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LONG-TIME BEHAVIOR OF SOLUTIONS TO A NONLOCAL QUASILINEAR PARABOLIC EQUATION

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Abstract. In this paper we consider a class of nonlinear nonlocal parabolic equations involving p-Laplacian operator where the nonlocal quantity is present in the diffusion coefficient which depends on L^p-norm of the gradient and the nonlinear term is of polynomial type. We first prove the existence and uniqueness of weak solutions by combining the compactness method and the monotonicity method. Then we study the existence of global attractors in various spaces for the continuous semigroup generated by the problem. Finally, we investigate the existence and exponential stability of weak stationary solutions to the problem.

1. Introduction

One of the most important problems in modern mathematical physics is to understand the asymptotic behavior of the trajectories of infinite-dimensional dynamical systems induced by PDEs. One way to attack this problem for dissipative dynamical systems is to study the existence and properties of their global attractors. In the past decades, the existence of the global attractors has been proved for a large class of local PDEs, see the monographs [9, 16, 20, 26, 29], in particular, parabolic equations involving p-Laplacian operators, see e.g. [3–5, 17, 18, 23, 24]. The existence of global attractors has been also studied recently for some nonlocal parabolic equations in [6, 10–12, 17, 19, 21, 27].

In this paper we study the following nonlinear parabolic equation with a nonlocal diffusion term

$$(1)\quad \begin{cases} u_t - \operatorname{div}\left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla u\right) + f(u) = g(x), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

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where $p \geq 2$, $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary $\partial\Omega$, the nonlinearity f, the diffusion coefficient a and the external force g satisfy the following conditions:

(H1) $a \in C(\mathbb{R}, \mathbb{R}_+)$ and there are two positive constants m and M such that

$$0 < m \leq a(s) \leq M, \quad \forall s \in \mathbb{R}. \tag{2}$$

Moreover, the mapping a is such that

$$s \mapsto a(s^p)s^{p-1} \text{ is nondecreasing.} \tag{3}$$

(H2) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$c_1|u|^q - c_0 \leq f(u)u \leq c_2|u|^q + c_0, \tag{4}$$

$$f'(u) \geq -c_3 \tag{5}$$

for some $q \geq 2$, where c_0, c_1, c_2, c_3 are positive constants.

(H3) $g \in L^2(\Omega)$.

The investigation in the present paper is motivated by the fact that the appearance of nonlocal terms and p-Laplacian operator in a variety of physical fields such as fluid dynamics and other phenomena, like nonlinear elasticity, nonlinear filtration or distribution of magnetic fields, see [7, 8, 23, 24]. Besides, the nonlocal terms might give more accurate results since in reality the measurements are not made pointwise, but through some local average. For instance, in population dynamics, the diffusion coefficient a is then supposed to depend upon the entire population in the domain rather than on a local density. For more details and other kinds of nonlocal terms, we refer the interested readers to [1, 2, 17, 19, 31] and references therein. However, this leads to a number of mathematical difficulties which make the analysis of the problem particularly interesting.

Let us review some related results in the case $a \equiv 1$, that is, the case without nonlocal term. The existence of global attractors for the p-Laplacian equations has been studied extensively by many authors in the last years, see e.g. [9, 13, 15, 20, 29] and references therein. In [9], Babin and Vishik established the existence of an $(L^2(\Omega), (W_0^{1,p}(\Omega) \cap L^q(\Omega))_w)$-global attractor; in Temam [29] only the special case $f = ku$ was discussed; in [13], Carvalho, Cholewa and Dlotko considered the existence of global attractors for problems with monotone operators, and as such an application, they got the existence of an $(L^2(\Omega), L^2(\Omega))$-global attractor for the p-Laplacian equation, see also Cholewa and Dlotko [20]. Recently, Carvalho and Gentile in [15], combining with their comparison results developed in [14], obtained that the corresponding semigroup has an $(L^2(\Omega), W_0^{1,p}(\Omega))$-global attractor. However they need some additional assumptions, i.e., either assume that $p > \frac{N}{2}$ and $f = f_1 + f_2$, where f_1 satisfies $(f_1, u) \geq 0$ and f_2 is a global $(L^2(\Omega), L^2(\Omega))$-Lipschitz mapping, or assume that f satisfies some growth condition such that it can be dominated by the p-Laplacian operator. The most general results were obtained in [30] where the authors proved the existence of an $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractor. We also refer the interested reader to [3–5] for results for parabolic equations involving weighted p-Laplacian operators.

The problem (1) was first studied by Chipot and Savitska in [17] in the case $f = 0$, $g \in W^{-1,p'}(\Omega)$ and $u_0 \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ with $1 < p < \infty$, $1/p + 1/p' = 1$. In this paper, we will extend the work of Chipot and Savitska in [17] by studying problem (1) in the case that the nonlinearity $f(u)$ satisfies a polynomial type condition of arbitrary order $q \geq 2$ and the external force g belongs to $L^2(\Omega)$. We will study the existence and long-time behavior of solutions in terms of existence of global attractors. Since the nonlinearity f is superlinear and the diffusion term depends upon the L^p-norm of the gradient of unknown function, this will generate some essential difficulties when studying the problem. The obtained results are extensions of some previous results for p-Laplacian parabolic equations and nonlocal parabolic equations.

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of a global weak solution to problem (1) by combining the compactness and the monotonicity methods. In Section 3, we show the existence of global attractors in various spaces for the semigroup associated to problem (1). Due to *a priori* estimates of the solutions in $W_0^{1,p}(\Omega)$ and the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, we immediately obtain the existence of an $(L^2(\Omega), L^2(\Omega))$-global attractor. However, the main difficulties actually arise when we prove the existence of $(L^2(\Omega), L^q(\Omega))$- and $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractors. Indeed, under the hypotheses **(H1), (H2)**, and **(H3)**, the solutions are at most in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$, and there are no compact embedding results in this case, so we need to prove the asymptotic compactness for the semigroup. In order to overcome these difficulties, we exploit the approach used in [3, 30, 32], which is so-called the *a priori* estimate method and has been used for some kind of parabolic partial differential equations. In the last section, we prove the existence of a weak stationary solution to problem (1) and we give a sufficient condition that ensures the uniqueness and exponential stability of the weak stationary solution.

2. Existence and uniqueness of weak solutions

In this section, we will study the existence and uniqueness of weak solutions to problem (1). Denote

$$\begin{aligned}
&\Omega_T := \Omega \times (0, T),\\
&V := L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q(\Omega_T),\\
&V^* := L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^{q'}(\Omega_T),
\end{aligned}$$

where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

Definition. Let $u_0 \in L^2(\Omega)$. A function u is called a weak solution of problem (1) on the interval $(0,T)$ if and only if

$$u \in V, \ \ \frac{du}{dt} \in V^*,$$

$$u|_{t=0} = u_0 \ \text{ a.e. in } \Omega,$$

and

$$\int_{\Omega_T} \left(u_t v + a\left(\|\nabla u\|^p_{L^p(\Omega)}\right) |\nabla u|^{p-2}\nabla u \cdot \nabla v + f(u)v - gv \right) dxdt = 0 \tag{6}$$

for all test functions $v \in V$.

It is known that if $u \in V$ and $du/dt \in V^*$, then $u \in C([0,T]; L^2(\Omega))$ (see e.g. [4]). This makes the initial condition in problem (1) meaningful.

We have the Poincaré type inequality

$$\lambda_1 \|u\|^p_{L^p(\Omega)} \le \|u\|^p_{W_0^{1,p}(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega), \tag{7}$$

where $\lambda_1 > 0$ is the first eigenvalue of operator $-\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ in $W_0^{1,p}(\Omega)$.

