = \alpha i^2(\alpha i^2 + i - 2) \tag{14}$$

For the equation $F(\theta) = 0$ to have a root, the maximum value F_{max} must be greater than or equal to zero.

$$\alpha i^2 + i - 2 \geq 0 \tag{15}$$

If $\alpha i^2 + i - 2 < 0$, that is, $\alpha i^2 < 2 - i$, there is no steady state temperature distribution. Even if the temperature rises above critical value, the conductor recovers the superconducting state. In this condition, the stable region is larger than the cryogenic stability condition $\alpha i^2 < 1$.

If $\alpha i^2 = 1$, $\theta_{min} = \theta_{max} = 1$. That is, the function $F(\theta)$ has an inflection point at $\theta = 1$. That is, the function $F(\theta)$ becomes a monotone decreasing function when $\alpha i^2 < 1$.

In this paper, it is assumed that the convective cooling coefficient h is constant. This assumption helps to conceptually understand the problem of superconductor stability. However, the superconductor immersed in liquid helium or liquid nitrogen causes boiling heat transfer at the surface, and the heat transfer coefficient of boiling depends on boiling modes such as nucleate boiling, transition boiling, and film boiling.

From the given Stekly parameter (α), the minimum current ratio (i_{mpz}) that makes up the MPZ can be obtained from $\alpha i^2 + i - 2 = 0$.

$$i_{mpz} = \frac{-1 + \sqrt{1 + 8\alpha}}{2\alpha} \tag{16}$$

The maximum temperature at $i = i_{mpz}$,

$$\theta_{mpz} = 2 - i_{mpz} \tag{17}$$

If Stekly parameter $\alpha = 10$, the minimum current ratio to form MPZ becomes i_{mpz}=0.4. The maximum value of the MPZ temperature becomes $\theta_{mpz} = 1.6$.

Now, let us look at the root of $F(\theta) = 0$ in the condition of $F_{max} > 0$. First, to find the roots in the condition of $1 < \theta_m < \theta_{mpz}$, the following equation derived from Eqs. (9) and (10) must be solved.

$$2\alpha i^2 \left(\frac{i}{2} + \theta - 1\right) - \theta^2 = 0 \tag{18}$$

The quadratic equation for θ is solved as follows.

$$\theta_m = \alpha i^2 \left[1 - \left(1 - \frac{2 - i}{\alpha i^2}\right)^{1/2}\right] \tag{19}$$

We can obtain $\alpha i^3 < 1$ by using $1 < \theta_m$, which means that the maximum temperature is 1 or more if $i < 1/\alpha^3$ for a given α. We can also confirm that $\theta_m = \theta_{mpz}$ by substituting $i = i_{mpz}$ in Eq. (19) and using $\alpha i_{mpz}^2 = 2 - i_{mpz}$.

For $1 > \theta_m$, the equations (9) and (10) become

$$2\alpha i^2 \frac{(\theta + i - 1)^2}{2i} - \theta^2 = 0 \tag{20}$$

and the solution

$$\theta_m = \frac{(1 - i)\sqrt{\alpha i}}{\sqrt{\alpha i} - 1} \tag{21}$$

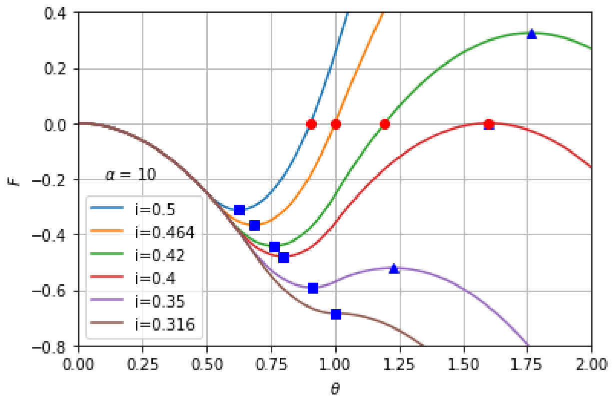

Fig. 2. Plot of stability function $F(\theta)$ for Stekly parameter $\alpha = 10$ with different current ratios.

TABLE I

DIMENSIONLESS MAXIMUM TEMPERATURE AND ENERGY FOR DIFFERENT CURRENT RATIOS $i = I/I_c$ WITH STEKLY PARAMETER $\alpha = 10$.

i	θ_{min}	θ_m	θ_{max}	αi^3	ε
0.316	1	-	1	-	-
0.35	0.91	-	1.225	0.429	-
0.4	0.8	1.6	1.6	0.64	∞
0.42	0.761	1.194	1.764	0.741	1.988
0.464	0.683	1	2.154	1	1.398
0.5	0.625	0.905	2.5	1.25	1.243

can be obtained. The maximum temperature is less than 1, so that $\alpha i^3 > 1$ can be obtained. That is, if $i > 1/\alpha^3$for a given α, the MPZ maximum temperature is less than 1. When the maximum temperature is 1, it occurs under the condition of $\alpha i^3 = 1$. If $\alpha i^3 = 1$ is substituted into Equation (21), $\theta_m = 1$ can be confirmed.

