Commun. Korean Math. Soc. **34** (2019), No. 4, pp. 1329–1334
https://doi.org/10.4134/CKMS.c180365
pISSN: 1225-1763 / eISSN: 2234-3024

A NOTE ON EXPANSIVE $\mathbb{Z}^k$-ACTION AND GENERATORS

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Abstract. We define the concept of a generator for a $\mathbb{Z}^k$-action T and show that T is expansive if and only it has a generator. Further, we prove several properties of a $\mathbb{Z}^k$-action including that the least upper bound of the set of expansive constants is not an expansive constant.

1. Introduction

A homeomorphism $h : X \longrightarrow X$ defined on a metric space X gives a natural action of the additive group of integers, $(\mathbb{Z}, +)$, on X given by $(n, x) \rightsquigarrow h^n(x)$, $n \in \mathbb{Z}$ and $x \in X$. The classical theory of dynamical systems is more inclined to the study of this $\mathbb{Z}$-action, if the system is a discrete dynamical system. If the group $\mathbb{Z}$ is replaced by the additive group of real numbers, $(\mathbb{R}, +)$, then we study the dynamics of flows (i.e., continuous time evolution systems). However, in last two decades there has been significant development in the study of topological aspect of various dynamical properties of $\mathbb{Z}^k$ or $\mathbb{R}^k$, $k > 1$, actions. We wish to note here that the study of algebraic aspect of these actions is much older and is now a well-developed part of dynamical systems.

A homeomorphism $h : X \longrightarrow X$ defined on a metric space X is called *expansive* provided there exists a real number $e > 0$ such that whenever $x, y \in X$ with $x \neq y$ then there exists an integer n (depending on x, y) satisfying $d(h^n(x), h^n(y)) > e$. Constant e is called an *expansive constant* for h. Note that expansivity corresponds to the concept of the hyperbolic set for a diffeomorphism in the sense that hyperbolic diffeomorphism are always expansive. In 1950, Utz, [10], introduced the notion of expansivity with the name unstable homeomorphism. Later, in 1958 Gottschalk and Hedlund suggested the term expansive homeomorphism for such maps. Since its inception expansivity has been extensively studied in the area of Topological Dynamics, Ergodic Theory, Continuum Theory, and Symbolic Dynamics. One of the good monograph for the study of expansive homeomorphism is [1].

Received August 28, 2018; Accepted November 23, 2018.
2010 *Mathematics Subject Classification.* Primary 37B99; Secondary 54H20.
Key words and phrases. expansive homeomorphisms, generators, $\mathbb{Z}^k$-actions.

In 2005, Shi and Zhou showed that there exists no expansive $\mathbb{Z}^k$-action on graphs [8]. Later, in 2007, they showed the non existence of expansive $\mathbb{Z}^k$-actions on Peano continua that contains θ-curves [9]. Using k-many commuting self homeomorphism on a compact metric space X, Mouron constructed an m-dimensional continuum X^k_m, such that X^k_m admits an expansive $\mathbb{Z}^{k+1}$-action but does not admit an expansive $\mathbb{Z}^k$-action [4]. In 2003, Pilyugin and Tikhomirov introduced the notion shadowing property for $\mathbb{Z}^k$-actions [6]. Oprocha in [5], proved the Spectral Decomposition Theorem (Smale type) for expansive $\mathbb{Z}^k$-actions, $k \geq 1$, having shadowing property. Recently, in 2014 [3], Kim and Lee proved the Spectral Decomposition Theorem for $\mathbb{Z}^2$-actions. In this paper we study generators for expansive $\mathbb{Z}^k$-actions.

In Section 2 of the paper we define the concept of generators for $\mathbb{Z}^k$-actions and characterize expansive $\mathbb{Z}^k$-actions in terms of generators. We study several properties of expansive $\mathbb{Z}^k$-actions in Section 3. We also show that the upper bound of expansive constants for an expansive $\mathbb{Z}^k$-action is not an expansive constant.