Using the Hölder inequality, the inequality (7), the Young inequality and the embedding $L^p(\Omega) \hookrightarrow L^2(\Omega), p \ge 2$, we have the following inequality for any $\varepsilon > 0$:

$$\left| \int_\Omega g u dx \right| \le \varepsilon \|u\|^p_{W_0^{1,p}(\Omega)} + \frac{|\Omega|^{\frac{(p-2)p'}{2p}}}{p'(p\varepsilon\lambda_1)^{\frac{p'}{p}}} \|g\|^{p'}_{L^2(\Omega)}, \ \forall u \in W_0^{1,p}(\Omega). \tag{8}$$

We need the following lemma.

Lemma 2.1 ([17])**.** *Under the assumption* **(H1)**, $-\mathrm{div}\left(a(\|\nabla u\|^p_{L^p(\Omega)}) |\nabla u|^{p-2}\nabla u \right)$ *is a monotone operator in* $W_0^{1,p}(\Omega)$*, i.e.,*

$$\begin{aligned}\Big\langle -\mathrm{div}\left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right) |\nabla u|^{p-2}\nabla u \right) \\ + \mathrm{div}\left(a\left(\|\nabla v\|^p_{L^p(\Omega)}\right) |\nabla v|^{p-2}\nabla v \right), u - v \Big\rangle \ge 0\end{aligned} \tag{9}$$

for all $u, v \in W_0^{1,p}(\Omega)$.

We are now ready to prove the following theorem.

Theorem 2.2. *Under the assumptions* **(H1)**, **(H2)**, *and* **(H3)**, *for each* $u_0 \in L^2(\Omega)$ *given, problem* (1) *has a unique weak solution* $u(\cdot)$ *satisfying*

$$u \in C([0,\infty); L^2(\Omega)) \cap L^p_{loc}(0,\infty; W_0^{1,p}(\Omega)) \cap L^q_{loc}(0,\infty; L^q(\Omega)),$$

$$\frac{du}{dt} \in L^{p'}_{loc}(0,\infty; W^{-1,p}(\Omega)) + L^{q'}_{loc}(0,\infty; L^{q'}(\Omega)).$$

Moreover, the mapping $u_0 \mapsto u(t)$ *is* $(L^2(\Omega), L^2(\Omega))$*-continuous.*

Proof. **i) Existence**. We consider approximate solutions $u_n(t)$ in the form

$$u_n(t) = \sum_{k=1}^{n} u_{nk}(t)e_k,$$

where $\{e_j\}_{j=1}^{\infty}$ is a basis of $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ consisting of eigenvalues of the operator $\Delta_p u$. Without loss of generality, one can suppose that $\{e_j\}_{j=1}^{\infty}$ is orthonormal in $L^2(\Omega)$. Therefore, u_n can be obtained by solving the following problem

$$\begin{cases} \int_\Omega \left[\frac{du_n}{dt} e_k + a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right) |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla e_k + f(u_n)e_k \right] dx \\ = \int_\Omega g(x)e_k dx, \\ \sum_{k=1}^{n} u_{nk}(0)e_k \to u_0 \text{ strongly in } L^2(\Omega) \ \text{ as } \ n \to \infty. \end{cases} \tag{10}$$

Since $f \in C^1(\mathbb{R})$, it follows from the Peano theorem that the Cauchy problem (10) possesses a unique (local) solution. We now establish some *a priori* estimates for u_n.

Multiplying (10) by $u_{nk}(t)$ and summing these relations from $k = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right) \int_\Omega |\nabla u_n|^p dx + \int_\Omega f(u_n)u_n dx = \int_\Omega g u_n dx. \tag{11}$$

In view of (2) and (4), we have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + m\int_\Omega |\nabla u_n|^p dx + c_1\int_\Omega |u_n|^q dx \le c_0|\Omega| + \int_\Omega g u_n dx.$$

Hence, it follows from (8) with $\varepsilon = m/2$ that

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + c_4\left(\int_\Omega |\nabla u_n|^p dx + \int_\Omega |u_n|^q dx\right) \le c_5, \tag{12}$$

where $c_4 = \min\{m, 2c_1\}$ and

$$c_5 = 2\frac{|\Omega|^{\frac{(p-2)p'}{2p}}}{p'(pm\lambda_1/2)^{\frac{p'}{p}}}\|g\|^{p'}_{L^2(\Omega)} + 2c_0|\Omega|.$$

Using the inequality (7), we get from (12) that

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + m\lambda_1\|u_n\|^p_{L^p(\Omega)} \le c_5. \tag{13}$$

Noting that for $p \ge 2$, there exists a constant $c_6 > 0$ such that

$$m\lambda_1\|u_n\|^p_{L^p(\Omega)} \ge m\lambda_1\|u_n\|^2_{L^2(\Omega)} - c_6.$$

Hence (13) becomes

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + m\lambda_1\|u_n\|^2_{L^2(\Omega)} \le c_5 + c_6. \tag{14}$$

Using the Gronwall inequality to (14), we obtain

$$\|u_n(t)\|^2_{L^2(\Omega)} \le \|u_n(0)\|^2_{L^2(\Omega)} e^{-m\lambda_1 t} + \frac{1}{m\lambda_1}(c_5 + c_6)\left(1 - e^{-m\lambda_1 t}\right). \tag{15}$$

This estimate implies that the solution $u_n(t)$ of (10) can be extended to $[0, +\infty)$.

Let T be an arbitrary positive number, integrating both sides of (12) from 0 to T, we obtain

$$\|u_n(T)\|^2_{L^2(\Omega)} + c_4 \left(\int_{\Omega_T} |\nabla u_n|^p dxdt + \int_{\Omega_T} |u_n|^q dxdt \right) \le \|u_n(0)\|^2_{L^2(\Omega)} + Tc_5.$$

This inequality yields

$$\begin{aligned}
&\{u_n\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\\
&\{u_n\} \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)),\\
&\{u_n\} \text{ is bounded in } L^q(\Omega_T).
\end{aligned}$$

Notice that $-\text{div}\left(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n\right)$ defines an element of $W^{-1,p'}(\Omega)$ given by the duality

$$\begin{aligned}
&\left\langle -\text{div}\left(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n\right), v\right\rangle\\
&= a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx
\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$.

By using (2) and the boundedness of $\{u_n\}$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we have

$$\begin{aligned}
&\left|\int_0^T \left\langle -\text{div}\left(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n\right), v\right\rangle dt\right|\\
&= \left|\int_{\Omega_T} a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dxdt\right|\\
&\le M\int_{\Omega_T} |\nabla u_n|^{p-1}|\nabla v| dxdt\\
&\le M\|u_n\|^{p/p'}_{L^p(0,T;W_0^{1,p}(\Omega))}\|v\|_{L^p(0,T;W_0^{1,p}(\Omega))}
\end{aligned}$$

for any $v \in L^p(0,T;W_0^{1,p}(\Omega))$. We deduce that

$$\left\{-\text{div}\left(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n\right)\right\} \text{ is bounded in } L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

On the other hand, it follows from (4) that

$$|f(u)| \le C(|u|^{q-1} + 1).$$

Using this together with the boundedness of $\{u_n\}$ in $L^q(\Omega_T)$, one can shows that $\{f(u_n)\}$ is bounded in $L^{q'}(\Omega_T)$. We rewrite the equation as

$$\frac{du_n}{dt} = g + \text{div}\left(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n\right) - f(u_n). \tag{16}$$

