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CHARACTERIZATION OF WARPED PRODUCT SUBMANIFOLDS OF LORENTZIAN CONCIRCULAR STRUCTURE MANIFOLDS

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Abstract. Recently Hui et al. ([8, 9]) studied contact CR-warped product submanifolds and also warped product pseudo-slant submanifolds of a $(LCS)_n$-manifold $\overline{M}$. The characterization for both these classes of warped product submanifolds have been studied here. It is also shown that there do not exists any proper warped product bi-slant submanifold of a $(LCS)_n$-manifold. Although the existence of a bi-slant submanifold of $(LCS)_n$-manifold is ensured by an example.

1. Introduction

A Lorentzian concircular structure manifold of dimension n was introduced by Shaikh [13]. It is simply written as $(LCS)_n$-manifold. This manifold has many applications, see [2], [15], [16]. Many authors have studied such manifolds, see [6], [14], [17, 18, 20]. Throughout the paper we denote an $(LCS)_n$-manifolds by $\overline{M}$ and denote a submanifold of $\overline{M}$ by M. In this connection it is mentioned that different classes of M are studied in [1], [7–11], [21].

Warped product submanifolds were introduced by Bishop and O'Neill [4]. It is the generalized class of Riemannian product. Recently Hui et al. ([8, 9]) studied contact CR-warped product submanifolds and also warped product pseudo-slant submanifolds of $\overline{M}$. In this paper, we characterize both these class of warped product submanifolds. An example of bi-slant submanifold of $\overline{M}$ is constructed. However, it is also shown that there do not exists any proper warped product bi-slant submanifold of $\overline{M}$.

2. Preliminaries

An $(LCS)_n$-manifold is a Lorentzian manifold $\overline{M}$ of dimension n endowed with the unit timelike concircular vector field ξ, its associated 1-form η and an

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$(1,1)$ tensor field ϕ such that

$$\bar{\nabla}_X \xi = \alpha \phi X, \tag{2.1}$$

where α is a non-zero scalar function such that

$$\bar{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.2}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$ and $\overline{\nabla}$ is the Levi-Civita connection of the Lorentzian metric g. In an $(LCS)_n$-manifold $(n > 2)$ $\overline{M}$, the following relations hold [13]:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.4}$$

$$(\overline{\nabla}_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}, \tag{2.5}$$

$$(\overline{\nabla}_X \phi)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.6}$$

$$(X\rho) = d\rho(X) = \beta\eta(X) \tag{2.7}$$

for all $X,\ Y,\ Z \in \Gamma(T\bar{M})$ and $\beta = -(\xi\rho)$ is a scalar function.

Let g be the induced metric on M. The induced connection on the tangent bundle TM and normal bundle $T^{\perp}M$ are ∇ and $\nabla^{\perp}$ resp. Then we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.8}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.9}$$

where h and A_V are the second fundamental form and the shape operator of M, respectively, such that $g(h(X,Y),V) = g(A_V X, Y)$ for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we can write

$$\text{(a) } \phi X = PX + QX, \quad \text{(b) } \phi V = bV + cV, \tag{2.10}$$

where $PX,\ bV$ are the tangential components and $QX,\ cV$ are the normal components of ϕX and ϕV, respectively.

Now M is said to be invariant if $\phi(T_pM) \subseteq T_pM$ for every $p \in M$ and anti-invariant if $\phi T_pM \subseteq T_p^{\perp}M$ for every $p \in M$. Also M is said to be a CR-submnaifold [3] if there is a differential distribution $\mathcal{D} : p \to \mathcal{D}_p \subseteq T_pM$ such that $\mathcal{D}$ is an invariant distribution and the orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant.

The normal space of a CR-submanifold M is decomposed as $T^{\perp}M = Q\mathcal{D}^{\perp} \oplus \nu$, where ν is the ϕ-invariant normal subbundle of M.

Again M is said to be slant if for each non-zero vector $X \in T_pM$ the angle between ϕX and T_pM is a constant, i.e., it does not depend on the choice of $p \in M$ and M is said to be a pseudo-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ such that

(i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}^{\perp} \oplus \mathcal{D}^{\theta}$,

(ii) The distribution $\mathcal{D}^{\perp}$ is anti-invariant,

(iii) The distribution $\mathcal{D}^{\theta}$ is slant with slant angle $\theta \neq 0, \frac{\pi}{2}$.

From the definition it is clear that if $\theta = 0$, then M is a CR-submanifold. We say that a pseudo-slant submanifold is proper if $\theta \neq 0, \frac{\pi}{2}$. The normal space of a pseudo-slant submanifold M is decomposed as $T^{\perp}M = Q\mathcal{D}^{\theta} \oplus \phi\mathcal{D}^{\perp} \oplus \nu$.

