Commun. Korean Math. Soc. **34** (2019), No. 4, pp. 1353–1364
https://doi.org/10.4134/CKMS.c180350
pISSN: 1225-1763 / eISSN: 2234-3024

MODIFIED LAGRANGE FUNCTIONAL FOR SOLVING ELASTIC PROBLEM WITH A CRACK IN CONTINUUM MECHANICS

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Abstract. Modified Lagrange functional for solving an elastic problem with a crack is considered. Two formulations of a crack problem are investigated. The first formulation concerns a problem where a crack extending to the outer boundary of the domain. In the second formulation, we consider a problem with an internal crack. Duality ratio is established for initial and dual problem in both cases.

1. Introduction

The classical statement of an equilibrium problem of an elastic body where a crack is to assume that on the crack faces zero stress conditions are given [2,6]. These conditions do not exclude the possibility of penetration of the crack faces into each other, which is unnatural in terms of continuum mechanics. In recent papers on the crack theory, the models with boundary conditions such as inequalities on the crack faces are considered [3,8,9]. These models provide a mutual non-penetration of the crack faces and can be formulated as variational problems of minimization of a convex functional on a closed convex subset of the initial Hilbert space or as variational inequalities. It is important to construct efficient approximate methods for solving such kind of variational inequalities. In the present paper, to solve a plain elasticity crack problem with mutual non-penetration between the crack faces we use a duality scheme based on modified Lagrange functionals. As a rule, there are two main statements of the problem with a crack. The statement of the problem with an internal crack and problem with a crack extending to the outer boundary. Modified Lagrangian functionals for a problem with an internal crack are studied in detail in [7]. In the present paper, the method of solving the problem with a crack extending

Received August 15, 2018; Accepted September 17, 2018.
2010 *Mathematics Subject Classification.* 65F10, 65K10, 49M15, 74G15, 74G65.
Key words and phrases. elasticity, duality scheme, crack problem, modified Lagrange functional.
The first two authors were supported by the Russian Foundation for Basic Research (Project 17-01-00682 A).

to the outer boundary is investigated. With the similarity of the theorems, their justifications are carried out using various functional spaces. Thus, for these two formulations of the problem with a crack, it is possible to construct a general duality scheme based on the modified Lagrange functionals.

2. Elastic problem with a crack extending to the outward boundary of the domain

Let $\Omega \subset R^2$ be a bounded domain with a regular boundary Γ, and let $\gamma \subset \Omega$ be a cut(crack) with edge lying on the outward boundary. Assume that $\Gamma = \Gamma_0 \bigcup \Gamma_1$, where Γ_0, Γ_1 are nonempty open disjoint subsets of Γ and $\Gamma_1 = \Gamma_1^+ \bigcup \Gamma_1^- \bigcup \Gamma_1^*$ (see Fig. 1).

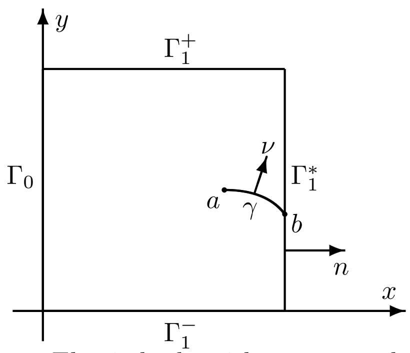


FIGURE 1. Elastic body with an external crack.

Assume that the boundary is smooth except finite points. We assume that γ is a smooth curve, without self-intersections, leaving one endpoint of γ under a nonzero angle to Γ. This assumption is important to study the duality scheme based on the modified Lagrange functional. Denote $\Omega_\gamma = \Omega \setminus \overline{\gamma}$, where $\overline{\gamma} = \gamma \cup \{a\} \cup \{b\}$ and a, b are the crack vertices. Let ν be the vector of unit normal on γ. In this case on the crack γ, denote the positive (upper) face by γ^+ and the negative (lower) face γ^-. Consider the following elasticity theory crack problem.

For the displacement vector $v = (v_1, v_2)$, define the deformation tensor

$$\varepsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \ i, j = 1, 2,$$

and the stress tensor

$$\sigma_{ij}(v) = c_{ijkm}\varepsilon_{km}(v),$$

where $c_{ijkm} = c_{jimk} = c_{kmij}$, $i, j, k, m = 1, 2$, and summation is implied over the repeated indices.