Therefore, $\left\{\frac{du_n}{dt}\right\}$ is bounded in V^*. In addition, we have the following chain of embeddings

$$W_0^{1,p}(\Omega) \subset\subset L^p(\Omega) \subset W^{-1,p'}(\Omega) + L^{q'}(\Omega).$$

We deduce from the Aubin-Lions lemma that $\{u_n\}$ is compact in $L^p(0,T;L^p(\Omega))$. Now applying the diagonalization procedure and using Lemma 1.3 in [25, p. 12] and Theorem 4.18 in [26, p. 105], we obtain (up to a subsequence) that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in} \quad L^p(0,T;W_0^{1,p}(\Omega)),\\ u_n &\to u \quad \text{in} \quad L^p(0,T;L^p(\Omega)),\\ \frac{du_n}{dt} &\rightharpoonup \frac{du}{dt} \quad \text{in} \quad V^*,\\ u_n(T) &\to u(T) \quad \text{in} \quad L^2(\Omega), \end{aligned}$$

and

$$f(u_n) \rightharpoonup f(u) \quad \text{in} \quad L^{q'}(\Omega_T), \tag{17}$$

$$-\operatorname{div}\left(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n\right) \rightharpoonup -\chi \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{18}$$

Now, passing to the limit in (16), one has in the distributional sense in Ω_T

$$u_t - \chi + f(u) = g. \tag{19}$$

Integrating (11) from 0 to T leads to

$$\begin{aligned} \int_0^T a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u_n|^p dxdt = \int_{\Omega_T} g u_n dxdt - \int_{\Omega_T} f(u_n)u_n dxdt\\ + \frac{\|u_n(0)\|^2_{L^2(\Omega)}}{2} - \frac{\|u_n(T)\|^2_{L^2(\Omega)}}{2}. \end{aligned} \tag{20}$$

Since $\lim_{n\to\infty} \|u_n(T)\|^2_{L^2(\Omega)} = \|u(T)\|^2_{L^2(\Omega)}$ and $\lim_{n\to\infty} \|u_n(0)\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega)}$, we deduce from (20) that

$$\begin{aligned} \lim_{n\to\infty}\int_0^T a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u_n|^p dxdt = \int_{\Omega_T} g u dxdt - \int_{\Omega_T} f(u)u dxdt\\ + \frac{\|u_0\|^2_{L^2(\Omega)}}{2} - \frac{\|u(T)\|^2_{L^2(\Omega)}}{2}. \end{aligned} \tag{21}$$

On the other hand, from Lemma 2.1 we have

$$\begin{aligned} \int_{\Omega_T} \Big(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n\\ - a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)|\nabla v|^{p-2}\nabla v\Big)\cdot\nabla(u_n - v)dxdt \geq 0. \end{aligned}$$

We derive by taking limit for any $v \in L^p(0,T;W_0^{1,p}(\Omega))$

$$\lim_{n\to\infty}\int_0^T a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u_n|^p dxdt + \int_0^T \langle \chi, v\rangle\, dt$$

$$-\int_{\Omega_T} a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)|\nabla v|^{p-2}\nabla v\cdot\nabla(u-v)dxdt \geq 0.$$

Therefore, in view of (21) and the last inequality, we have

$$\begin{aligned}(22)\qquad &\int_{\Omega_T} gudxdt - \int_{\Omega_T} f(u)udxdt + \frac{\|u_0\|^2_{L^2(\Omega)}}{2} - \frac{\|u(T)\|^2_{L^2(\Omega)}}{2}\\ &+\int_0^T \langle \chi, v\rangle\, dt - \int_{\Omega_T} a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)|\nabla v|^{p-2}\nabla v\cdot\nabla(u-v)dxdt \geq 0.\end{aligned}$$

On the other hand, by integrating (19) from 0 to T after taking inner product with u, we obtain

$$\begin{aligned}(23)\qquad -\int_0^T \langle \chi, u\rangle\, dt = &\int_{\Omega_T} gudxdt - \int_{\Omega_T} f(u)udxdt\\ &+\frac{\|u_0\|^2_{L^2(\Omega)}}{2} - \frac{\|u(T)\|^2_{L^2(\Omega)}}{2}.\end{aligned}$$

Combining (22) with (23), we have

$$\begin{aligned}&\int_0^T \left\langle \chi - \operatorname{div}\left(a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)|\nabla v|^{p-2}\nabla v\right), u-v\right\rangle dt \leq 0,\\ &\forall v\in L^p(0,T;W^{1,p}_0(\Omega)).\end{aligned}$$

Choosing $v = u-\delta\varphi$, we see that

$$\int_0^T \left\langle \chi - \operatorname{div}\left(a\left(\|\nabla (u-\delta\varphi)\|^p_{L^p(\Omega)}\right)|\nabla (u-\delta\varphi)|^{p-2}\nabla (u-\delta\varphi)\right), \varphi\right\rangle dt \leq 0,$$

if $\delta>0$ and

$$\int_0^T \left\langle \chi - \operatorname{div}\left(a\left(\|\nabla (u-\delta\varphi)\|^p_{L^p(\Omega)}\right)|\nabla (u-\delta\varphi)|^{p-2}\nabla (u-\delta\varphi)\right), \varphi\right\rangle dt \geq 0,$$

if $\delta<0$, for all $\varphi\in L^p(0,T;W^{1,p}_0(\Omega))$. Letting $\delta\to 0$, we get

$$\int_0^T \left\langle \chi - \operatorname{div}\left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla u\right), \varphi\right\rangle dt = 0,\ \forall \varphi\in L^p(0,T;W^{1,p}_0(\Omega)).$$

This implies that $\chi = \operatorname{div}\left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla u\right)$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, which completes the proof of the existence of the solution.

Analogously to (15) we have

$$(24)\qquad \|u(t)\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)}e^{-m\lambda_1 t} + \frac{1}{m\lambda_1}(c_5+c_6)\left(1-e^{-m\lambda_1 t}\right).$$

This implies that the solution u exists globally in time.

ii) Uniqueness and continuous dependence on the initial data. Let u, v be two weak solutions to (1) with initial data $u_0, v_0\in L^2(\Omega)$, respectively.

Taking $w = u - v$, and then the following equations are directly obtained from (1) by subtraction

$$(25)\qquad \begin{cases} w_t - \operatorname{div}\left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla u\right) \\ + \operatorname{div}\left(a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)|\nabla v|^{p-2}\nabla v\right) + f(u) - f(v) = 0, \\ w(0) = u_0 - v_0. \end{cases}$$

Multiplying (25) by w and integrating over Ω, one gets

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} \\ &+ \int_\Omega \left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla u - a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)|\nabla v|^{p-2}\nabla v\right)\cdot\nabla(u-v)dx \\ &+ \int_\Omega (f(u)-f(v))(u-v)dx = 0. \end{aligned}$$

It follows from (5) and Lemma 2.1 that

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} \le 2c_3\|w\|^2_{L^2(\Omega)}.$$

Applying the Gronwall inequality, we obtain

$$\|w\|^2_{L^2(\Omega)} \le \|w(0)\|^2_{L^2(\Omega)}e^{2c_3 t}.$$

This completes the proof. □

3. Existence of global attractors

3.1. The $(L^2(\Omega), L^2(\Omega))$-global attractor

Theorem 2.2 allows us to construct a continuous (nonlinear) semigroup $S(t): L^2(\Omega) \to L^2(\Omega)$ associated to problem (1) as follows

$$S(t)u_0 := u(t),$$

where $u(\cdot)$ is the unique global weak solution of (1) with the initial datum u_0.