Theorem 2.1 ([1]). *Let M be a submanifold of $\overline{M}$ such that $\xi \in \Gamma(TM)$. Then M is slant if and only if $\exists\ \lambda \in [0,1]$ such that*

$$P^2 = \lambda\{I + \eta \otimes \xi\}. \tag{2.11}$$

Moreover if θ is slant angle, then $\lambda = \cos^2\theta$.

Also for a slant submanifold, from (2.10) and (2.11) we have

$$bQX = \sin^2\theta(X + \eta(X)\xi) \quad \text{and} \quad cQX = -QPX. \tag{2.12}$$

For a Riemannian manifold $\overline{M}$ of dimension n and a smooth function f on $\overline{M}$, $\boldsymbol{\nabla} f$, the gradient of f is defined by

$$g(\boldsymbol{\nabla} f, X) = X(f)(\text{or} = (Xf)) \tag{2.13}$$

for any $X \in \Gamma(TM)$.

The warped product of two Riemannian manifolds (N_1, g_1) and (N_2, g_2) is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$ such that [4]

$$g = g_1 + f^2 g_2, \tag{2.14}$$

where $f \in C^{\infty}(N_1)$. This f is known as warping function. If f is constant, then the warped product is trivial Riemannian product. In a warped product manifold $N_1 \times_f N_2$ we have [12]

$$\nabla_U X = \nabla_X U = (X \ln f)U \tag{2.15}$$

for any $X \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$.

3. Contact CR-warped product submanifolds

In [8], it is shown that contact CR-warped product submanifolds of $\overline{M}$ of the form $N_{\perp} \times_f N_T$ exists if $\xi \in \Gamma(TN_{\perp})$ and does not exists if $\xi \in \Gamma(TN_T)$, where N_T and $N_{\perp}$ are invariant and anti-invariant submanifolds of $\overline{M}$ respectively. Now we find a characterization for a submanifold M of $\overline{M}$ to be contact CR-warped product of the form $N_{\perp} \times_f N_T$ such that $\xi \in \Gamma(TN_{\perp})$. First we prove the following Lemma:

Lemma 3.1. *Let $M = N_{\perp} \times_f N_T$ be a warped product submanifold of $\overline{M}$ such that $\xi \in \Gamma(TN_{\perp})$. Then*

$$g(h(X,Y), \phi Z) = -\alpha\eta(Z)g(X,Y) - (Z \ln f)g(\phi X, Y) \tag{3.1}$$

for $X,\ Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_{\perp})$.

Proof. For $X,\ Y \in \Gamma(TN_T)$ and $Z, \xi \in \Gamma(TN_\perp)$, we have

$$g(h(X,Y),\phi Z) = g(\overline{\nabla}_X \phi Y, Z) - g((\overline{\nabla}_X \phi)Y, Z). \tag{3.2}$$

Using (2.6) and (2.8) in (3.2), we obtain

$$g(h(X,Y),\phi Z) = -g(\phi Y, \nabla_X Z) - \alpha g(X,Y)\eta(Z). \tag{3.3}$$

By virtue of (2.15), (3.3) yields (3.1). □

Now interchanging X by ϕX and Y by ϕY in (3.1), we get the following respective relations

$$g(h(\phi X,Y),\phi Z) = -\alpha\eta(Z)g(\phi X,Y) - Z(\ln f)g(X,Y), \tag{3.4}$$

$$g(h(X,\phi Y),\phi Z) = -\alpha\eta(Z)g(\phi X,Y) - Z(\ln f)g(X,Y), \tag{3.5}$$

$$g(h(\phi X,\phi Y),\phi Z) = -\alpha\eta(Z)g(X,Y) - Z(\ln f)g(\phi X,Y). \tag{3.6}$$

Corollary 3.1. *Let $M = N_\perp \times_f N_T$ be a warped product submanifold of $\overline{M}$ such that $\xi \in \Gamma(TN_\perp)$. Then*

$$g(h(\phi X,Y),\phi Z) = g(h(X,\phi Y),\phi Z)$$
$$\text{and} \quad g(h(\phi X,\phi Y),\phi Z) = g(h(X,Y),\phi Z)$$

for $X,\ Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\perp)$.

Now we prove the following:

Theorem 3.1. *Let M be a contact CR-submanifold of $\overline{M}$ such that ξ is tangent to the anti-invariant distribution $\mathcal{D}^\perp$. Then M is locally a warped product submanifold if and only if*

$$A_{\phi Z}X = -\alpha\eta(Z)X - (Z\mu)\phi X \tag{3.7}$$

for any $X \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^\perp)$ and for some smooth function μ on M such that $(Y\mu) = 0$ for any $Y \in \mathcal{D}$.

