Commun. Korean Math. Soc. **34** (2019), No. 4, pp. 1335–1352
https://doi.org/10.4134/CKMS.c180301
pISSN: 1225-1763 / eISSN: 2234-3024

THE LOGARITHMIC KUMARASWAMY FAMILY OF DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

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Abstract. In this article, a new family of lifetime distributions by adding two additional parameters is introduced. The new family is called, the logarithmic Kumaraswamy family of distributions. For the proposed family, explicit expressions for some mathematical properties are derived. Maximum likelihood estimates of the model parameters are also obtained. This method is applied to develop a new lifetime model, called the logarithmic Kumaraswamy Weibull distribution. The proposed model is very flexible and capable of modeling data with increasing, decreasing, unimodal or modified unimodal shaped hazard rates. To access the behavior of the model parameters, a simulation study has been carried out. Finally, the potentiality of the new method is proved via analyzing two real data sets.

1. Introduction

Broadly speaking, the use of statistical distributions are very important in predicting and describing real-world phenomena. The quality of the results heavily depend on the specification of the right model for the data under consideration. Statistical distributions such as Rayleigh, exponential, gamma, beta or Weibull distributions are frequently used in modeling lifetime data.

However, these distributions have certain limitations. For example, the exponential distribution offers data modeling with constant failure rate only, the Rayleigh distribution is useful in data modeling having the increasing failure rate. Whereas, the gamma and beta distributions do not have closed form solutions of the cumulative distribution functions causing difficulties in estimating the parameters. Among these distributions, the Weibull model is the most prominent one, and offer data modeling with monotonic (increasing, decreasing or constant) failure rates. But, unfortunately, the Weibull model is not suitable to use in data modeling having non-monotonic (unimodal, modified unimodal or bathtub) failure rates. Therefore, to improve the characteristics

Received July 15, 2018; Revised December 29, 2018; Accepted June 4, 2019.
2010 *Mathematics Subject Classification.* Primary 60E05, 62F10.
Key words and phrases. family of distributions, Weibull distribution, moments, order statistic, residual life function, maximum likelihood estimation.

of these traditional distributions, researchers have been developing various extensions and modified forms of these distributions.

However, in the recent literature, researchers have shown a deep interest in proposing new families of distributions. The literature is filled with such methods that are quite rich and still growing rapidly. This has been done through many different approaches.

An interesting idea of extending a distribution, known in the literature by exponentiation, is proposed by Mudholkar and Srivastava [13]. The cumulative distribution function (cdf) of the exponentiated random variable is given by

$$G_E(x) = F(x;\xi)^a, \qquad a,\xi > 0,\ x \in \mathbb{R}, \tag{1}$$

where, ξ is a parameter vector of the baseline distribution $F(x;\xi)$. Using (1), several lifetime distributions have been proposed. For example, [3], [7], [8], [11], [14] and [15], among others.

Another prominent approach in this domain, is Marshall-Olkin family of distributions proposed by Marshall and Olkin [12] defined by the cdf

$$G_{MO}(x) = \frac{F(x;\xi)}{1-(1-\sigma)(1-F(x;\xi))}, \qquad \sigma,\xi > 0,\ x \in \mathbb{R}. \tag{2}$$

Using (2), Marshall and Olkin [12] studied the Marshall Olkin exponential (MOE) and Marshall Olkin Weibull (MOW) distributions. Since then, using (2), numerous life distributions have been studied. For example, see [5], [6], [16] and [17].

Taking inspiration from [9], the Kumaraswamy-G class of distributions appeared in [2] whose cdf is given by

$$G_{KuG}(x) = 1-\left(1-F(x;\xi)^\beta\right)^\gamma, \quad x \in \mathbb{R}, \tag{3}$$

where, $\beta,\gamma > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weights.

Recently, [18] proposed the exponentiated Kumaraswamy G-logarithmic (EKuG-L) class of distributions given by

$$G(x) = 1-\frac{\log\left\{1-(1-p)\left[1-G_{KuG}(x)^\theta\right]\right\}}{\log(p)}, \qquad x \in \mathbb{R}, \tag{4}$$

where $\theta > 0$ and $p \in (0,1)$. For $\theta = 1$, the cdf of the EKuG-L reduces to the new Kumaraswamy-G-logarithmic (KG-L) family with cdf given by

$$G(x) = 1-\frac{\log\{1-(1-p)[1-G_{KuG}(x)]\}}{\log(p)}, \qquad x \in \mathbb{R},$$

where $\xi > 0$ and $p \in (0,1)$. In the EKuG-L and KG-L families, the parametric space of p is restricted to $(0,1)$. Due to the restricted parametric space of p, these two families may not be flexible enough to counter complex forms of data. Furthermore, the EKuG-L family has two additional parameters. Due to the higher number of parameters, the estimation of parameters as well as computation of many distributional characteristics becomes very difficult.

Therefore, in this paper, a more flexible class of distributions, called the logarithmic Kumaraswamy (LKu) family is proposed by reparametrizing (4). The new family is introduced by keeping constant $\theta = 1$ (to reduce the number of parameters) and reparametrizing $p = e^{\alpha}$ (to relax the upper limit of the parametric space of p), where $\alpha > 0$. Due to unrestricted upper bound, the proposed distribution would be quite flexible in modeling complex forms of data.

Thus the motivation for introducing this new model is to relax the boundary conditions of the parametric values to provide more flexibility in the shape of the hazard rate function than the classical monotone behavior and to improve description which call for complexity by adding the parameters in Kumaraswamy-G class of distributions which gives us more information about the behaviour of the hazard rate function in the tail end, and how skewed the distribution is.