2. Generators for $\mathbb{Z}^k$-actions

We recall the definition of a $\mathbb{Z}^k$-action. Elements of $\mathbb{Z}^k$ are denoted by $\overline{n}$, $\overline{m}$, etc.

Definition. Let (X, d) be a metric space. A continuous map $T : \mathbb{Z}^k \times X \longrightarrow X$ is said to be a $\mathbb{Z}^k$*-action* if the following are satisfied:

(1) $T(\overline{n} + \overline{m},\ x) = T(\overline{n}, T(\overline{m},\ x))$ for all $\overline{n},\ \overline{m} \in \mathbb{Z}^k, x \in X$.
(2) $T(\overline{0}, x) = x$ for all $x \in X$. Here $\overline{0} = (0, 0, \ldots, 0)$ (k-tuple).

Note that for $\overline{n} \in \mathbb{Z}^k$, $T(\overline{n}, \cdot) : X \longrightarrow X$ is a homeomorphism. We denote $T(\overline{n}, x)$ also by $T^{\overline{n}}(x)$. On observing the definition of generators given by Keynes and Robertson in [2] we give the following definition of generators for $\mathbb{Z}^k$-actions.

Definition. Let (X, d) be a compact metric space, $T : \mathbb{Z}^k \times X \longrightarrow X$ be a $\mathbb{Z}^k$-action and let $\mathcal{U}$ be a finite open cover of X. Then $\mathcal{U}$ is said to be a *generator for* T if for any collection $\left\{A_{\overline{n}} : \overline{n} \in \mathbb{Z}^k\right\}$ of members of $\mathcal{U}$,

$$\bigcap_{\overline{n} \in \mathbb{Z}^k} T^{-\overline{n}}(cl(A_{\overline{n}}))$$

contains at most one point. Here $cl(A)$ denotes the closure of a set A in X.

If $k = 1$, then the above definition coincides with the definition of generators for $\mathbb{Z}$-actions. We now recall the definition of an expansive $\mathbb{Z}^k$-action.

Definition. Let (X, d) be a metric space and $T : \mathbb{Z}^k \times X \longrightarrow X$ be a $\mathbb{Z}^k$-action. Then T is said to be an *expansive* $\mathbb{Z}^k$*-action* if there exists a positive real number e such that for any $x, y \in X$ with $x \neq y$ there exists $\overline{n} \in \mathbb{Z}^k$ such that $d\left(T^{\overline{n}}(x), T^{\overline{n}}(y)\right) > e$; e is called an *expansive constant* for T.

Thus if T is expansive with an expansive constant e, then for each $\overline{n} \in \mathbb{Z}^k$, $d(T^{\overline{n}}(x), T^{\overline{n}}(y)) < e$ implies $x = y$. In the following theorem we obtain characterization of expansive $\mathbb{Z}^k$-actions in terms of generators.

Theorem 2.1. *Let (X, d) be a compact metric space and $T : \mathbb{Z}^k \times X \longrightarrow X$ be a $\mathbb{Z}^k$-action. Then T is expansive if and only if T has a generator.*

Proof. Suppose T is expansive with an expansive constant e. Let $\mathcal{U}$ be a finite open cover of X with diameter of each element in $\mathcal{U}$ less than $e/2$. Let $\left\{A_{\overline{n}} : \overline{n} \in \mathbb{Z}^k\right\}$ be any collection of elements of $\mathcal{U}$. If possible suppose $\bigcap_{\overline{n}\in\mathbb{Z}^k} T^{-\overline{n}}(cl(A_{\overline{n}}))$ contains more than one point. Let

$$x, y \in \bigcap_{\overline{n}\in\mathbb{Z}^k} T^{-\overline{n}}(cl(A_{\overline{n}})).$$

This implies for every $\overline{n} \in \mathbb{Z}^k$,

$$\begin{aligned} &x, y \in T^{-\overline{n}}(cl(A_{\overline{n}})) = T(-\overline{n}, cl(A\overline{n})) \\ \Longrightarrow\ & T(\overline{n}, x), T(\overline{n}, y) \in T\left(\overline{n}, T(-\overline{n}, cl(A\overline{n}))\right). \end{aligned}$$