Let us specify vector-functions of the body and surface forces $f = (f_1, f_2)$ and $p = (p_1, p_2)$, respectively. The boundary value problem is formulated as follows [2, 7]:

$$(1)\qquad \begin{gathered} -\frac{\partial \sigma_{ij}}{\partial x_j} = f_i \text{ in } \Omega_\gamma,\ i = 1, 2, \\ u = 0 \text{ on } \Gamma_0, \\ \sigma_{ij} n_j = p_i \text{ on } \Gamma_1,\ i = 1, 2, \end{gathered}$$

where $n = (n_1, n_2)$ is the unit outward normal vector to Γ.

The following conditions are set on γ:

$$(2)\qquad \begin{gathered} [u_\nu] \geq 0,\ [\sigma_\nu(u)] = 0,\ \sigma_\nu(u)[u_\nu] = 0 \text{ on } \gamma, \\ \sigma_\nu(u) \leq 0,\ \sigma_\tau(u) = 0 \text{ on } \gamma^\pm. \end{gathered}$$

Here $u_\nu = u\nu$, $[u_\nu] = u_\nu^+ - u_\nu^-$, $\sigma_\nu(u) = \sigma_{ij}(u)\nu_i\nu_j$, $[\sigma_\nu(u)] = \sigma_\nu^+(u) - \sigma_\nu^-(u)$, $\sigma_\tau(u) = \sigma(u) - \sigma_\nu\nu$, where $\sigma(u) = (\sigma_1(u), \sigma_2(u))$, $\sigma_i(u) = \sigma_{ij}(u)\nu_j$, $i = 1, 2$.

We observe a variational problem for a domain with a crack corresponding to the boundary value problem (1), (2). As mentioned in [2], we introduce the set of admissible displacements

$$K = \left\{ v \in [H^1(\Omega_\gamma)]^2 :\ [v_\nu] \geq 0 \text{ on } \gamma,\ v = 0 \text{ on } \Gamma_0 \right\},$$

where, as before, $[v_\nu] = v_\nu^+ - v_\nu^-$ is a jump of the function $v_\nu = v\nu$ on γ, $v_\nu^\pm \in H^{1/2}(\gamma)$ (see [2, p. 12]). The norm in the space $H^{1/2}(\gamma)$ is defined as

$$\|v_\nu\|^2_{H^{1/2}(\gamma)} = \|v_\nu\|^2_{L_2(\gamma)} + \int_\gamma \int_\gamma \frac{|v_\nu(x) - v_\nu(y)|^2}{|x - y|^2} dx dy.$$

The boundary value problem (1), (2) corresponds to the following variational problem [2, 7]:

$$(3)\qquad \begin{cases} J(v) = \dfrac{1}{2} a(v, v) - \displaystyle\int_{\Omega_\gamma} f_i v_i \, d\Omega - \int_{\Gamma_1} p_i v_i \, d\Gamma \to min, \\ v \in K, \end{cases}$$

where $a(u, v) = \int_{\Omega_\gamma} c_{ijpm} \varepsilon_{pm}(u) \varepsilon_{ij}(v) \, d\Omega$, $f \in [L_2(\Omega_\gamma)]^2$, $p \in [L_2(\Gamma_1)]^2$.

The problem (3) is equivalent to variational inequality [1]

$$(4)\quad u \in K :\ a(u, v - u) - \int_{\Omega_\gamma} f_i(v_i - u_i) \, d\Omega - \int_{\Gamma_1} p_i(v_i - u_i) \, d\Gamma \geq 0 \ \forall v \in K.$$

3. Elastic problem with an internal crack

Let Ω be the same domain as in Section 2. Crack γ is an internal crack, so we assuming that the end points a, b are crack vertices that do not reach the outer boundary Γ (see Fig. 2).