The logarithmic Kumaraswamy (LKu) distributions is defined by cdf

$$G(x) = 1 - \frac{\log\left[e^{\alpha} - \left\{1 - \left(1 - F(x;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha} - 1)\right]}{\alpha}, \quad x \in \mathbb{R}, \tag{5}$$

where, $\alpha, \gamma, \beta, \xi > 0$, and $F(x;\xi)$ is cdf of the baseline random variable depending on the vector parameter ξ. The probability density function (pdf), sf and hazard rate function (hrf) of the LKu family are given (6)-(8), respectively by

$$g(x) = \frac{\gamma\beta(e^{\alpha} - 1) f(x;\xi) F(x;\xi)^{\beta-1}\left(1 - F(x;\xi)^{\beta}\right)^{\gamma-1}}{\alpha\left[e^{\alpha} - \left\{1 - \left(1 - F(x;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha} - 1)\right]}, \qquad x \in \mathbb{R}, \tag{6}$$

$$S(x) = \frac{\log\left[e^{\alpha} - \left\{1 - \left(1 - F(x;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha} - 1)\right]}{\alpha}, \qquad x \in \mathbb{R}, \tag{7}$$

$$\begin{aligned} h(x) = {} & \frac{\gamma\beta(e^{\alpha} - 1) f(x;\xi) F(x;\xi)^{\beta-1}\left(1 - F(x;\xi)^{\beta}\right)^{\gamma-1}}{\left[\log\left[e^{\alpha} - \left\{1 - \left(1 - F(x;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha} - 1)\right]\right]} \\ & \times \frac{1}{\left[e^{\alpha} - \left\{1 - \left(1 - F(x;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha} - 1)\right]}, \qquad x \in \mathbb{R}. \end{aligned} \tag{8}$$

The new pdf is most tractable when $F(x,\xi)$ and $f(x,\xi)$ of the baseline variable have simple analytic expressions. Henceforth, a random variable X with pdf (6) is denoted by$X \sim LKu(x;\Theta)$, where $\Theta = (\alpha, \gamma, \beta, \xi)$. The key motivations for using the LKu family in practice are the following:

(1) A very simple and convenient method of adding additional parameters to modify the existing distributions.
(2) To improve the characteristics and flexibility of the existing distributions.
(3) To introduce the extended version of the baseline distribution having closed forms for cdf, sf as well as hrf.
(4) To provide better fits than the other modified models.
(5) To introduce a distribution with increasing, decreasing, unimodal and increasing-decreasing-increasing (also called modified unimodal) shaped hazard functions.

This rest of this article is organized in the following way. Shapes of the density and hazard functions are discussed in Section 2. In Section 3, a special sub-model of the proposed family is discussed. Some mathematical properties are obtained in Section 4. Maximum likelihood estimates of the model parameters are obtained in Section 5. A simulation study is conducted in Section 6. Section 7, is devoted to analyze two real life applications. Finally, concluding remarks are provided in Section 8.

2. The shape of the density and Hazard function

Here, we give a general description of the shape of the distribution analytically. Consider (6) the critical points can be obtained by solving the equation

$$\frac{\partial \ln g\left(x;\Theta\right)}{\partial x}=0.$$

The above equation may have more than one root. If $x=x_0$ is a root of the above equation, then it corresponds to a local maximum if

$$\frac{\partial^2 \ln g\left(x;\Theta\right)}{\partial x^2}<0,$$

and a local minimum if

$$\frac{\partial^2 \ln g\left(x;\Theta\right)}{\partial x^2}>0,$$

and a point of inflection if

$$\frac{\partial^2 \ln g\left(x;\Theta\right)}{\partial x^2}=0.$$

In a similar way the shape of the hazard function can also be described analytically.

3. Sub-model description

In this section, a special sub-model of the new family, called the logarithmic Kumaraswamy Weibull (LKu-W) distribution is introduced. Let $F\left(x\right)$ be the cdf of the one parameter Weibull model given by $F(x)=1-e^{-x^{\theta}}$, $x,\theta>0$, with pdf $f(x)$ given by $f(x) = \theta x^{\theta-1} e^{-x^{\theta}}$. Then, the cdf of the LKu-W distribution has the following expression

$$G(x;\Theta) = 1 - \frac{\log\left[e^{\alpha} - \left\{1 - \left(1 - \left(1 - e^{-x^{\theta}}\right)^{\beta}\right)^{\gamma}\right\}\left(e^{\alpha}-1\right)\right]}{\alpha}, \quad x > 0, \tag{9}$$

where, $\alpha, \gamma, \beta, \theta > 0$. The pdf, sf and hrf of the LKu-W distribution are given (10)-(12), respectively by

$$g(x;\Theta) = \frac{\beta\gamma\theta(e^{\alpha}-1)x^{\theta-1}e^{-x^{\theta}}\left(1-e^{-x^{\theta}}\right)^{\beta-1}\left(1-\left(1-e^{-x^{\theta}}\right)^{\beta}\right)^{\gamma-1}}{\alpha\left[e^{\alpha}-\left\{1-\left(1-\left(1-e^{-x^{\theta}}\right)^{\beta}\right)^{\gamma}\right\}\left(e^{\alpha}-1\right)\right]}, \quad x > 0, \tag{10}$$

$$S(x;\Theta) = \frac{\log\left[e^{\alpha}-\left\{1-\left(1-\left(1-e^{-x^{\theta}}\right)^{\beta}\right)^{\gamma}\right\}\left(e^{\alpha}-1\right)\right]}{\alpha}, \quad x > 0, \tag{11}$$