Fig. 2 is the result of calculating the stability function for various current ratios when Stekly parameter is 10. For each current ratio, the stability function has two extreme points, and the required zero point can be located between them. In Fig. 2, the triangular mark indicates the maximum value and the square mark indicates the minimum value, and the circle mark indicates a point of $F = 0$.
Table 1 shows the results of calculating the values shown in Fig. 2.

The cryogenic stability condition occurs when $i = 0.316$ by $\alpha i^2 = 1$, where both the maximum and minimum values of the stability function appear at $\theta_{min} = \theta_{max} = 1$ and become inflection point. For $0 < i < 0.316$, the stability function becomes a monotone decreasing function. Therefore, the derivative of the stability function, $F'(\theta) = 2[\alpha i^2 g(\theta) - \theta]$, is negative, that is, $\alpha i^2 g(\theta) < \theta$ for any values of θ. The maximum value of $g(\theta)$ is 1 when $\theta = 1$, that is, $\alpha i^2 < 1$, so unconditional stability criterion is obtained.

The region where the current ratio is $0.316 < i < 0.4$ is the cold-end recovery region in which there is no solution satisfying $F(\theta) = 0$ because the maximum value is less than 0. For example, $i = 0.35$, in the Equation (15), there is no MPZ temperature.

The minimum i where MPZ exists at $\alpha = 10$ is obtained as $i_{mpz} = 0.4$ in Eq. (16). From this value the maximum value of stability function becomes 0. The maximum value of the MPZ temperature is obtained as $\theta_{mpz} = 2 - i_{mpz} = 1.6$.

If the current ratio i is greater than i_{mpz}, the maximum value of stability function is greater than 0 and there exists a root satisfying $F(\theta) = 0$ in the interval $\theta_{min} < \theta < \theta_{max}$.

If the current ratio is in the interval of $i_{mpz} < i < 1/\alpha^3$, the root of the stability function F exists between $1 < \theta_m < 2 - i_{mpz}$. This example is shown in Fig. 2 for $i = 0.42$, and the calculated maximum temperature is 1.194.

The value of i with $\theta_m = 1$ is obtained from $\alpha i^3 = 1$. This condition is satisfied at a current ratio of 0.464. When the current ratio is greater than 0.464, the maximum temperature of MPZ becomes less than 1. The maximum temperature at current ratio 0.5 is 0.905. In Fig. 2, the red circle mark indicates the maximum temperature.

The superconductor stability is summarized as follows. For a given Stekly parameter α, we can calculate i_s, i_{mpz}, i_1 as below

$$\alpha i_s^2 = 1$$
$$\alpha i_{mpz}^2 + i_{mpz} - 2 = 0$$
$$\alpha i_1^3 = 1$$

$0 < i < i_s$:	cryogenic stability
$i_s < i < i_{mpz}$:	cold-end recovery
$i_{mpz} < i < i_1$:	MPZ ($\theta_m > 1$)
$i_1 < i < 1$:	MPZ ($\theta_m < 1$)

Now let's look at the MPZ temperature distribution and temperature gradient. Using the given Stekly parameter α and current ratio i, we can obtain the roots of stability function. Using $s = 0$, $\theta = \theta_m$ at $x = 0$, the following equation is derived:

$$x = \int_{\theta_m}^{\theta} \frac{d\theta}{s} \tag{22}$$

The energy in conductor is $E = \int_0^\infty (T - T_b) 2CAdz$, where C is the heat capacity per unit volume. By using the definition of x and θ,

$$E = 2CA(T_c - T_b)\sqrt{kA/hP} \int_0^\infty \theta dx \tag{23}$$

and using $dx = dx/d\theta \, d\theta = d\theta/s$, the dimensionless energy becomes

$$\varepsilon = \frac{E}{2CA(T_c - T_b)\sqrt{kA/hP}} = \int_0^{\theta_m} \frac{\theta}{s} d\theta \tag{24}$$

The MPZ formation energy is solved analytically in the literature [4] for $\theta_m < 1$, however, there is no analytical solution for $\theta_m > 1$. In this paper, the equations (22) and (24) are solved numerically.