For the sake of brevity, in the following propositions, we just give some formal calculations, their rigorous proofs are done by use of Galerkin approximations and Lemma 11.2 in [26].

We see from (24) that the ball $B_0 = B(\sqrt{\rho_0})$ with $\rho_0 = \frac{2}{m\lambda_1}(c_5 + c_6)$, is an $(L^2(\Omega), L^2(\Omega))$-bounded absorbing set of $\{S(t)\}_{t\ge 0}$, i.e., for any bounded set B in $L^2(\Omega)$, there exists $T_0 = T_0(B)$ depending only on the L^2-norm of B such that

$$\|S(t)u_0\|^2_{L^2(\Omega)} \le \rho_0$$

for all $t \ge T_0$, $u_0 \in B$.

Proposition 3.1. *The semigroup $\{S(t)\}_{t\ge 0}$ has an $(L^2(\Omega), W_0^{1,p}(\Omega))$-bounded absorbing set B_1.*

Proof. First, as in (12) we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + c_4\left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^q dx\right) \le c_5.$$

Integrating the above inequality from t to $t+1$, for $t \ge T_0$, and using $u(t) \in B_0$ we have

$$\int_t^{t+1} \|\nabla u(s)\|^p_{L^p(\Omega)} ds \le \frac{c_5+\rho_0}{c_4}. \tag{26}$$

Now, multiplying the first equation in (1) by $-\Delta_p u$, we get

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\nabla u\|^p_{L^p(\Omega)} + \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} + a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\|\Delta_p u\|^2_{L^2(\Omega)}\\ &= \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} - \int_\Omega f'(u)|\nabla u|^p dx - \langle g, \Delta_p u\rangle .\end{aligned}$$

Putting this together with (2) and (5) leads to

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\nabla u\|^p_{L^p(\Omega)} + \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} + m\|\Delta_p u\|^2_{L^2(\Omega)}\\ &\le (\frac{1}{p} + c_3)\|\nabla u\|^p_{L^p(\Omega)} - \langle g, \Delta_p u\rangle .\end{aligned} \tag{27}$$

On the other hand, using the Cauchy inequality we have

$$\begin{aligned}&(\frac{1}{p}+c_3)\|\nabla u\|^p_{L^p(\Omega)} - \langle g, \Delta_p u\rangle = -(\frac{1}{p}+c_3)\langle u, \Delta_p u\rangle - \langle g, \Delta_p u\rangle\\ &\le m\|\Delta_p u\|^2_{L^2(\Omega)} + \frac{1}{2m}\|g\|^2_{L^2(\Omega)} + \frac{(1/p+c_3)^2}{2m}\|u\|^2_{L^2(\Omega)}.\end{aligned} \tag{28}$$

In view of (27) and (28) with note that $u(t) \in B_0$ for all $t \ge T_0$, we have

$$\frac{d}{dt}\|\nabla u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)} \le R_1 := \frac{p}{2m}\|g\|^2_{L^2(\Omega)} + \frac{(1+pc_3)^2\rho_0}{2m}. \tag{29}$$

Applying the uniform Gronwall inequality to (26) and (29) we have

$$\|\nabla u(t)\|^p_{L^p(\Omega)} \le R_1 + \frac{c_5+\rho_0}{c_4}, \quad \forall t \ge T_0 + 1.$$

Hence, the ball $B_1 = B_{W_0^{1,p}(\Omega)}(\rho_1^{-p})$ with $\rho_1 = (1+1/\lambda_1)\left(R_1 + \frac{c_5+\rho_0}{c_4}\right)$ is a bounded absorbing set in $W_0^{1,p}(\Omega)$ for the semigroup $\{S(t)\}_{t\ge 0}$, i.e., for any bounded set B in $L^2(\Omega)$, there exists $T_1 = T_1(B) := T_0 + 1$ depending only on the L^2-norm of B such that

$$\|S(t)u_0\|^p_{W_0^{1,p}(\Omega)} \le \rho_1 \tag{30}$$

for all $t \ge T_1$, $u_0 \in B$. □

As a direct result of Proposition 3.1 and the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, we get the following result.

Theorem 3.2. *Assume that the hypotheses* **(H1)**, **(H2)**, *and* **(H3)** *are satisfied. Then the semigroup* $S(t)$ *generated by problem* (1) *has an* $(L^2(\Omega), L^2(\Omega))$*-global attractor* $\mathcal{A}_2$.

3.2. The $(L^2(\Omega), L^q(\Omega))$-global attractor

In this and the next subsections, we will prove the existence of $(L^2(\Omega), L^q(\Omega))$- and $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractors, respectively. To do this, we assume furthermore that

(H1bis) The diffusion coefficient a is continuously differentiable, nondecreasing and satisfies condition **(H1)**.

First, we prove the existence of a bounded absorbing set in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ for the semigroup $S(t)$.

Proposition 3.3. *Assume that the assumptions* **(H1bis)**, **(H2)**, *and* **(H3)** *hold. Then the semigroup* $\{S(t)\}_{t\geq 0}$ *has an* $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$*-bounded absorbing set* B_2*, that is, there is a positive constant* ρ_2 *such that for any bounded subset* B *in* $L^2(\Omega)$*, there is a positive constant* T_2 *depending only on* L^2*-norm of* B *such that*

$$\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^q dx \leq \rho_2$$

for all $t \geq T_2$ *and* $u_0 \in B$*, where* u *is the unique weak solution of* (1) *with the initial datum* u_0.

Proof. Multiplying the first equation in (1) by u and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\|u\|^p_{W_0^{1,p}(\Omega)} + \int_\Omega f(u)udx = \int_\Omega gudx.$$

Then integrating this inequality over $[t, t+1]$ with $t \geq T_1$, we derive

$$\int_t^{t+1}\Big[a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u|^p dx + \int_\Omega f(u)udx - \int_\Omega gudx\Big]ds \leq \frac{\rho_0}{2} \tag{31}$$

for all $t \geq T_1$. We define

$$F(u) = \int_0^u f(s)ds.$$

Due to (4) and (5), it fulfills the bounds for some positive constants c_7, c_8

$$c_7|u|^q - c_8 \leq F(u) \leq uf(u) + \frac{c_3}{2}|u|^2. \tag{32}$$

Therefore,

$$\int_\Omega F(u)dx \leq \int_\Omega f(u)udx + \frac{c_3\rho_0}{2}. \tag{33}$$

We deduce from (31) and (33) that

$$\int_t^{t+1}\Big[\frac{1}{p}a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u|^p dx + \int_\Omega F(u)dx - \int_\Omega gudx\Big]ds \tag{34}$$

$$\leq \frac{\rho_0(c_3+1)}{2}.$$

On the other hand, multiplying (1) by u_t, we obtain

$$\|u_t\|^2_{L^2(\Omega)} + a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u_t dx + \int_\Omega f(u)u_t dx - \int_\Omega g u_t dx = 0.$$

We can rewrite the last equality as

$$\|u_t\|^2_{L^2(\Omega)} + \frac{d}{dt}\left[\frac{1}{p}a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u|^p dx + \int_\Omega F(u)dx - \int_\Omega gudx\right] \tag{35}$$
$$= \frac{1}{p}a'(\|\nabla u\|^p_{L^p(\Omega)})\,\|\nabla u\|^p_{L^p(\Omega)}\,\frac{d}{dt}\|\nabla u\|^p_{L^p(\Omega)}.$$