$$h(x;\Theta) = \frac{\beta\gamma\theta(e^{\alpha}-1)x^{\theta-1}e^{-x^{\theta}}\left(1-e^{-x^{\theta}}\right)^{\beta-1}\left(1-\left(1-e^{-x^{\theta}}\right)^{\beta}\right)^{\gamma-1}}{\left[\log\left[e^{\alpha}-\left\{1-\left(1-\left(1-e^{-x^{\theta}}\right)^{\beta}\right)^{\gamma}\right\}\left(e^{\alpha}-1\right)\right]\right]} \times \frac{1}{\left[e^{\alpha}-\left\{1-\left(1-\left(1-e^{-x^{\theta}}\right)^{\beta}\right)^{\gamma}\right\}\left(e^{\alpha}-1\right)\right]}, \quad x > 0. \tag{12}$$

For $\beta = 1$ and different values of α, γ and θ, plots of the pdf and hrf of the LKu-W distribution are sketched in Figures 1 and 2, respectively.

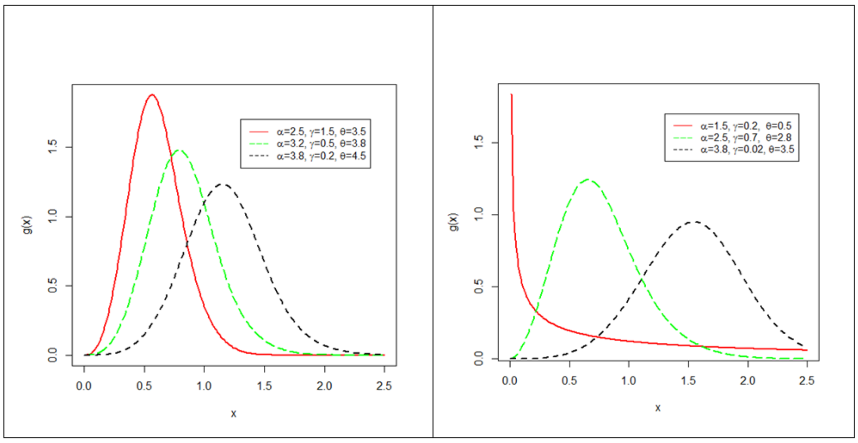

FIGURE 1. Graphical sketching of the LKu-W density function for some parameter values.

4. Basic mathematical properties

In this section, some statistical properties for the proposed family are derived.

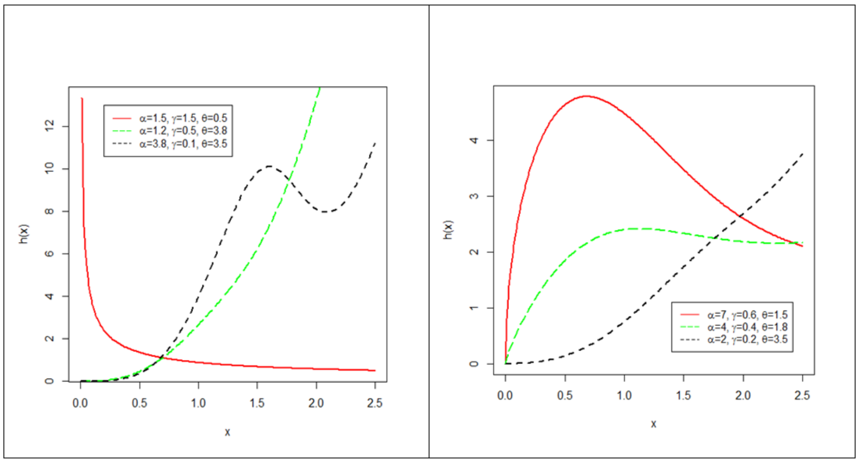

FIGURE 2. Graphical sketching of the LKu-W hazard rate function for some parameter values.

4.1. Quantile function

Let X be the LKu random variable with pdf (6), the quantile function of X, say $Q(u)$ is

$$Q\left(u\right)=F^{-1}\left\{1-\left\{1-\left(\frac{e^{\alpha}-e^{\alpha(1-u)}}{\left(e^{\alpha}-1\right)}\right)\right\}^{1/\gamma}\right\}^{1/\beta}, \tag{13}$$

where, u has the uniform distribution on the interval (0,1) and $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$.

4.2. Moments

Moments are very important and play an essential role in statistical analysis, especially in the applications. It helps to capture the important features and characteristics of the distribution (e.g., central tendency, dispersion, skewness and kurtosis). The rth moment of the LKu family of distributions is given by

$$\mu_r^{/}=\int_{-\infty}^{\infty}x^r g\left(x;\Theta\right)dx, \tag{14}$$

using (6) in (14), one may have

$$\begin{aligned}\mu_r^{/}=&\frac{\beta\gamma}{\alpha}\sum_{i,j,k=0}^{\infty}\left(-1\right)^{j+k}\left(\begin{array}{c}i\\j\end{array}\right)\left(\begin{array}{c}\gamma\left(j+1\right)-1\\k\end{array}\right)\\&\times\left(\frac{e^{\alpha}-1}{e^{\alpha}}\right)^{i+1}\eta_{r,\beta(k+1)-1},\end{aligned} \tag{15}$$

where,

$$\eta_{r,\beta(k+1)-1} = \int_{-\infty}^{\infty} x^r f(x;\xi) F(x;\xi)^{\beta(k+1)-1} dx.$$