The boundary statement of the problem with an internal crack looks just like for the problem of Section 2. The variational problem (3) and the variational

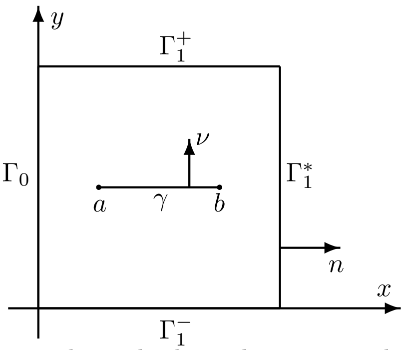


FIGURE 2. Elastic body with an internal crack.

inequality (4) remain in force. However, in order to justify the applicability of the duality method, here we need an additional function space

$$H_{00}^{1/2}(\gamma) = \left\{ w \in H^{1/2}(\gamma) : \ \frac{w}{\sqrt{\rho}} \in L_2(\gamma) \right\}$$

with the norm

$$\|w\|^2_{H_{00}^{1/2}(\gamma)} = \|w\|^2_{H^{1/2}(\gamma)} + \left\| \frac{w}{\sqrt{\rho}} \right\|^2_{L_2(\gamma)} ,$$

where $\rho(x) = dist(x, \partial\gamma)$ (see [7]).

4. General duality scheme

To solve the crack problems introduced in Sections 2 and 3 we construct a general duality scheme based on a modified Lagrange functional. At the same time, all necessary substantiations will be carried out for the problem of Section 2 only. The same substantiations for problem of Section 3 one can find in [7].

Define the space

$$W = \left\{ v \in [H^1(\Omega_\gamma)]^2 : \ v = 0 \text{ on } \Gamma_0 \right\}.$$

For an arbitrary $m \in L_2(\gamma)$, construct the set

$$K_m = \{ v \in W : \ -[v_\nu] \leq m \text{ on } \gamma \}.$$

It is easy to show that K_m is a closed convex set in the $H^1(\Omega_\gamma)$ norm.

On the space L_2, we define a sensitivity functional

$$\chi(m) = \begin{cases} \inf\limits_{v \in K_m} J(v), \ \text{if } K_m \neq \varnothing, \\ +\infty, \ \text{otherwise}. \end{cases}$$

Since the crack leaves at point b on the outer boundary Γ under a nonzero angle to Γ, then the Korn inequality is fulfilled [2]

$$\int_{\Omega_\gamma} \varepsilon_{ij}(v)\varepsilon_{ij}(v)\, d\Omega \geq c\|v\|^2_{H^1_{\Omega_\gamma}} \ \forall \ v \in W,$$

where $c > 0$ is a constant. The Korn inequality implies the property of strong convexity (and hence coercivity) of the functional $J(v)$ on the space W. Therefore the problem $\inf_{v \in K_m} J(v)$ under the condition $K_m \neq \varnothing$ is solvable. If the function $m \in L_2(\gamma) \setminus H^{1/2}(\gamma)$, the set K_m can be empty [7, 12].

The functional $\chi(m)$ is a proper convex functional on $L_2(\gamma)$, but its effective domain $dom\chi = \{m \in L_2(\gamma) : \ \chi(m) < +\infty\}$ does not coincide with $L_2(\gamma)$. Also note the $dom\chi$ is a convex but not closed in $L_2(\gamma)$, and in our case $\overline{dom\chi} = L_2(\gamma)$.

On the space $W \times L_2(\gamma) \times L_2(\gamma)$, we define the functional

$$Q(v,l,m) = \begin{cases} J(v) + \displaystyle\int_\gamma lmd\Gamma + \frac{r}{2}\int_\gamma m^2 d\Gamma, \text{ if } -[v_\nu] \leq m \text{ a.e. on } \gamma, \\ +\infty, \text{ otherwise.} \end{cases}$$

and a modified Lagrange functional $M(v,l)$ on the space $W \times L_2(\gamma)$:

$$M(v,l) = \inf_{m \in L_2(\gamma)} Q(v,l,m) = J(v) + \frac{1}{2r}\int_\gamma \left\{ \left[(l - r[v_\nu])^+\right]^2 - l^2 \right\} d\Gamma,$$

where $(l - r[v_\nu])^+ \equiv \max\{0, l - r[v_\nu]\}$, $r > 0$ is a constant.