Setting $L = \sup_{0\leq s\leq \rho_1} |a'(s)|$, then from (29), (30) and (35), we have

$$\frac{d}{dt}\left[\frac{1}{p}a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u|^p dx + \int_\Omega F(u)dx - \int_\Omega gudx\right] \leq \frac{LR_1^2}{p}. \tag{36}$$

Therefore, from (34) and (36), by using the uniform Gronwall inequality, we get

$$\frac{1}{p}a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u|^p dx + \int_\Omega F(u)dx - \int_\Omega gudx \leq \frac{\rho_0(c_3+1)}{2} + \frac{LR_1^2}{p}. \tag{37}$$

Using (2), (4) and the Cauchy inequality for the term $\int_\Omega gudx$, we deduce from (37) and (32) that for all $t \geq T_2 = T_1 + 1$:

$$\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^q dx \leq \rho_2 := \frac{c_8|\Omega| + \rho_0(1+c_3)/2 + LR_1^2/p + \|g\|^2_{L^2(\Omega)}/2}{\min\left\{\frac{m}{p}; c_7\right\}}.$$

This ends the proof. □

Proposition 3.4. *The semigroup $\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous on $S(B_2)$, where B_2 is the $(L^2(\Omega), W_0^{1,p}(\Omega)\cap L^q(\Omega))$-bounded absorbing set obtained in Proposition 3.3.*

Proof. Choosing $Y = L^2(\Omega)$, $X = W_0^{1,p}(\Omega)\cap L^q(\Omega)$, the conclusion follows immediately from Theorem 3.2 in [32]. □

The set B_2 obtained in Proposition 3.3 is also of course an $(L^2(\Omega), L^q(\Omega))$-bounded absorbing set for the semigroup $S(t)$. To prove the existence of a global attractor in $L^q(\Omega)$, we will use the following results.

Lemma 3.5 ([32])**.** *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on $L^2(\Omega)$ and has an $(L^2(\Omega), L^2(\Omega))$-global attractor. Then $\{S(t)\}_{t\geq 0}$ has an $(L^2(\Omega), L^q(\Omega))$-global attractor provided that the following conditions holds:*

(i) *$\{S(t)\}_{t\geq 0}$ has an $(L^2(\Omega), L^q(\Omega))$-bounded absorbing set;*

(ii) *for any $\varepsilon > 0$ and any bounded subset B of $L^2(\Omega)$, then there exist positive constants $M = M(\varepsilon)$ and $T = T(\varepsilon, B)$ such that for all $u_0 \in B$ and $t \geq T$:*

$$\int_{\Omega(|S(t)u_0| \geq M)} |S(t)u_0|^q < \varepsilon.$$

Lemma 3.6 ([32])**.** *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on $L^r(\Omega)$ $(r \geq 1)$ and suppose that $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^r(\Omega)$. Then for any $\varepsilon > 0$ and any bounded subset $B \subset L^r(\Omega)$, there exist positive constants $T = T(B)$ and $M = M(\varepsilon)$ such that $|\Omega(|S(t)u_0| \geq M)| \leq \varepsilon$ for all $u_0 \in B$ and $t \geq T$.*

Theorem 3.7. *Assume that the hypotheses* **(H1bis)**, **(H2)**, *and* **(H3)** *are satisfied. Then the semigroup $S(t)$ associated to* (1) *has an $(L^2(\Omega), L^q(\Omega))$-global attractor $\mathcal{A}_q$.*

Proof. We know that $\{S(t)\}_{t\geq 0}$ has an $(L^2(\Omega), L^q(\Omega))$-bounded absorbing set B_2 and $\{S(t)\}_{t\geq 0}$ has an $(L^2(\Omega), L^2(\Omega))$-global attractor. By Lemma 3.5, it is sufficient to prove that for any $\varepsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist two positive constants $T = T(\varepsilon, B)$ and $M = M(\varepsilon)$ such that

$$\int_{\Omega(|u| \geq M)} |u|^q < C\varepsilon$$

for all $u_0 \in B$ and $t \geq T$, where the constant C is independent of ε and B, where $\Omega(u \geq M) := \{x \in \Omega : u(x) - M \geq 0\}$. It follows from Lemma 3.6 that for any fixed $\varepsilon > 0$, there exist $\delta > 0$, $T = T(B)$ and $M = M(\varepsilon)$ such that the Lebesgue measure $|\Omega(|S(t)u_0| \geq M)| \leq \delta$ for all $u_0 \in B$ and $t \geq T$ and

$$\int_{\Omega(|S(t)u_0| \geq M)} |g|^2 < \varepsilon. \tag{38}$$

We now multiply the first equation in (1) by $(u - M)_+^{q-1}$ to get that

$$\begin{aligned} &u_t(u - M)_+^{q-1} - \operatorname{div}\left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla u\right)(u - M)_+^{q-1} \\ &\quad + f(u)(u - M)_+^{q-1} = g(x)(u - M)_+^{q-1}, \end{aligned} \tag{39}$$

where $(u - M)_+$ denotes the positive part of $(u - M)$, that is,

$$(u - M)_+ = \begin{cases} u - M, & \text{if} \quad u \geq M, \\ 0, & \text{if} \quad u < M, \end{cases}$$

and M is a positive constant. We deduce from (4) that $f(u) \geq \widetilde{c}|u|^{q-1}$ with $u \geq M$ and M is large enough. Thus

$$\begin{aligned} f(u)(u - M)_+^{q-1} &\geq \widetilde{c}|u|^{q-1}(u - M)_+^{q-1} \\ &= \frac{\widetilde{c}}{2}|u|^{q-1}(u - M)_+^{q-1} + \frac{\widetilde{c}}{2}|u|^{q-1}(u - M)_+^{q-1} \\ &\geq \frac{\widetilde{c}}{2}(u - M)_+^{2(q-1)} + \frac{\widetilde{c}}{2}|u|^{q-2}(u - M)_+^{q} \end{aligned} \tag{40}$$

$$\geq \frac{\widetilde{c}}{2}(u-M)_+^{2(q-1)} + \frac{\widetilde{c}}{2}M^{q-2}(u-M)_+^q.$$

In addition

$$g(u-M)_+^{q-1} \leq \frac{\widetilde{c}}{2}(u-M)_+^{2(q-1)} + \frac{|g|^2}{2\widetilde{c}}. \tag{41}$$

It follows from (39), (40) and (41) that

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_{\Omega(u\geq M)}(u-M)_+^q dx \\
&+ (q-1)\int_{\Omega(u\geq M)} a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^p (u-M)_+^{q-2}dx \\
&+ \frac{\widetilde{c}}{2}M^{q-2}\int_{\Omega(u\geq M)}(u-M)_+^q dx \leq \frac{1}{2\widetilde{c}}\int_{\Omega(u\geq M)}|g|^2 dx,
\end{aligned}$$

and then

$$\frac{d}{dt}\int_{\Omega(u\geq M)}(u-M)_+^q dx + \frac{\widetilde{c}}{2}qM^{q-2}\int_{\Omega(u\geq M)}(u-M)_+^q dx \leq \frac{q}{2\widetilde{c}}\int_{\Omega(u\geq M)}|g|^2 dx.$$

By the Gronwall inequality, we have

$$\begin{aligned}
\int_{\Omega(u\geq M)}(u-M)_+^q dx \leq{}& e^{-\frac{\widetilde{c}}{2}qM^{q-2}t}\int_{\Omega(u\geq M)}(u(0)-M)_+^q dx \\
&+ \frac{1-e^{-\frac{\widetilde{c}}{2}qM^{q-2}t}}{\widetilde{c}^2 M^{q-2}}\int_{\Omega(u\geq M)}|g|^2 dx.
\end{aligned}$$

If we take M large enough, taking (38) into account, the last inequality leads to

$$\int_{\Omega(u\geq M)}(u-M)_+^q dx < \varepsilon. \tag{42}$$

Repeating the same steps above, just taking $(u+M)_-$ instead of $(u-M)_+$ where

$$(u+M)_- = \begin{cases} u+M & \text{if} \quad u \leq -M, \\ 0 & \text{if} \quad u > -M, \end{cases}$$

we also obtain

$$\int_{\Omega(u\leq -M)}|(u+M)_-|^q dx < \varepsilon, \tag{43}$$

where $\Omega(u\leq -M) := \{x\in\Omega : u(x)+M \leq 0\}$.