Furthermore, a general expression for moment generating function (mgf) of the LKu random variable X is

$$M_x(t) = \frac{\beta\gamma}{r!\alpha} \sum_{i,j,k=0}^{\infty} (-1)^{j+k} t^r \binom{i}{j} \binom{\gamma(j+1)-1}{k} \times \left(\frac{e^{\alpha}-1}{e^{\alpha}}\right)^{i+1} \eta_{r,\beta(k+1)-1}. \tag{16}$$

4.3. Residual & reverse residual life

The residual life offer wider applications in reliability theory and risk management. The residual lifetime of X denoted by $R_{(t)}$ is derived as

$$R_{(t)}(x) = \frac{S(x+t)}{S(t)},$$

$$R_{(t)}(x) = \frac{\log\left[e^{\alpha} - \left\{1 - \left(1 - F(x+t;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha}-1)\right]}{\log\left[e^{\alpha} - \left\{1 - \left(1 - F(t;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha}-1)\right]}.$$

Additionally, the reverse residual life of the LKu random variable denoted by $\bar{R}_{(t)}$ can be derived as

$$\bar{R}_{(t)} = \frac{S(x-t)}{S(t)},$$

$$\bar{R}_{(t)}(x) = \frac{\log\left[e^{\alpha} - \left\{1 - \left(1 - F(x-t;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha}-1)\right]}{\log\left[e^{\alpha} - \left\{1 - \left(1 - F(t;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha}-1)\right]}.$$

4.4. Order statistics

Order statistics are among the essential tools in inferencial and non-parametric statistics. The applications of these statistics appears in the study of reliability and life testing. Consider $X_1, X_2, \ldots, X_k$ be a random sample of size k taken independently from LKu distribution. Let $X_{1:k}, X_{2:k}, \ldots, X_{k:k}$ be the corresponding order statistics. Then, from [4], the density of $X_{r:k}$ for $(r = 1, 2, \ldots, k)$ is given by

$$g_{r:k}(x) = \frac{g(x;\Theta)}{B(r,\ k\ -\ r+\ 1)} \sum_{i=0}^{k-r} \binom{k-r}{i} (-1)^i \left[G(x;\Theta)\right]^{i+r-1}. \tag{17}$$

The expression (22) provides the density of the $X_{r:k}$.

5. Estimation

In this section, the estimation of unknown parameters of LKu family via the method of maximum likelihood is discussed. Let $X_1, X_2, \ldots, X_k$ be a random sample from LKu family with parameters $(\alpha, \gamma, \beta, \theta)$. The log-likelihood function of this sample is

$$\begin{aligned}\log L(x;\Theta) = & -k\log\alpha + k\log\gamma + k\log\beta + k\log\left(e^{\alpha}-1\right) \\ & + \sum_{i=1}^{k}\log\left[f(x_i;\xi)\right] + (\beta-1)\sum_{i=1}^{k}\log\left[F(x_i;\xi)\right] \\ & - \sum_{i=1}^{k}\log\left[e^{\alpha} - \left\{1-\left(1-F(x_i;\xi)^{\beta}\right)^{\gamma}\right\}\left(e^{\alpha}-1\right)\right] \\ & + (\gamma-1)\sum_{i=1}^{k}\log\left[1-F(x_i;\xi)^{\beta}\right]. \end{aligned} \tag{18}$$

Obtaining the partial derivatives of (18), one may get

$$\begin{aligned}\frac{\partial}{\partial\alpha}\log L(x;\Theta) = & -\sum_{i=1}^{k}\frac{e^{\alpha} - \left\{1-\left(1-F(x_i;\xi)^{\beta}\right)^{\gamma}\right\}e^{\alpha}}{\left[e^{\alpha} - \left\{1-\left(1-F(x_i;\xi)^{\beta}\right)^{\gamma}\right\}\left(e^{\alpha}-1\right)\right]} \\ & - \frac{k}{\alpha} + \frac{ke^{\alpha}}{e^{\alpha}-1}, \end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\partial}{\partial\beta}\log L(x;\Theta) = & \sum_{i=1}^{k}\log\left[F(x_i;\xi)\right] - (\gamma-1)\sum_{i=1}^{k}\frac{\left[\log\left\{F(x_i;\xi)\right\}\right]F(x_i;\xi)^{\beta}}{1-F(x_i;\xi)^{\beta}} \\ & + \frac{k}{\beta} - \sum_{i=1}^{k}\frac{\left(1-w_{\beta,\xi}\right)^{\gamma}\left(e^{\alpha}-1\right)}{e^{\alpha} - \left\{1-\left(1-F(x_i;\xi)^{\beta}\right)^{\gamma}\right\}\left(e^{\alpha}-1\right)}, \end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial}{\partial\gamma}\log L(x;\Theta) = & \frac{k}{\gamma} - \sum_{i=1}^{k}\frac{\left(1-w_{\beta,\xi}\right)^{\gamma}\left(e^{\alpha}-1\right)}{e^{\alpha} - \left\{1-\left(1-F(x_i;\xi)^{\beta}\right)^{\gamma}\right\}\left(e^{\alpha}-1\right)} \\ & + \sum_{i=1}^{k}\log\left[1-F(x_i;\xi)^{\beta}\right], \end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial}{\partial\xi}\log L(x;\Theta) = & \beta(\gamma-1)\sum_{i=1}^{k}\frac{F(x_i;\xi)^{\beta-1}\left(\partial F(x_i;\xi)/\partial\xi\right)}{1-F(x_i;\xi)^{\beta}} \\ & \sum_{i=1}^{k}\frac{\partial f(x_i;\xi)/\partial\xi}{f(x_i;\xi)} + (\beta-1)\sum_{i=1}^{k}\frac{\partial F(x_i;\xi)/\partial\xi}{F(x_i;\xi)} \end{aligned} \tag{22}$$