A modified dual functional is defined as follows

$$\underline{M}(l) = \inf_{v \in W} M(v,l) = \inf_{v \in W} \left\{ J(v) + \frac{1}{2r}\int_\gamma \left\{ \left[(l - r[v_\nu])^+\right]^2 - l^2 \right\} d\Gamma \right\}.$$

Since $\inf_{v \in W} \inf_{m \in L_2(\gamma)} Q(v,l,m) = \inf_{m \in L_2(\gamma)} \inf_{v \in W} Q(v,l,m)$ for $\underline{M}(l)$, we also have the following representation [12]:

$$\underline{M}(l) = \inf_{m \in L_2(\gamma} \left\{ \chi(m) + \int_\gamma lmd\Gamma + \frac{r}{2}\int_\gamma m^2 d\Gamma \right\}.$$

It is easy to see that the following estimate is valid for any $l \in L_2(\gamma)$:

$$\underline{M}(l) \leq \chi(0) = \inf_{v \in K} J(v). \tag{5}$$

For the functional $\underline{M}(l)$, we define a dual problem:

$$\begin{cases} \underline{M}(l) - \sup, \\ l \in L_2(\gamma). \end{cases} \tag{6}$$

For a problem with a crack it is natural to assume only $[H^1(\Omega_\gamma)]^2$ regularity of the solution. In this case, the dual problem (6) may be unsolvable. We consider a dual method for solving a problem with a crack, extending to the outer boundary, in which the solvability of the dual problem (6) is not assumed in advance. A similar study of the dual method for solving an elastic problem with an internal crack was performed in [7].

Let us investigate the sensitivity functional $\chi(m)$ and the corresponding dual functional $\underline{M}(l)$. We show that the sensitivity functional $\chi(m)$ is weakly lower semicontinuous on $L_2(\gamma)$. Since $\chi(m)$ is a convex functional, it is sufficient to prove that epigraph of the functional is closed. Closure of the epigraph follows from the following two lemmas.

Lemma 4.1. *Let $\overline{m} \in L_2(\gamma)$ do not belong to $dom\chi$. Then for any sequence $\{m_i\} \subset dom\chi$ such that $\lim_{i\to\infty} \|m_i - \overline{m}\|_{L_2(\gamma)} = 0$ we have a limit equality $\lim_{i\to\infty} \chi(m_i) = +\infty$.*

Proof. For a function $\overline{m} \notin dom\chi$, we consider an arbitrary sequence $\{m_i\} \subset dom\chi$ such that $\lim_{i\to\infty} \|m_i - \overline{m}\|_{L_2(\gamma)} = 0$. Since $K_{m_i} \neq \varnothing$ and the functional $J(v)$ is coercive on W, there exists a unique element $v^i = \arg\min_{v \in K_{m_i}} J(v)$, $(i = 1, 2, \ldots)$. Let us show that $\lim_{i\to\infty} \left\|v^i\right\|_W = +\infty$.

Assume the contrary, that is, let the sequence $\{v^i\}$ have a bounded subsequence $\{v^{i_j}\}$, $\left\|v^{i_j}\right\|_W \leq c$ for all i_j, where $c > 0$ is a constant. It follows from the trace theorem that $\left\|[v_\nu^{i_j}]\right\|_{H^{1/2}(\gamma)} \leq c_1$, where $c_1 > 0$ is a constant [2]. Then $\{[v_\nu^{i_j}]\}$ is a compact subsequence in $L_2(\gamma)$. Let $t \in H^{1/2}(\gamma)$ be a weak limit point of this sequence. Without loss of generality, the number t can be considered a weak limit $\{[v_\nu^{i_j}]\}$ in $H^{1/2}(\gamma)$. Then $\{[v_\nu^{i_j}]\}$ converges to t in the norm in $L_2(\gamma)$. Since $-[v_\nu^{i_j}] \leq m_i$, we have $-t \leq \overline{m}$ on γ. Hence, $K_{\overline{m}} \neq \varnothing$ or $\overline{m} \in dom\chi$. This contradiction shows that $\lim_{i\to\infty} \left\|v^i\right\|_W = +\infty$. Since the functional $J(v)$ is coercive on W, we have $\lim_{i\to\infty} \chi(m_i) = \lim_{i\to\infty} J(v^i) = +\infty$. □