In both cases, we deduce from (42) and (43) that

$$\int_{\Omega(|u|\geq M)}(|u|-M)^q dx < \varepsilon$$

for M large enough. Therefore,

$$\int_{\Omega(|u|\geq 2M)}|u|^q dx = \int_{\Omega(|u|\geq 2M)}(|u|-M+M)^q dx$$

$$\leq 2^q \int_{\Omega(|u|\geq 2M)} (|u|-M)^q dx + 2^q \int_{\Omega(|u|\geq 2M)} M^q dx$$
$$\leq 2^{q+1} \int_{\Omega(|u|\geq 2M)} (|u|-M)^q dx$$
$$< C\varepsilon$$

for M large enough and C is independent of ε and B. As a consequence, the semigroup $S(t)$ has an $(L^2(\Omega), L^q(\Omega))$-global attractor $\mathcal{A}_q$. □

3.3. The $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractor

Lemma 3.8. *Assume that the assumptions* **(H1bis)**, **(H2)**, *and* **(H3)** *hold. Then for any bounded subset B in $L^2(\Omega)$, there exists a positive constant $T_3 = T_3(B)$ such that*

$$\|u_t(s)\|^2_{L^2(\Omega)} \leq \rho_3 \quad \textit{for all } u_0 \in B, \textit{and } s \geq T_3,$$

where $u_t(s) = \frac{d}{dt}(S(t)u_0)|_{t=s}$ and ρ_3 is a positive constant independent of u_0.

Proof. By differentiating the first equation in (1) in time and denoting $v = u_t$, we get

$$\begin{aligned} &v_t - \operatorname{div}\left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-2}\nabla v\right) \\ &\quad - (p-2)\operatorname{div}\left(a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)|\nabla u|^{p-4}(\nabla u\cdot\nabla v)\nabla u\right) \\ &\quad - p\operatorname{div}\left(a'(\|\nabla u\|^p_{L^p(\Omega)})\int_\Omega |\nabla u|^{p-2}(\nabla u\cdot\nabla v)dx|\nabla u|^{p-2}\nabla u\right) + f'(u)v = 0. \end{aligned}$$

Multiplying the above equality by v, integrating over Ω and using (5), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Omega)} + a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u|^{p-2}|\nabla v|^2 dx \\ &+ (p-2)a\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u|^{p-4}(\nabla u\cdot\nabla v)^2 dx \\ &+ pa'\left(\|\nabla u\|^p_{L^p(\Omega)}\right)\left(\int_\Omega \|\nabla u\|^{p-2}(\nabla u\cdot\nabla v)dx\right)^2 \leq c_3\|v\|^2_{L^2(\Omega)}. \end{aligned} \tag{44}$$

Since a is nondecreasing, it follows from (44) that

$$\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \leq 2c_3\|v\|^2_{L^2(\Omega)}. \tag{45}$$

On the other hand, we deduce from (29), (35), (36) and (37) that

$$\int_t^{t+1} \|u_t\|^2_{L^2(\Omega)} dx \leq C \tag{46}$$

for some positive constant C and $t \geq T_2$. Combining (45) with (46) and using the uniform Gronwall inequality we obtain

$$\|u_t\|^2_{L^2(\Omega)} \leq \rho_3$$

as $t \geq T_3 = T_2 + 1$, and ρ_3 is a some positive constant. The proof is complete. □

Lemma 3.9. *Let $p \geq 2$. Then under the assumption* **(H1bis)**, *we have for all $u_1, u_2 \in W_0^{1,p}(\Omega)$, that*

$$\begin{aligned}(47)\qquad &\Big\langle -\operatorname{div}\Big(a(\|\nabla u_1\|^p_{L^p(\Omega)})|\nabla u_1|^{p-2}\nabla u_1\Big) \\ &\qquad\qquad + \operatorname{div}\Big(a(\|\nabla u_2\|^p_{L^p(\Omega)})|\nabla u_2|^{p-2}\nabla u_2\Big), u_1 - u_2\Big\rangle \\ &= \int_\Omega \Big(a(\|\nabla u_1\|^p_{L^p(\Omega)})|\nabla u_1|^{p-2}\nabla u_1 \\ &\qquad\qquad - a(\|\nabla u_2\|^p_{L^p(\Omega)})|\nabla u_2|^{p-2}\nabla u_2\Big)\cdot \nabla(u_1-u_2)dx \\ &\geq c_p\|u_1-u_2\|^p_{W_0^{1,p}(\Omega)},\end{aligned}$$

where

$$c_p = \begin{cases} m & \text{if } p = 2,\\ \dfrac{m}{8.3^{p/2}} & \text{if } p > 2.\end{cases}$$

Proof. One sees that (47) is equivalent to proving that for $p \geq 2, x, y \in \mathbb{R}^N$, we have

$$(48)\qquad \big\langle a(|x|^p)|x|^{p-2}x - a(|y|^p)|y|^{p-2}y, x-y\big\rangle \geq c_p|x-y|^p.$$

Here $\langle\cdot,\cdot\rangle$ be the standard scalar product in $\mathbb{R}^N$.

Following the ideas in [22, Lemma 4.4], we have

$$\begin{aligned} I(p) &= \big\langle a(|x|^p)|x|^{p-2}x - a(|y|^p)|y|^{p-2}y, x-y\big\rangle \\ &= \Big\langle \int_0^1 \frac{d}{ds}\Big[a(|sx+(1-s)y|^p)|sx+(1-s)y|^{p-2}(sx+(1-s)y)\Big]ds, x-y\Big\rangle \\ &= |x-y|^2\int_0^1 a(|sx+(1-s)y|^p)|sx+(1-s)y|^{p-2}ds \\ &\quad + (p-2)\int_0^1 a(|sx+(1-s)y|^p)|sx+(1-s)y|^{p-4}|\langle(sx+(1-s)y), x-y\rangle|^2 ds \\ &\quad + p\int_0^1 a'(|sx+(1-s)y|^p)|sx+(1-s)y|^{2p-4}|\langle sx+(1-s)y, x-y\rangle|^2 ds \\ &\geq m|x-y|^2\int_0^1 |sx+(1-s)y|^{p-2}ds.\end{aligned}$$

- When $p = 2$ then we get (48) from the above inequality with $c_p = m$.
- Now, we consider the case $p > 2$.