$$-\sum_{i=1}^{k} \frac{(1-w_{\beta,\xi})^{\gamma}(e^{\alpha}-1)}{e^{\alpha}-\left\{1-\left(1-F(x_i;\xi)^{\beta}\right)^{\gamma}\right\}(e^{\alpha}-1)},$$

where $w_{\beta,\xi} = \left(1-F(x_i;\xi)^{\beta}\right)$. Setting $\frac{\partial}{\partial\alpha}\log L(x;\Theta)$, $\frac{\partial}{\partial\beta}\log L(x;\Theta)$, $\frac{\partial}{\partial\gamma}\log L(x;\Theta)$ and $\frac{\partial}{\partial\xi}\log L(x;\Theta)$ equal to zero and solving numerically these expressions simultaneously yields the maximum likelihood estimates of $(\alpha, \gamma, \beta, \xi)$.

6. Simulation study

In order to assess the performances of the maximum likelihood estimators, a small simulation study is carried out. The process is carried out as follow:

(1) The number of Monte Carlo replications was made 1000 times each with sample sizes $n = 30$, 50 and 100.
(2) Initial values for the parameters are selected as given in Table 1.
(3) Formulas used for calculating Bias and MSE are given by $Bias(\hat{\alpha}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\alpha}-\alpha)$ and $MSE(\hat{\alpha}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\alpha}-\alpha)^2$, respectively.
(4) Step (iii) is also repeated for the other parameters (γ, β, θ).

The empirical results are given in Tables 1 and 2.

TABLE 1. The parameter estimation from LKu-W distribution using MLE.

n	Par	Init	MLE	Bias	MSE	Init	MLE	Bias	MSE
	α	0.5	0.5145	0.0130	0.0072	0.75	0.7701	0.0209	0.0163
30	γ	0.5	0.5266	0.0254	0.0186	0.5	0.5231	0.0231	0.0160
	β	0.5	0.5146	0.0147	0.0081	0.5	0.5150	0.0150	0.0082
	θ	0.5	0.5329	0.0264	0.0182	0.5	0.5208	0.0228	0.0160
	α	0.5	0.5070	0.0070	0.0043	0.75	0.7581	0.0109	0.0085
50	γ	0.5	0.5218	0.0221	0.0108	0.5	0.5119	0.0113	0.0098
	β	0.5	0.5073	0.0079	0.0050	0.5	0.5070	0.0076	0.0046
	θ	0.5	0.5209	0.0239	0.0103	0.5	0.5091	0.0108	0.0089
	α	0.5	0.5017	0.0028	0.0019	0.75	0.7591	0.0099	0.0040
100	γ	0.5	0.5125	0.0132	0.0049	0.5	0.5043	0.0051	0.0043
	β	0.5	0.5023	0.0027	0.0035	0.5	0.5068	0.0072	0.0022
	θ	0.5	0.5189	0.0132	0.0050	0.5	0.5038	0.0095	0.0040

7. Applications

To prove the flexibility of the proposed family, two applications to real data sets are analyzed. The goodness of fits of the LKu-W distribution have been compared with the other lifetime models such as exponentiated Weibull (EW), Marshall-Olkin Weibull (MOW) and Kumaraswamy Weibull (Ku-W) distributions.

Table 2. The parameter estimation from LKu-W distribution using MLE.

n	Par	Init	MLE	Bias	MSE	Init	MLE	Bias	MSE
	α	1.5	1.5487	0.0488	0.0668	1.5	1.5586	0.0587	0.1208
30	γ	0.5	0.5770	0.0772	0.0413	0.5	0.5127	0.0129	0.0149
	β	0.5	0.5181	0.0183	0.0086	1.5	1.5086	0.0086	0.0201
	θ	0.5	0.5734	0.0767	0.0409	0.5	0.5122	0.0124	0.0143
	α	1.5	1.5270	0.0277	0.0436	1.5	1.5223	0.0220	0.0617
50	γ	0.5	0.5687	0.0691	0.0244	0.5	0.5044	0.0042	0.0088
	β	0.5	0.5106	0.0102	0.0058	1.5	1.4998	-0.0003	0.0107
	θ	0.5	0.5678	0.0679	0.0241	0.5	0.5034	0.0039	0.0079
	α	1.5	1.5171	0.0170	0.0184	1.5	1.5169	0.0169	0.0291
100	γ	0.5	0.5534	0.0533	0.0119	0.5	0.4930	-0.0047	0.0039
	β	0.5	0.5062	0.0056	0.0022	1.5	1.5019	0.0020	0.0052
	θ	0.5	0.5525	0.0528	0.0237	0.5	0.4928	0.0032	0.0034

The distribution functions of the competing models are as:

(1) The exponentiated Weibull is given by

$$G(x) = \left(1 - e^{-\eta x^{\theta}}\right)^{a}, \qquad x, a, \eta, \theta > 0.$$

(2) The Marshall-Olkin Weibull is

$$G(x) = \frac{\left(1 - e^{-\eta x^{\theta}}\right)}{1 - (1-\sigma)\left(1 - \left(1 - e^{-\eta x^{\theta}}\right)\right)}, \qquad x, \sigma, \eta, \theta > 0.$$

(3) The Kumaraswsamy Weibull is given by

$$G(x) = 1 - \left(1 - \left(1 - e^{-\eta x^{\theta}}\right)^{\beta}\right)^{\gamma}, \qquad x, \beta, \gamma, \eta, \theta > 0.$$

The analytical measures of goodness of fit including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Kolmogoro Smirnov (KS), Cramer–von Mises (CM) and Anderson-Darling (AD) statistics are considered to compare the proposed method with the other fitted models. In general, a model with smaller values of these analytical measure indicate better fit to the data. All the required computations have been carried out in the R-language using "Nelder-Mead" algorithm.