Lemma 4.2. *Let $\overline{m} \in L_2(\gamma)$ belong to $dom\chi$. Then for any sequence $\{m_i\} \subset dom\chi$ converging to $\overline{m}$ in $L_2(\gamma)$, the following inequality holds*

$$\underline{\lim}_{i\to\infty} \chi(m_i) \geq \chi(\overline{m}).$$

Proof. Let $\{m_i\} \subset dom\chi$ and $\lim_{i\to\infty} \|m_i - \overline{m}\|_{L_2(\gamma)} = 0$, where $\overline{m} \in dom\chi$. From the sequence $\{m_i\}$, we take a subsequence $\{m_{i_j}\}$ for which

$$\lim_{i\to\infty} \chi(m_{i_j}) = \underline{\lim}_{i\to\infty} \chi(m_i).$$

Consider a subsequence $\{v_{i_j}\}$, where $v^{i_j} = \arg\min_{v \in K_{m_{i_j}}} J(v)$. The sequence $\{v_{i_j}\}$ is bounded in W (otherwise $\lim_{i\to\infty} \chi(m_{i_j}) = +\infty$ and the required inequality is proved). Denote $\widetilde{\Gamma} = \Gamma \cup \gamma^+ \cup \gamma^-$ and let $[H^{1/2}(\widetilde{\Gamma})]^2$ be the space of traces of functions from the space $[H^1(\Omega_\gamma)]^2$ to $\widetilde{\Gamma}$. Let $[H^{1/2}(\Gamma)]^2$ be the space of functions that are the restrictions of functions from $[H^{1/2}(\widetilde{\Gamma})]^2$ to Γ. Since $W \subset [H^1(\Omega_\gamma)]^2 \subset [H^{1/2}(\widetilde{\Gamma})]^2$, we have $\|v_{i_j}\|_{[H^{1/2}(\Gamma)]^2} \leq c$, where $c > 0$ is a constant. In addition, $\{v_{i_j}\}$ is a compact sequence in $L_2(\Gamma)$. Let $\hat{v} \in [H^{1/2}(\Gamma)]^2$ be a weak limit point of this sequence. Without loss of generality $\hat{v}$ may be considered a weak limit $\{v_{i_j}\}$. Then $\{v_{i_j}\}$ converges to $\hat{v}$ in $L_2(\Gamma)$. It follows from the trace theorem [2] that the sequence $\{[v_{i_j}]\}$ is weakly compact in $H^{1/2}(\gamma)$. Let $t \in H^{1/2}(\gamma)$ be a weak limit point of this sequence. Without loss of generality $\{[v_{i_j}]\}$ can be considered a weakly converging sequence, that is, t is a weak limit of $\{[v_{i_j}]\}$ in $H^{1/2}(\gamma)$.

Since the space $H^{1/2}(\gamma)$ is compactly embedded into $L_2(\gamma)$ and $L_2(\gamma)$ is embedded into $H^{-1/2}(\gamma)$, $[v_{i_j}]$ converges to t in the norm $L_2(\gamma)$. Here $H^{-1/2}(\gamma)$ is the space that dual to $H^{1/2}(\gamma)$. From the convergence m_{i_j} to $\overline{m}$ in $L_2(\gamma)$, $[v_{i_j}]$ to t in $L_2(\gamma)$, and condition $-[v_{i_j}] \leq m_{i_j}$, we obtain $-t \leq \overline{m}$ on γ.

Let us denote $\widetilde{t} = \arg\min\limits_{v \in W_t} J(v)$, where $W_t = \{v \in W : \ [v_\nu] = t \text{ on } \gamma, \ v = \hat{v}$ on $\Gamma_1\}$. We have

$$\begin{aligned} J(v^{i_j}) - J(\widetilde{t}) &= a(\widetilde{t}, v^{i_j} - \widetilde{t}) - \int_{\Omega_\gamma} f_s(v_s^{i_j} - \widetilde{t}_s)d\Omega + \frac{1}{2}a(v^{i_j} - \widetilde{t}, v^{i_j} - \widetilde{t}) \\ &\quad - \int_{\Gamma_1} p_s(v_s^{i_j} - \hat{v}_s)d\Gamma \\ &= \langle \mu_1, v^{i_j} - \hat{v}\rangle + \langle \mu_2, [v_\nu^{i_j}] - t\rangle - \int_{\Gamma_1} p_s(v_s^{i_j} - \hat{v}_s)d\Gamma \\ &\quad + \frac{1}{2}a(v^{i_j} - \widetilde{t}, v^{i_j} - \widetilde{t}), \end{aligned}$$