But T is a $\mathbb{Z}^k$-action. Therefore $T\left(\overline{n}, T(-\overline{n}, cl(A\overline{n}))\right) = T(\overline{0}, cl(A_{\overline{n}})) = cl(A_{\overline{n}})$. Thus for every $\overline{n} \in \mathbb{Z}^k$, $d\left(T^{\overline{n}}(x), T^{\overline{n}}(y)\right) \leq e$, as $A_{\overline{n}} \in \mathcal{U}$. But T is expansive with an expansive constant e. Therefore, we obtain $x = y$. Hence $\bigcap_{\overline{n}\in\mathbb{Z}^k} T^{-\overline{n}}(cl(A_{\overline{n}}))$ contains at most one point.

Conversely, suppose $\mathcal{V}$ is a generator for T and let $e > 0$ be the Lebsegue number of $\mathcal{V}$. Note that such a number exists as $\mathcal{V}$ is a finite open cover of X. Suppose for $x, y \in X$ and for each $\overline{n} \in \mathbb{Z}^k$, $d\left(T^{\overline{n}}(x), T^{\overline{n}}(y)\right) \leq e/2$. Now e being Lebsegue number of $\mathcal{V}$, for each $\overline{n} \in \mathbb{Z}^k$, there is $A_{\overline{n}} \in \mathcal{V}$ such that

$$\begin{aligned} & T(\overline{n}, x),\ T(\overline{n}, y) \in A_{\overline{n}} \subset cl(A\overline{n}) \\ \Longrightarrow\ & T(-\overline{n},\ T(\overline{n}, x)),\ T(-\overline{n},\ T(\overline{n}, y)) \in T(-\overline{n},\ cl(A\overline{n})) \\ \Longrightarrow\ & x,\ y \in \bigcap_{\overline{n}\in\mathbb{Z}^k} T^{-\overline{n}}(cl(A_{\overline{n}})). \end{aligned} \tag{1}$$

Thus there is collection $\left\{A_{\overline{n}} : \overline{n} \in \mathbb{Z}^k\right\}$ of elements of $\mathcal{V}$ satisfying Equation (1). But $\mathcal{V}$ is a generator. Therefore the intersection in Equation (1) can contain at most one point, which implies, $x = y$. Hence T is expansive with an expansive constant $e/2$. □

3. Properties of expansive $\mathbb{Z}^k$-actions

In this section we prove some basic properties of expansive $\mathbb{Z}^k$-actions as a consequence of its characterization in terms of generators. We recall the definition of topological conjugacy for $\mathbb{Z}^k$-actions.

Definition. Let (X, d), (Y, ρ) be two metric spaces and let $T : \mathbb{Z}^k \times X \longrightarrow X$, $H : \mathbb{Z}^k \times Y \longrightarrow Y$ be two $\mathbb{Z}^k$-actions. Then T and H are said to be *topologically conjugate* if there exists a homeomorphism $f : X \longrightarrow Y$ such that $f \circ T^{\overline{n}} =$

$H^{\overline{n}} \circ f$ for all $\overline{n} \in \mathbb{Z}^k$. Map f is called the *topological conjugacy* between T and H.

Recall, a point $x \in X$ is said to be a *periodic point* of a $\mathbb{Z}^k$-action T, if the set $O_T(x) = \{T^{\overline{n}}(x) : \overline{n} \in \mathbb{Z}^k\}$ is a finite set. If $O_T(x) = \{x\}$, then x is said to be a *fixed point* of T. Thus, x is a fixed point of T if $T^{\overline{n}}(x) = x$ for all $\overline{n} \in \mathbb{Z}^k$. It is simple to observe that periodic points of T are preserved under topological conjugacy. Equivalently, if T and H are topologically conjugate by conjugacy f and if x_0 is a periodic point of T, then $f(x_0)$ is periodic point of H.