Data 1: The data set represents survival times of guinea pigs injected with the different amount of tubercle bacilli studied by [1]. The MLEs and the considered statistics are shown in Tables 3 and 4, respectively. Corresponding to data 1, the estimated pdf and cdf of the proposed model are plotted in Figure 3, pp-plot and Kaplan Meier survival plot are presented in Figure 4, while, the scale TTT-transform plot and estimated hazard rate plot are sketched in Figure 5.

TABLE 3. MLEs with their standard errors in brackets for data 1.

Dist.	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{\beta}$
LKu-W	3.943 (0.2901)		1.971 (0.3104)		0.075 (0.3509)	0.962 (0.2967)
MOW		0.210 (0.0255)	0.698 (0.3153)	1.770 (0.0771)		
EW		0.708 (0.3530)	1.171 (0.2830)		1.994 (0.9831)	
Ku-W		0.641 (0.5713)	1.062 (0.6322)		1.432 (1.0107)	2.310 (2.4604)

TABLE 4. The analytical measures of the fitted distributions using data 1.

Dist.	KS	CM	AD	AIC	BIC	CIAC	HQIC
LKu-W	0.095	0.090	0.591	209.70	216.53	210.05	212.42
MOW	0.106	0.146	0.906	213.45	220.28	213.80	216.17
EW	0.100	0.110	0.730	211.62	218.45	211.97	214.341
Ku-W	0.097	0.107	0.714	213.63	222.73	214.22	217.25

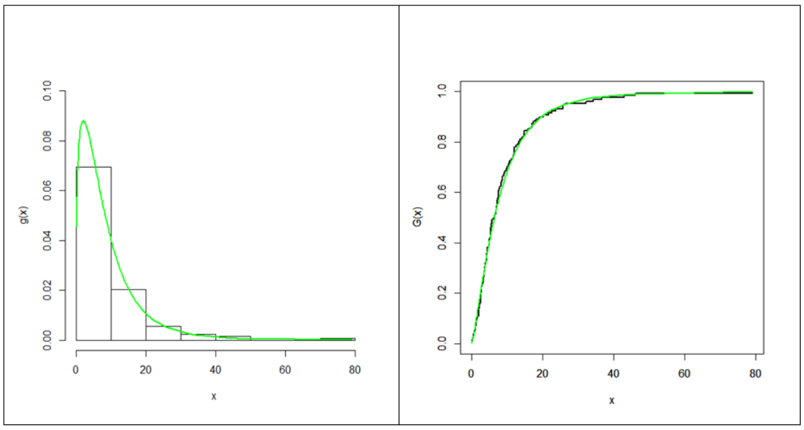

FIGURE 3. Plots of the estimated pdf and cdf of the LKu-W distribution for data 1.

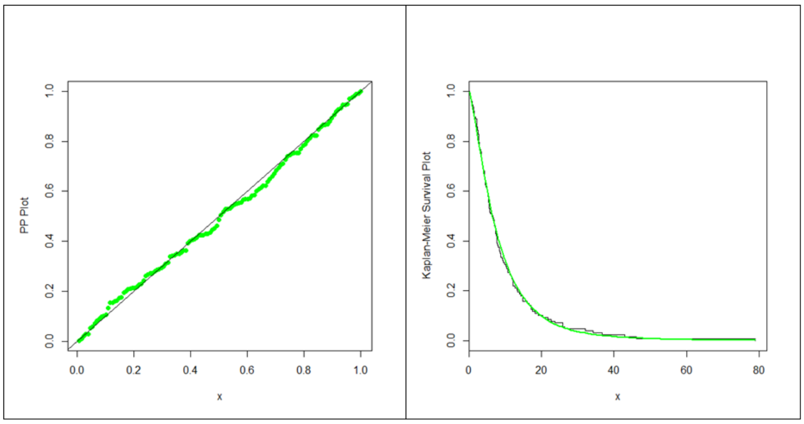

FIGURE 4. PP and Kaplan-Meir survival plots of the LKu-W distribution for data 1.

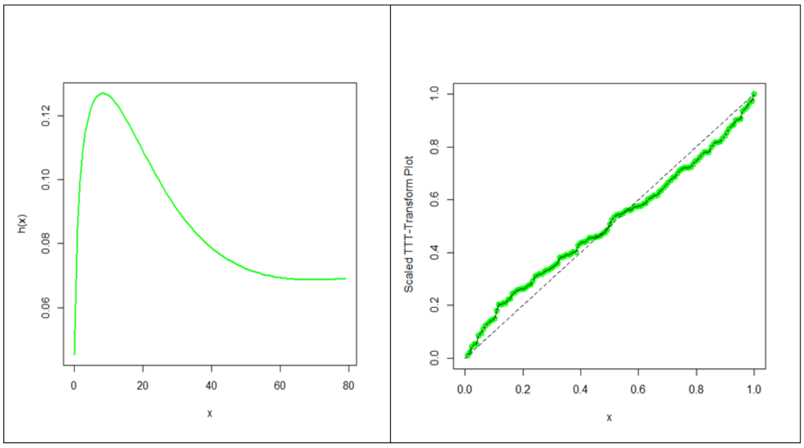

FIGURE 5. Scale TTT-transform plot and estimated hazard rate plot of the LKu-W distribution for data 1.