where $\mu_1 \in [H^{-1/2}(\Gamma)]^2$, $\mu_2 \in H^{-1/2}(\gamma)$. Here

$$\langle \mu_1, v^{i_j} - \hat{v}\rangle + \langle \mu_2, [v_\nu^{i_j}] - t\rangle = a(\widetilde{t}, v^{i_j} - \widetilde{t}) - \int_{\Omega_\gamma} f_s(v_s^{i_j} - \widetilde{t}_s)d\Omega$$

and $\mu_1 + \mu_2 \in ([H^{1/2}(\Gamma)]^2 \times H^{1/2}(\gamma))^*$, where $([H^{1/2}(\Gamma)]^2 \times H^{1/2}(\gamma))^*$ is the space that is dual to $[H^{1/2}(\Gamma)]^2 \times H^{1/2}(\gamma)$.

Since $\{v_{i_j}\}$ weakly converges to $\hat{v}$ in $[H^{1/2}(\Gamma)]^2$ and $\{[v_{i_j}]\}$ weakly converges to t in $H^{1/2}(\gamma)$, owing to the uniqueness of the weak limits we have

$$\lim_{j\to\infty} \langle \mu_1, v^{i_j} - \hat{v}\rangle + \lim_{j\to\infty} \langle \mu_2, [v_\nu^{i_j}] - t\rangle = 0.$$

Therefore, the following estimate is valid:

$$\lim_{j\to\infty} \chi(m_{i_j}) = \lim_{j\to\infty} J(v^{i_j}) \geq J(\widetilde{t}) \geq \chi(\overline{m})$$

and, hence,

$$\varliminf_{i\to\infty} \chi(m_i) \geq \chi(\overline{m}). \qquad \square$$

As was noted above, from the lemmas proved above, it follows the weak lower semicontinuity of a convex functional $\chi(m)$ or the closure of its epigraph.

For an arbitrary fixed $l \in L_2(\gamma)$, consider the functional

$$F_l(m) = \chi(m) + \int_\gamma lmd\Gamma + \frac{r}{2}\int_\gamma m^2 d\Gamma \ , \ r > 0 - const.$$

It is easy to see that $F_l(m)$ is a functional that is lower semicontinuous on $L_2(\gamma)$. Since $epi\chi$ is a convex closed set in $L_2(\gamma) \times R$, $R = (-\infty, +\infty)$, then according to the Mazur separability theorem [4], there exist $\alpha \in L_2(\gamma)$ and $\beta \in R$ such that

$$\chi(m) + \int_\gamma \alpha m d\Gamma + \beta \geq 0 \ \forall \ m \in dom\chi.$$

Hence, for the functional $F_l(m)$ the following lower estimate holds:

$$F_l(m) \geq -\int_\gamma \alpha m d\Gamma + \int_\gamma l m d\Gamma + \frac{r}{2}\int_\gamma m^2 d\Gamma \ -\beta \geq 0 \ \forall \ m \in L_2(\gamma).$$

Therefore, $F_l(m) \to +\infty$ as $\|m\|_{L_2(\gamma)} \to +\infty$, that is, $F_l(m)$ is coercive in $L_2(\gamma)$.

It follows from the weak semicontinuity and coercivity of $F_l(m)$ that for any $l \in L_2(\gamma)$ there exists an element $m(l) \in L_2(\gamma)$ such that

$$m(l) = \arg\min_{m \in L_2(\gamma)} F_l(m).$$

It follows from the strong convexity of $F_l(m)$ on $dom\chi$ [10] that for any $l \in L_2(\gamma)$ the element $m(l)$ is unique.

We formulate for the dual functional $\underline{M}(l)$ some characteristic statements that can be proved similarly to Theorem 24 in [13].

Theorem 4.3. *The dual functional $\underline{M}(l)$ is continuous in $L_2(\gamma)$.*

Theorem 4.4. *The dual functional $\underline{M}(l)$ is Gateaux differentiable in $L_2(\gamma)$ and its derivative $\bigtriangledown\underline{M}(l)$ satisfies a Lipschitz condition with a constant $\frac{1}{r}$, that is, the following inequality holds*:

$$\| \bigtriangledown \underline{M}(l^{'}) - \bigtriangledown\underline{M}(l^{''})\|_{L_2(\gamma)} \leq \frac{1}{r}\|l^{'} - l^{''}\|_{L_2(\gamma)} \ \forall \ l^{'}, l^{''} \in L_2(\gamma).$$

It can be shown that $\bigtriangledown\underline{M}(l) = m(l) = \max\{-[u_\nu], -\frac{l}{r}\} \ \forall \ l \in L_2(\gamma)$ [11].