If $|x| \geq |x-y|$, we have

$$|sx+(1-s)y| = |x-(1-s)(x-y)| \geq ||x|-(1-s)|x-y|| \geq s|x-y|.$$

Therefore,

$$I(p) \geq m|x-y|^p \int_0^1 s^{p-2} ds = \frac{m}{p-1}|x-y|^p.$$

If $|x| < |x-y|$, we have

$$|sx + (1-s)y| = |x + (1-s)(y-x)| \leq |x| + (1-s)|x-y| < (2-s)|x-y|.$$

Therefore,

$$\begin{aligned} I(p) &\geq m|x-y|^2 \int_0^1 \frac{(|sx+(1-s)y|^2)^{\frac{p}{2}}}{(2-s)|x-y|^2} ds \\ &\geq \frac{m}{2} \int_0^1 (|sx+(1-s)y|^2)^{\frac{p}{2}} ds \\ &\geq \frac{m}{2} \left(\int_0^1 |sx+(1-s)y|^2 ds \right)^{\frac{p}{2}} \\ &= \frac{m}{2} \frac{1}{3^{\frac{p}{2}}} \left(|x|^2 + \langle x, y \rangle + |y|^2 \right)^{\frac{p}{2}} \\ &\geq \frac{m}{8} \frac{1}{3^{\frac{p}{2}}} |x-y|^p. \end{aligned}$$

So we conclude for the case $p > 2$ that $I(p) \geq c_p|x-y|^p$ with $c_p = \frac{m}{8.3^{p/2}}$. □

To prove the existence of a global attractor in $W_0^{1,p}(\Omega)$, we will use the following result.

Theorem 3.10 ([32])**.** *Let X be a Banach space and Z be a metric space. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on X such that*:

(i) *$\{S(t)\}_{t\geq 0}$ has an (X, Z)-bounded absorbing set $\mathcal{A}$*;
(ii) *$\{S(t)\}_{t\geq 0}$ is (X, Z)-asymptotically compact*;
(iii) *$\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous on $S(\mathcal{A})$.*

Then $\{S(t)\}_{t\geq 0}$ has an (X, Z)-global attractor.

We are now in the position to state the main result of this section.

Theorem 3.11. *Assume that the assumptions* **(H1bis)**, **(H2)**, *and* **(H3)** *are satisfied. Then the semigroup $\{S(t)\}_{t\geq 0}$ associated to* (1) *has an $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractor $\mathcal{A}$.*

Proof. By Theorem 3.10 and Propositions 3.3-3.4, we only need to show that the semigroup $\{S(t)\}_{t\geq 0}$ is $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-asymptotically compact. This means that we take a bounded subset B of $L^2(\Omega)$, we have to show that for any $\{u_{0n}\} \subset B$ and $t_n \to +\infty$, $\{u_n(t_n)\}_{n=1}^{\infty}$ is precompact in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$, where $u_n(t_n) = S(t_n)u_{0n}$. By Theorem 3.7, it is sufficient to verify that $\{u_n(t_n)\}_{n=1}^{\infty}$ is precompact in $W_0^{1,p}(\Omega)$.

To do this, we will prove that $\{u_n(t_n)\}$ is a Cauchy sequence in $W_0^{1,p}(\Omega)$. Thanks to Theorems 3.2-3.7, one has that $\{u_n(t_n)\}$ is a Cauchy sequence in $L^2(\Omega)$ and in $L^q(\Omega)$. It follows from (47) that

$$\begin{aligned}
&c_p\|u_n(t_n)-u_m(t_m)\|^p_{W_0^{1,p}(\Omega)}\\
&\le \left\langle -\frac{d}{dt}u_n(t_n)-f(u_n(t_n))+\frac{d}{dt}u_m(t_m)+f(u_m(t_m)),u_n(t_n)-u_m(t_m)\right\rangle\\
&\le \int_\Omega |\frac{d}{dt}u_n(t_n)-\frac{d}{dt}u_m(t_m)|\,|u_n(t_n)-u_m(t_m)|dx\\
&\quad+\int_\Omega |f(u_n(t_n))-f(u_m(t_m))|\,|u_n(t_n)-u_m(t_m)|dx\\
&\le \|\frac{d}{dt}u_n(t_n)-\frac{d}{dt}u_m(t_m)\|_{L^2(\Omega)}\|u_n(t_n)-u_m(t_m)\|_{L^2(\Omega)}\\
&\quad+\|f(u_n(t_n))-f(u_m(t_m))\|_{L^{q'}(\Omega)}\|u_n(t_n)-u_m(t_m)\|_{L^q(\Omega)}.
\end{aligned}$$

It follows from Lemma 3.8 and the boundedness of $\{f(u_n(t_n))\}$ in $L^{q'}(\Omega)$ that $\{u_n(t_n)\}$ is a Cauchy sequence in $W_0^{1,p}(\Omega)$. This completes the proof. □

4. Existence and exponential stability of stationary solutions

An element $u^* \in W_0^{1,p}(\Omega)\cap L^q(\Omega)$ is said to be a weak stationary solution to problem (1) if

$$a(\|\nabla u^*\|^p_{L^p(\Omega)})\int_\Omega |\nabla u^*|^{p-2}\nabla u^*\cdot\nabla v dx+\int_\Omega f(u^*)v dx=\int_\Omega g v dx \tag{49}$$

for all test functions $v\in W_0^{1,p}(\Omega)\cap L^q(\Omega)$.

Theorem 4.1. *Under the hypotheses* **(H1)**, **(H2)**, *and* **(H3)**, *the problem* (1) *has at least one weak stationary solution u^* satisfying*

$$\|u^*\|^p_{W_0^{1,p}(\Omega)}+\|u^*\|^q_{L^q(\Omega)}\le \ell, \tag{50}$$

where

$$\ell=\frac{2p'c_0|\Omega|(pm\lambda_1)^{\frac{p'}{p}}+|\Omega|^{\frac{(p-2)p'}{2p}}\|g\|^{p'}_{L^2(\Omega)}2^{p'}}{\min\left\{1,\frac{2c_1}{m}\right\}mp'(pm\lambda_1)^{\frac{p'}{p}}}.$$

Moreover, if f is strictly increasing, i.e.,

$$f'(s)\ge\alpha>0 \;\; \textit{for all} \;\; s\in\mathbb{R}, \tag{51}$$

then for any solution u of (1), *we have*

$$\|u(t)-u^*\|^2_{L^2(\Omega)}\le\|u(0)-u^*\|^2_{L^2(\Omega)}e^{-2\alpha t} \;\textit{for all}\; t>0. \tag{52}$$

That is, the weak stationary solution of (1) *is unique and exponentially stable.*

Proof. **i) Existence.** We find an approximate stationary solution u_n by

$$u_n = \sum_{j=1}^{n} \gamma_{nj} e_j,$$

where $\{e_j\}_{j=1}^{\infty}$ is a basis of $W_0^{1,p}(\Omega) \cap L^q(\Omega)$. For each $n > 1$, we denote $V_n = \operatorname{span}\{e_1, e_2, \dots, e_n\}$. It follows from (49) that

$$a(\|\nabla u_n\|_{L^p(\Omega)}^p) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx + \int_\Omega f(u_n) v dx = \int_\Omega g v dx \tag{53}$$

for all test functions $v \in V_n$. We construct the operator $R_n : V_n \to V_n$ by

$$[R_n u, v] = a\left(\|\nabla u\|_{L^p(\Omega)}^p\right) \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \int_\Omega f(u) v dx - \int_\Omega g v dx$$

for all $u, v \in V_n$. Due to the Cauchy inequality and (8) with $\varepsilon = m/2$, it follows from (2) and (4) that