Data 2: The second data set representing the remission times (in months) of a random sample of 128 bladder cancer patients taken from [10].The MLEs and the considered statistics are shown in Tables 5 and 6, respectively. Corresponding to data 2, the estimated pdf and cdf of the proposed model are plotted in Figure 6, pp-plot and Kaplan Meier survival plot are presented in Figure 7, while the scale TTT-transform plot and estimated hazard rate plot sketched in Figure 8.

Table 5. MLEs with their standard errors in brackets for data 1.

Dist.	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{\gamma}$	$\hat{\beta}$
LKu-W	4.265 (0.8934)		1.268 (0.7023)		0.016 (0.7023)	0.893 (0.6980)
MOW		0.877 (0.5205)	0.564 (0.1308)	11.829 (1.286)		
EW		0.720 (0.5492)	0.541 (0.1883)		4.332 (3.5347)	
Ku-W		0.487 (0.4800)	0.520 (0.9073)		1.988 (1.2719)	3.712 (1.9700)

Table 6. The analytical measures of the fitted distributions using data 2.

Dist.	KS	CM	AD	AIC	BIC	CIAC	HQIC
LKu-W	0.037	0.038	0.255	826.37	834.93	826.57	829.85
MOW	0.075	0.150	0.884	834.98	843.54	835.18	838.46
EW	0.046	0.046	0.324	828.21	836.77	828.41	831.69
Ku-W	0.041	0.040	0.271	829.20	840.61	829.53	833.84

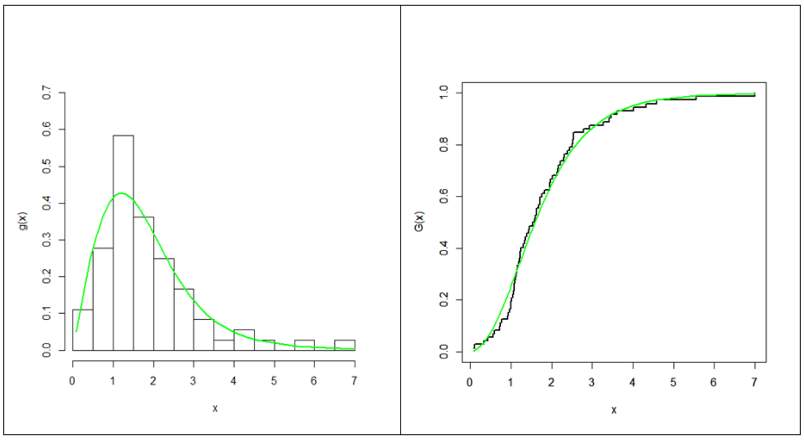

Figure 6. Plots of the estimated pdf and cdf of the LKu-W distribution for data 2.

8. Concluding remarks

In this article, a new method is adopted to extend the existing distributions. This effort leads to a new family of life distributions, called the LKu family of distributions. General expressions for some of the mathematical properties of the new family are investigated. The estimation of the of model parameters

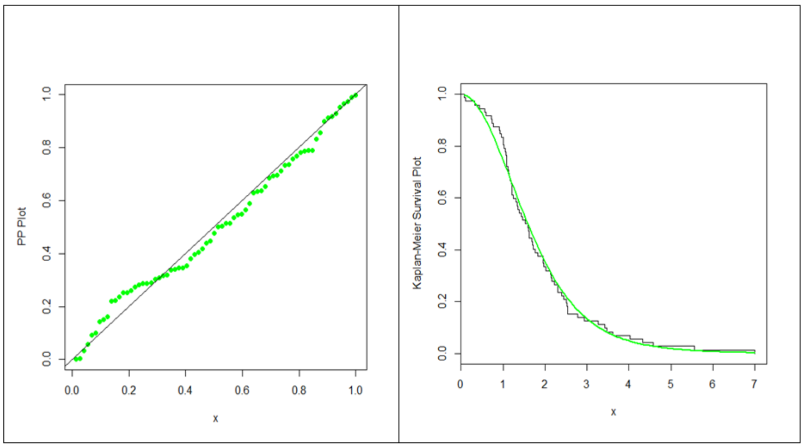

Figure 7. PP and Kaplan-Meir survival plots of the LKu-W distribution for data 2.

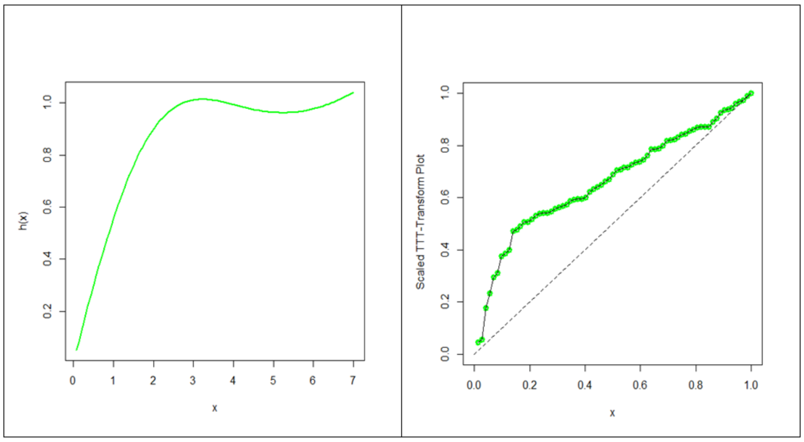

Figure 8. Scale TTT-transform plot and estimated hazard rate plot of the LKu-W distribution for data 2.

through maximum likelihood method is discussed.There are certain advantages of using the proposed method like its cdf has a closed form solution and facilitating data modeling with monotonic and non-monotonic failure rates. A special sub-model of the new family, called LKu-Weibulldistribution is considered and two real applications are analyzed. In simulation study, the consistency and proficiency of the maximum likelihood estimators of the proposed model are illustrated. The practical applications of the proposed model reveal better fits to real-life data than the other well-known existing distributions.