To solve the dual problem (6), consider a gradient method [7]

$$l^{k+1} = l^k + \theta_k m(l^k), \ k = 1, 2, \ldots, \tag{7}$$

with any initial value $l^0 \in L_2(\gamma)$, $\theta_k \in [\tau, 2r-\tau], \ \tau \in (0, r]$.

Theorem 4.5. *For the sequence $\{l^k\}$ constructed by the method* (7) *we have a limit equality*

$$\lim_{k\to\infty} \|m(l^k)\|_{L_2(\gamma)} = 0.$$

The gradient method (7) generates the following algorithm of a Uzawa-type method for solving the problem (3) [7]. At the initial step, $k = 0$, specify an arbitrary function $l^0 \in L_2(\gamma)$ and for every $k = 0, 1, 2, \ldots$ subsequently calculate:

$$\text{(i) } u^{k+1} = \arg\min_{v \in W}\left\{J(v) + \frac{1}{2r}\int_\gamma \left\{\left[(l^k - r[v_\nu])^+\right]^2 - (l^k)^2\right\} d\Gamma\right\}; \tag{8}$$

$$\text{(ii) } l^{k+1} = l^k + \theta_k \max\{-[u_\nu^{k+1}], -\frac{l^k}{r}\}, \ \theta_k \in [\tau, 2r-\tau], \ \tau \in (0, r]. \tag{9}$$

Theorem 4.6. *The following duality relation holds* [7]

$$\sup_{l\in L_2(\gamma)} \underline{M}(l) = \inf_{v\in K} J(v).$$

Note that when the dual problem (6) is solvable, it can be proved that the sequence $\{l^k\}$ is bounded in $L_2(\gamma)$ [7, 14]. According to Theorem 4.5, this means that the following equality is valid:

$$\lim_{k\to\infty} \int_\gamma l^k m(l^k) d\Gamma = 0.$$

Hence, the method (8), (9) converges for the functional of the (3), that is,

$$\lim_{k\to\infty} \chi(m(l^k)) = \lim_{k\to\infty} J(u^{k+1}) = J(u),$$

where u is the solution to the problem (3).

5. Numerical experiment

The domain Ω is taken as a unit square. Consider two cracks:

$$\gamma_1 = \{(x, y) : 0.75 \le x \le 1, y = \sqrt{0.3125^2 - (x - 0.75)^2} + 0.1875\},$$
$$\gamma_2 = \{(x, y) : 0.2 \le x \le 0.8, y = 0.5\},$$

where γ_1 is a crack, leaving right endpoint on Γ under a nonzero angle, γ_2 is a crack inside the body. To find a solution to the problem (8), we use a finite element method.

We introduce the following notation: h - edge length on γ, n - number of all triangulation nodes, n_γ - number of triangulation nodes on γ, W_h is the linear shell of the basis functions $\varphi_i(x, y)$, $u_h = (u_1^h, u_2^h)$ is the piecewise linear approximation of the exact solution u:

$$u_1^h(x, y) = \sum_{j=1}^{n} t_j \varphi_j(x, y),\ u_2^h(x, y) = \sum_{j=n+1}^{2n} t_j \varphi_{j-n}(x, y),\ t_j \in \mathbb{R}. \tag{10}$$

Since Ω is a polygon, the embedding $W_h \subset W$ is guaranteed. Thus, the problem (8) is replaced by the finite element problem

$$u^{k+1} = \operatorname*{arg\,min}_{v\in W_h} \left\{ J(v) + \frac{1}{2r} \int_\gamma \left\{ \left[(l^k - r[v_\nu])^+ \right]^2 - (l^k)^2 \right\} d\Gamma \right\}. \tag{11}$$

We approximate the boundary integral of γ using the trapezium quadrature rule. Let $t = (t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_{2n})$, then the minimization problem (11) is reduced to finding the optimal values of t_i. For this, we use generalized Newton method [5].