$$\begin{aligned}
[R_n u, u] &= a\left(\|\nabla u\|_{L^p(\Omega)}^p\right) \int_\Omega |\nabla u|^p dx + \int_\Omega f(u) u dx - \int_\Omega g u dx \\
&\geq \frac{m}{2} \|u\|_{W_0^{1,p}(\Omega)}^p + c_1 \|u\|_{L^q(\Omega)}^q - c_0 |\Omega| - \frac{|\Omega|^{\frac{(p-2)p'}{2p}}}{p'(pm\lambda_1/2)^{\frac{p'}{p}}} \|g\|_{L^2(\Omega)}^{p'} \\
&= \frac{m}{2} \Big[\|u\|_{W_0^{1,p}(\Omega)}^p + \frac{2c_1}{m} \|u\|_{L^q(\Omega)}^q \\
&\qquad - \frac{2p' c_0 |\Omega| (pm\lambda_1)^{\frac{p'}{p}} + |\Omega|^{\frac{(p-2)p'}{2p}} \|g\|_{L^2(\Omega)}^{p'} 2^{p'}}{mp'(pm\lambda_1)^{\frac{p'}{p}}} \Big] \\
&\geq \frac{m}{2} \min\left\{1, \frac{2c_1}{m}\right\} \left[\|u\|_{W_0^{1,p}(\Omega)}^p + \|u\|_{L^q(\Omega)}^q - \ell \right] \qquad (54)
\end{aligned}$$

for all $u \in V_n$, where

$$\ell = \frac{2p' c_0 |\Omega| (pm\lambda_1)^{\frac{p'}{p}} + |\Omega|^{\frac{(p-2)p'}{2p}} \|g\|_{L^2(\Omega)}^{p'} 2^{p'}}{\min\left\{1, \frac{2c_1}{m}\right\} mp'(pm\lambda_1)^{\frac{p'}{p}}}.$$

We deduce from (54) that $[R_n u, u] \geq 0$ for all $u \in V_n$ satisfying $\|u\|_{W_0^{1,p}(\Omega)}^p + \|u\|_{L^q(\Omega)}^q = \ell$. Consequently, by a corollary of the Brouwer fixed point theorem (see [28, Chapter 2, Lemma 1.4]), there exists $u_n \in V_n$ such that $R_n(u_n) = 0$ with

$$\|u_n\|_{W_0^{1,p}(\Omega)}^p + \|u_n\|_{L^q(\Omega)}^q \leq \ell. \tag{55}$$

Therefore, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$. By the compactness of the injection $W_0^{1,p}(\Omega) \cap L^q(\Omega) \hookrightarrow L^2(\Omega)$, we can extract a subsequence of $\{u_n\}$ (relabeled the same) that converges weakly in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ and strongly in $L^2(\Omega)$ to an element $u^* \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$. Thus, it has a.e. convergent subsequence in Ω. Moreover, $f(u_n)$ is bounded in $L^{q'}(\Omega)$, $f \in C^1(\mathbb{R})$ and $-\text{div}\left(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n\right)$ is bounded in $W^{-1,p'}(\Omega)$. An application of diagonalization procedure and using [25, Lemma 1.3, p. 12] and [26, Chapter 4, Theorem 4.18], it follows that (up to a subsequence)

$$f(u_n) \rightharpoonup f(u^*) \text{ in } L^{q'}(\Omega),$$

$$-\text{div}\left(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n\right) \rightharpoonup -\chi \text{ in } W^{-1,p'}(\Omega). \tag{56}$$

Replacing $v = u_n$ in (53) leads to

$$a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u_n|^p dx = \int_\Omega g u_n dx - \int_\Omega f(u_n)u_n dx.$$

Hence

$$\lim_{n\to\infty} a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u_n|^p dx = \int_\Omega g u^* dx - \int_\Omega f(u^*)u^* dx. \tag{57}$$

Using (9), we have

$$\int_\Omega \Big(a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n - a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)|\nabla v|^{p-2}\nabla v\Big)\cdot \nabla(u_n - v)dx \geq 0$$

for all $v \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$. Therefore

$$a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u_n|^p dx - \int_\Omega a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx - a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot \nabla(u_n - v)dx \geq 0.$$

We derive by taking limit for any $v \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$ that

$$\lim_{n\to\infty} a\left(\|\nabla u_n\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla u_n|^p dx + \langle \chi, v\rangle - a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(u^* - v)dx \geq 0. \tag{58}$$

In view of (57) and (58), one gets that

$$\int_\Omega g u^* dx - \int_\Omega f(u^*)u^* dx + \langle \chi, v\rangle - a\left(\|\nabla v\|^p_{L^p(\Omega)}\right)\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(u^* - v)dx \geq 0. \tag{59}$$

In addition, we deduce the ‘limit equation’ from (53) and (56) that

$$-\chi + f(u^*) = g.$$

This implies that

$$-\langle \chi, u^* \rangle = \int_\Omega g u^* dx - \int_\Omega f(u^*) u^* dx. \tag{60}$$

Putting (59) and (60) together, we obtain

$$\left\langle \chi - \operatorname{div}\left(a\left(\|\nabla v\|^p_{L^p(\Omega)} \right) |\nabla v|^{p-2} \nabla v \right), u^* - v \right\rangle \le 0$$

for all $v \in W^{1,p}_0(\Omega) \cap L^q(\Omega)$. Taking $v = u^* - \delta w$, and then let $\delta \to 0$, we have

$$\chi = \operatorname{div}\left(a\left(\|\nabla u^*\|^p_{L^p(\Omega)} \right) |\nabla u^*|^{p-2} \nabla u^* \right).$$

Taking everything into consideration, we infer that $u^* \in W^{1,p}_0(\Omega) \cap L^q(\Omega)$ is the weak stationary solution to problem (1). The inequality (50) is obtained directly from (55) as n tends to infinity.

ii) Uniqueness and exponential stability. Denote $w(t) = u(t) - u^*$, one gets

$$\begin{aligned} \int_\Omega w_t v dx + \int_\Omega \Big(a\left(\|\nabla u\|^p_{L^p(\Omega)} \right) |\nabla u|^{p-2} \nabla u \\ - a\left(\|\nabla u^*\|^p_{L^p(\Omega)} \right) |\nabla u^*|^{p-2} \nabla u^* \Big) \cdot \nabla v dx \\ + \int_\Omega (f(u) - f(u^*)) v dx = 0 \end{aligned}$$

for all test functions $v \in W^{1,p}_0(\Omega) \cap L^q(\Omega)$. In particular, choosing $v = w$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \int_\Omega (f(u) - f(u^*))(u - u^*) dx \\ + \int_\Omega \Big(a\left(\|\nabla u\|^p_{L^p(\Omega)} \right) |\nabla u|^{p-2} \nabla u \\ - a\left(\|\nabla u^*\|^p_{L^p(\Omega)} \right) |\nabla u^*|^{p-2} \nabla u^* \Big) \cdot \nabla (u - u^*) dx = 0. \end{aligned}$$

In view of (51) and

$$\begin{aligned} \int_\Omega \Big(a\left(\|\nabla u\|^p_{L^p(\Omega)} \right) |\nabla u|^{p-2} \nabla u \\ - a\left(\|\nabla u^*\|^p_{L^p(\Omega)} \right) |\nabla u^*|^{p-2} \nabla u^* \Big) \cdot \nabla (u - u^*) dx \ge 0. \end{aligned}$$

We infer that

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + 2\alpha \|w\|^2_{L^2(\Omega)} \le 0.$$

This concludes the proof by using the Gronwall inequality. □

Remark 4.2. If $p = 2$, a satisfies **(H1bis)** and $m\lambda_1 > c_3$, it is easily verified that the weak stationary solution u^* is unique and exponentially stable. Moreover, for any solution u to (1), we have

$$\|u(t) - u^*\|^2_{L^2(\Omega)} \leq \|u(0) - u^*\|^2_{L^2(\Omega)} e^{-2(m\lambda_1 - c_3)t} \text{ for all } t > 0.$$

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