Appendix

1. Code for computing the analytical results

```
data=c( Insert the Data Set Here) pdf_proposed <-function(par,x)
{
########## Write down the parameters as mentioned below
α= Insert here the estimated value
β= Insert here the estimated value
θ= Insert here the estimated value
γ= Insert here the estimated value
######### Write down the PDF of the Model here
}
cdf_proposed <-function(par,x)
{
α= Insert here the estimated value
β= Insert here the estimated value
θ= Insert here the estimated value
γ= Insert here the estimated value
######### Write down the CDF of the Model here
}
set.seed(0)
goodness.fit(pdf=pdf_proposed,
cdf=cdf_proposed,
starts = c(Set initial values of the parameters here ), data = data,
method=”Nelder-Mead”, domain=c(0,Inf),mle=NULL)
```

2. Codes for plotting the empirical pdf of the model

```
x=c( Insert the Data Set Here)
x=sort(x)
########## Write down the parameters of the model and the corre-
sponding estimated values
α= Insert here the estimated value
β= Insert here the estimated value
θ= Insert here the estimated value
γ= Insert here the estimated value
pdf= Write the pdf of the Model here
f=pdf
x=sort(x)
yrange=c(Insert the Range for Y-axis here)
xrange=c(Min value of the data, Max value of the data)
hist(x, freq=FALSE, breaks=10, xlim=xrange, ylim=yrange, ylab=”f(x)”,
xlab=”x”, main=” ”)
par(new=TRUE)
lines(x, f, xlim=xrange, lty=1, ylab=” ”, ylim=yrange, lwd=2,col=”green”,
```

```
xlab=””)
par(new=TRUE)
```

3. Codes for plotting the empirical cdf of the model

```
x=c(Insert the Data Set Here)
m=length(x)
########## Write down the parameters of the model and the corresponding estimated values
α= Insert here the estimated value
β= Insert here the estimated value
θ= Insert here the estimated value
γ= Insert here the estimated value
x<-sort(x)
F1<-ecdf(x) #### ecdf stands for the estimated cdf
ecdf<-F1(c(x))
LKucdf<- #### Insert the cdf of the Model here
plot(x ,ecdf, lty=1, lwd=2.5, type=”s”, xlab=”x”, ylab=”F(x)”, ylim=c(0,1),
xlim=c(Min value of the data, Max value of the data), col=”black”)
par(new=TRUE)
plot(x, LKucdf, lty=1, lwd=2.5, type=”l”, xlab=”x”,
ylab=”F(x)”, ylim=c(0,1),
xlim=c(Min value of the data, Max value of the data), col=”green”)
par(new=TRUE)
```

4. Codes for plotting Kaplan-Meier survival plot of the model

```
x=c( Insert the Data Set Here)
library(survival)
delta=rep(1,length(x))
x<-sort(x)
km = survfit(Surv(x,delta)˜1)
plot(km, conf.int=FALSE, ylab=”Kaplan-Meier Survival Plot”, xlab=”x”)
########## Write down the parameters of the model and the corresponding estimated values
α= Insert here the estimated value
β= Insert here the estimated value
θ= Insert here the estimated value
γ= Insert here the estimated value
ss <-function(x)
{
########## Write the survival function of the Model here
}
lines(seq(Min value of the data, Max value of the data, length.out=100),
ss(seq(Min value of the data, Max value of the data, length.out=100)),
col=”green”, lwd=2)
```

5. Codes for plotting PP-plot of the model

```
x=c( Insert the Data Set Here)
cdfLku=function(x, a, b, g, t)
{
##### Insert the cdf of the model here
}
##### After that we write the code for PP-plot
x=sort(x)
n=length(x)
##### Emperical Distribution Function
Fn=seq(1,n)/n
plot(Fn, Lku(x, insert the estimated values of the parameters),xlab=”x”,
ylab=”PP Plot”, pch=21, col=”green”, bg=”green”)
abline(0,1)
##### Adding a legend if desired
```

6. Codes for plotting scaled TTT-transform plot

```
data=c( Insert the Data Set Here)
ti<-sort(data)
n<-length(ti)
t<-c(1:n)
aux<-0
for(i in 1:n)
{
t[i]<-ti[i]+aux
aux<-t[i]}
r<-c(1:n)
f<-t+(n-r)*ti
s<-sum(ti)
f<-f/s
plot(r/n, f, xlim=c(0,1), ylim=c(0,1), xlab=”x”,
ylab=” Scaled TTT-Transform Plot”, lwd=3, col=”green”)
lines(r/n,f)
x<-c(0,1)
y<-c(0,1)
lines(x,y,lty=2)
```

Acknowledgement. The author is grateful to the Associate Editor and anonymous Referees for their meaningful comments on the earlier version of the paper which greatly improved the presentation of the paper. The author would also like to knowledge Dr. Clement Boateng Ampadu (USA) for his help and special guidance in revising the paper.

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