The iterations of generalized Newton method terminate when the following criterion is met:

$$\|t^{m+1} - t^m\|_\infty < \varepsilon_t, \varepsilon_t = 10^{-12}.$$

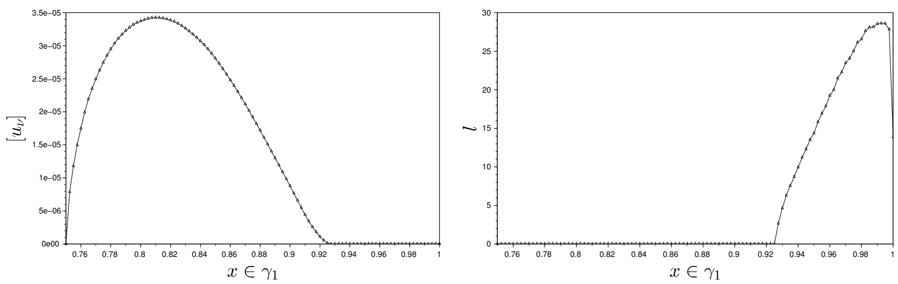
The stop criterion for the Uzawa method has the following form:

$$\|\alpha^{k+1} - \alpha^k\|_\infty < \varepsilon_\alpha, \varepsilon_\alpha = 10^{-8},$$

where $\alpha^k = (\alpha_1^k, \alpha_2^k, \ldots, \alpha_{n_\gamma}^k)$ - approximate value of the dual variable l^k.

Let us present the results of numerically solving the problem. The parameter values are as follows: $f = (f_1, f_2) = (0, 0)$, the right side surface force $p_1|_{\Gamma_1^*} = -27 \cdot g(y)$ MPa, $p_2|_{\Gamma_1^*} = 0$ MPa, on the upper side $p_1|_{\Gamma_1^+} = 0$ MPa, $p_2|_{\Gamma_1^+} = -1$ MPa and on the lower side $p_1|_{\Gamma_1^-} = 0$ MPa, $p_2|_{\Gamma_1^-} = 1$ MPa, Young's elasticity modulus $E = 73000$ MPa, the Poisson coefficient $\mu = 0.34$, the constant $r = 10^{10}$.

The results of the numerical solution with crack γ_1 are presented graphically in Figure 3(a), 3(b). Here $g(y) = 1$ and $h = 0.003$. The graphs show that the jump $[u_\nu] \geq 0$ on the crack, so there is no mutual penetration between the crack faces into each other. In addition, it can be seen from Figure 3(b) that the value of the dual variable is greater than zero at points where crack faces are stuck together. This indicates the presence of a normal stress in these nodes.

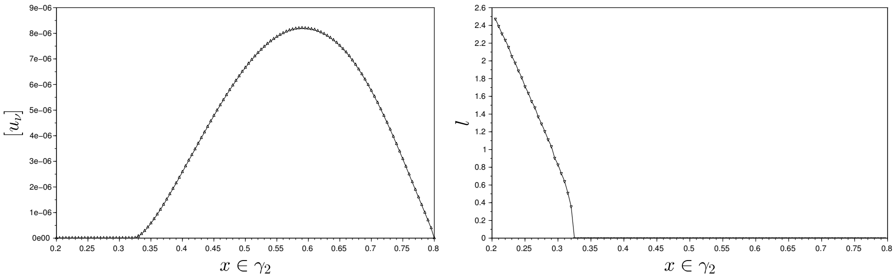


(a) Jump of the function u_ν (b) Value of the dual variable l

FIGURE 3. Results with crack γ_1

Figure 4(a), 4(b) show solution of the problem with crack γ_2, where $g(y) = 1 - |2y - 1|$, $(0 \leq y \leq 1)$ and $h = 0.005$. As in the previous example, there is no mutual penetration between the crack faces.

A small number of steps (ii) provides fast convergence of the Uzawa method by dual variable l. In the first example, only 5 iterations by dual variable are performed. The number of iterations for the generalized Newton method at step (i) also turned out to be relatively small. On the first step of the Uzawa method, 9 iterations by primal variable are executed, and on the next steps only 2 iterations. Similar for the second example with inner crack: 4 iterations by dual variable, 8 iterations by primal variable on first step (i). Thus, the numerical calculations confirm that modified Lagrange functionals make it possible to efficiently solve mathematical models with nonlinear boundary conditions in the form of inequalities.



(a) Jump of the function u_ν (b) Value of the dual variable l

FIGURE 4. Results with crack γ_2

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