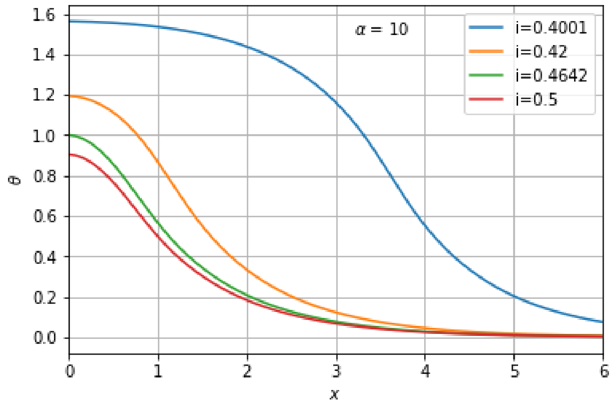

Fig. 3. Unstable equilibrium dimensionless temperature profile for Stekly parameter $\alpha = 10$ and different current ratio $i = I/I_c$.

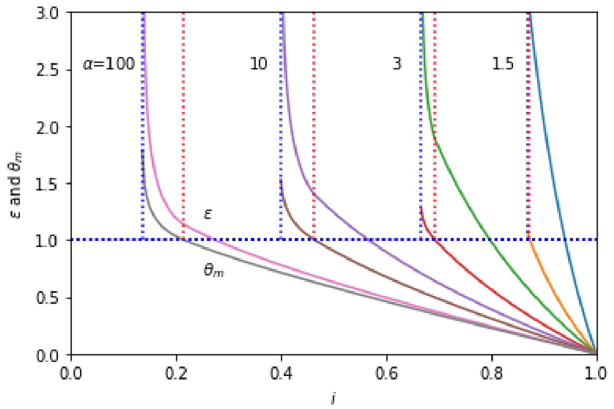

Fig. 4. A plot of dimensionless formation energy ε and dimensionless maximum temperature θ_m of the MPZ plotted against $i = I/I_c$ with the Stekly number α as parameter.

Fig. 3 shows the temperature distribution for $\alpha = 10$ and various current ratios. When $i = 0.4001$, which is very close to $i_{mpz} = 0.4$, it can be seen that the maximum temperature is close to $\theta_{mpz} = 1.6$. Calculations show that the MPZ maximum temperature for current ratio 0.4001 is 1.563. This maximum temperature decreases rapidly as i increases. $\alpha i^3 = 1$ gives the current ratio 0.4642. The maximum temperature at this value is 1. Fig. 3 shows the temperature distribution for this case. If the current ratio is greater than 0.4642, the MPZ maximum temperature will be less than one.

Fig. 4 shows the dimensionless MPZ formation energy ε and MPZ maximum temperature θ_m for different Stekly parameter α and current ratio i. The vertical blue dotted line indicates i_{mpz} and the red dotted line indicates i_1 for a given α. There are two curves for a given α, the upper curve represents the dimensionless energy and the lower curve represents the maximum temperature.

For large Stekly parameter, the Joule heat generation is larger than the heat release, the cold-end recovery condition is satisfied only at the low current ratio. Also, if the Stekly number is large, there is a lot of room for the MPZ to have a maximum temperature of 1 or more. Fig. 4 shows that the distance between the vertical blue dotted line and the red dotted line becomes wider as the Stekly parameter increases. In the case of $\alpha = 100$, $i_1 - i_{mpz} = 0.079$. On the other hand, when $\alpha = 1.5$, $i_1 - i_{mpz} = 0.005$, which is 1/15 compared with $\alpha = 100$. i_1 is the current ratio with a maximum temperature of 1.

When the current ratio is 1, it means that the operating current is equal to the critical current of the superconductor, MPZ cannot exist. When the current ratio is smaller than 1, MPZ can be formed and the forming energy is also increased.

Formation energy slowly increases until the MPZ peak temperature reaches 1, but increases rapidly when peak temperature is greater than 1, resulting in infinity at $\theta_{mpz} = 2 - i_{mpz}$.

3. CONCLUSION

The superconductor stability theories are consistently described by the integral formula. The case where there is no minimum value and maximum value of the developed integral function is the cryogenic stability domain. Although the maximum and minimum values of the stability function exist, the case where the maximum value is less than 0 is the cold-end recovery region. And, when the maximum value of the stability function is 0 or more, the root of the stability function can be obtained, and there is an unstable equilibrium temperature or MPZ temperature distribution. The MPZ region can be divided into two regions where the dimensionless maximum temperature is 1 or more or 1 or less. The MPZ formation energy was numerically calculated for the MPZ temperature profile. Particularly, the dimensionless energy is obtained when the dimensionless MPZ maximum temperature is greater than 1.

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