In [7], Robinson showed that there exists a $\mathbb{Z}^2$-action which is expansive without periodic points and therefore an expansive action may not have any fixed point. Let $Fix\,T$ denote the set of the fixed points of T. Note that $Fix\,T$ is a closed subset of X. In the following we show that if T is expansive, then $Fix\,T$ is a finite set.

Theorem 3.1. *Let (X, d) be a compact metric space and $T : \mathbb{Z}^k \times X \longrightarrow X$ be an expansive $\mathbb{Z}^k$-action. Then $Fix\,T$ is a finite subset of X.*

Proof. If $Fix\ T = \phi$ we are through. Suppose $Fix\ T \neq \phi$. Assume $Fix\ T$ is infinite. Let $\{x_m\}_{m\geq 0}$ be sequence of distinct points in $Fix\ T$ and let x be a limit point of $\{x_m\}$. Now for each $\overline{n} \in \mathbb{Z}^k$, $T^{\overline{n}}(x_m) = x_m$. Also, each $T^{\overline{n}} = T(\overline{n}, \cdot) : X \longrightarrow X$ is a homeomorphism. Therefore $x \in Fix\,T$.

Let e be an expansive constant for T. By Theorem 3.1, T has a generator. Suppose $\mathcal{U}$ is a generator with a Lebsegue number δ, where $0 < \delta < e/4$. Let $x_k,\ x_l$ be two distinct points in $B(x, e/8)$. Note that there exists $A \in \mathcal{U}$ such that $B(x, e/8) \subset A$. Also, for each $\overline{n} \in \mathbb{Z}^k$,

$$T^{\overline{n}}(x_k),\ T^{\overline{n}}(x_l) \in A.$$

For each $\overline{n} \in \mathbb{Z}^k$, set $A_{\overline{n}} = A$. Then the collection $\left\{A_{\overline{n}} : \overline{n} \in \mathbb{Z}^k\right\}$ by elements of $\mathcal{U}$ satisfies

$$x_k, x_l \in \bigcap_{\overline{n}\in\mathbb{Z}^k} T^{-\overline{n}}(cl(A_{\overline{n}})).$$

But this contradicts that $\mathcal{U}$ is a generator for T. Therefore our supposition is wrong and hence $Fix\,T$ is finite. □

Following is an immediate corollary to the above the theorem.

Corollary 3.2. *Let (X, d) be a compact metric space and $T : \mathbb{Z}^k \times X \longrightarrow X$ be an expansive $\mathbb{Z}^k$-action. Then the set of periodic points of T, $Per\,T$ is an at most countable subset of X.*

Using the concept of generators we show that expansivity is preserved under conjugancy.

Theorem 3.3. *Let (X, d), (Y, ρ) be compact metric spaces and let $T : \mathbb{Z}^k \times X \longrightarrow X$, $H : \mathbb{Z}^k \times Y \longrightarrow Y$ be two $\mathbb{Z}^k$-actions. Suppose T is topologically conjugate to H by conjugacy f. If T is expansive, then so is H.*

Proof. Let e be an expansive constant for T and $\mathcal{U}$ be a finite open cover of X such that for each $U \in \mathcal{U}$, $diam\, U < e/2$. Then it can be easily proved that $\mathcal{U}$ is a generator for T. In order to show that H is expansive we show that H has a generator. Since $f : X \longrightarrow Y$ is a homeomorphism, it follows that for $e/4 > 0$, there exists δ, $\delta > 0$, such that

$$\rho(y_1, y_2) < \delta \Longrightarrow d\left(f^{-1}(y_1), f^{-1}(y_2)\right) < e/4. \tag{2}$$

Let $\mathcal{V}$ be an open cover of Y such that for each $V \in \mathcal{V}$, $diam\, V < \delta$. We complete the proof by showing that $\mathcal{V}$ is a generator for Y. Set

$$\mathcal{U} = f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}.$$

Then $\mathcal{U}$ is a finite open cover of X and for each $U \in \mathcal{U}$, $diam\, U < e/2$. This follows using Equation 2. Note that $\mathcal{U}$ is a generator for T. Let $\left\{A_{\overline{n}} : \overline{n} \in \mathbb{Z}^k\right\}$ be a collection of sets from $\mathcal{V}$. Then $\{f^{-1}(A_{\overline{n}})\}$ is the collection of sets from $\mathcal{U}$. But T is expansive. Therefore

$$\bigcap_{\overline{n} \in \mathbb{Z}^k} T^{-\overline{n}}\left(f^{-1}(cl(A_{\overline{n}}))\right)$$

contains at most one point, which further implies that

$$\bigcap_{\overline{n} \in \mathbb{Z}^k} f\left(T^{-\overline{n}}\left(f^{-1}(cl(A_{\overline{n}}))\right)\right) = \bigcap_{\overline{n} \in \mathbb{Z}^k} H^{-\overline{n}}\left((cl(A_{\overline{n}}))\right)$$

contains at most one point. Hence $\mathcal{V}$ is a generator for H and therefore H is expansive. □

If e is an expansive constant for T, an expansive $\mathbb{Z}^k$-action on a compact metric space X, then any δ, $0 < \delta < e$, is also an expansive constant for T. Let $\mathcal{F}$ be the set of all expansive constants for T. Then $\mathcal{F}$ is a bounded set of positive real numbers and hence least upper bound of $\mathcal{F}$ exists.

Theorem 3.4. *Let X be a compact metric space and let $T : \mathbb{Z}^k \times X \longrightarrow X$ be an expansive action. If β is the least upper bound of $\mathcal{F}$, the set of all expansive constants for T, then β is not an expansive constant for T.*

Proof. For every $m \in \mathbb{N}$, $\beta + \frac{1}{m}$ is not an expansive constant for T and therefore there is $x'_m \neq y'_m$ in X such that for all $\overline{n} \in \mathbb{Z}^k$

$$d(T^{\overline{n}}(x'_m), T^{\overline{n}}(y'_m)) < \beta + \frac{1}{m}.$$

Let $e > 0$ be an expansive constant for T. Then for each $m \in \mathbb{N}$, using expansivity of T there is $\overline{k}_m \in \mathbb{Z}^k$ such that

$$d(T^{\overline{k}_m}(x'_m), T^{\overline{k}_m}(y'_m)) > e.$$

Let $x_m = T^{\overline{k}_m}(x'_m)$ and $y_m = T^{\overline{k}_m}(y'_m)$. We assume that the sequence $\{x_m\}$ and $\{y_m\}$ converge to say, x and y respectively. Then $x \neq y$. Further, for

$\overline{r} \in \mathbb{Z}^k$ and $\alpha > 0$ choose p, $q \in \mathbb{N}$ and $\eta > 0$ such that $\frac{1}{p} < \frac{\alpha}{3}$, $d(x, x_q) < \eta$, $d(y, y_q) < \eta$ and satisfying

$$d(a,b) < \eta \Longrightarrow d(T^{\overline{k}}(a), T^{\overline{k}}(b)) < \frac{\alpha}{3}. \tag{3}$$

Therefore, for $\overline{r} \in \mathbb{Z}^k$

$$\begin{aligned} d(T^{\overline{r}}(x), T^{\overline{r}}(y)) &\leq d(T^{\overline{r}}(x), T^{\overline{r}}(y'_q)) + d(T^{\overline{r}}(y'_q), T^{\overline{r}}(x'_q)) + d(T^{\overline{r}}(x'_q), T^{\overline{r}}(y)) \\ &\leq \alpha + \beta. \end{aligned}$$

Above inequality, implies that there exist $x \neq y$ in X such that for any $\overline{r} \in \mathbb{Z}^k$, $d(T^{\overline{r}}(x), T^{\overline{r}}(y)) < \beta$ and therefore β is not expansive constant for T. □

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