Proof. If M is a contact CR-warped product submanifold, then for any $X \in \Gamma(TN_T)$ and $Z,\ W \in \Gamma(TN_\perp)$, we have

$$g(A_{\phi Z}X, W) = g(\overline{\nabla}_W X, \phi Z) = g(\overline{\nabla}_W \phi X, Z) - g((\overline{\nabla}_W \phi)X, Z). \tag{3.8}$$

Using (2.6) and (2.8) in (3.8), we get

$$g(A_{\phi Z}X, W) = g(\nabla_W \phi X, Z). \tag{3.9}$$

In view of (2.15), (3.9) yields $g(A_{\phi Z}X, W) = 0$. Therefore $A_{\phi Z}X$ has no component in $\Gamma(TN_\perp)$ and hence by virtue of (3.1), the relation (3.7) follows.

Conversely, let M be a contact CR-submanifold of $\overline{M}$ with the invariant and anti-invariant distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ such that the relation (3.7) holds. Then for any $X \in \Gamma(\mathcal{D})$ and $Z,\ W \in \Gamma(\mathcal{D}^\perp)$, we have from (2.6) and (2.8) that

$$g(\nabla_Z W, \phi X) = g(\overline{\nabla}_Z \phi W, X) = -g(A_{\phi W}X, Z). \tag{3.10}$$

Using (3.7) in (3.10), we find $g(\nabla_Z W, \phi X) = 0$, i.e., $\mathcal{D}^\perp$ is integrable and its leaves are totally geodesic in M. Again for any $X,\ Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we get

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X \phi Y, \phi Z) + \eta(Z) g(Y, \overline{\nabla}_X \xi). \tag{3.11}$$

Using (2.1) in (3.11), we get

$$g(\nabla_X Y, Z) = g(A_{\phi Z} X, \phi Y) + \alpha\eta(Z) g(Y, \phi X). \tag{3.12}$$

Interchanging X and Y in (3.12), we find

$$g(\nabla_Y X, Z) = g(A_{\phi Z} Y, \phi X) + \alpha\eta(Z) g(X, \phi Y). \tag{3.13}$$

From (3.12) and (3.13), we have

$$g([X,Y], Z) = g(A_{\phi Z} X, \phi Y) - g(A_{\phi Z} Y, \phi X). \tag{3.14}$$

Using (3.7) in (3.14), we obtain $g([X,Y],Z) = 0$ and therefore $\mathcal{D}$ is integrable.

Let us consider a leaf N_T of $\mathcal{D}$ and h^T be the second fundamental form of N_T in M, then we have from (2.6) and (2.8) that

$$g(h^T(X,Y), Z) = g(\overline{\nabla}_Y \phi X, \phi Z) + \eta(Z) g(X, \overline{\nabla}_Y \xi). \tag{3.15}$$

By virtue of (2.1) and (2.9), (3.15) yields

$$g(h^T(X,Y), Z) = g(A_{\phi Z} Y, \phi X) + \alpha\eta(Z) g(X, \phi Y). \tag{3.16}$$

From (3.7) and (3.16), we obtain

$$g(h^T(X,Y), Z) = -(Z\mu) g(X,Y). \tag{3.17}$$

Using (2.13) in (3.17), we get

$$h^T(X,Y) = -(\boldsymbol{\nabla}\mu) g(X,Y), \tag{3.18}$$

where $\boldsymbol{\nabla}\mu$ is the gradient of μ and therefore N_T is totally umbilical in M with mean curvature $-\boldsymbol{\nabla}\mu$. Since $(Y\mu) = 0$, for any $Y \in \mathcal{D}$, $\boldsymbol{\nabla}\mu$ is parallel along N_T, thus the leaves of $\mathcal{D}$ are extrinsic sphere in M. Hence by Hiepko's theorem (see [5]) M is locally a warped product $N_\perp \times_f N_T$, where N_T and $N_\perp$ denote the integral manifolds of $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively. Hence the theorem is proved. □

4. Warped product pseudo slant submanifolds

Warped product pseudo-slant submanifolds of $\overline{M}$ are studied in [9]. Here we obtain a characterization for a submanifold M of $\overline{M}$ to be a warped product pseudo-slant submanifold of the form $N_\theta \times_f N_\perp$, where N_θ is a slant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$.

Lemma 4.1. *Let M be a proper pseudo-slant submanifold of $\overline{M}$ with anti-invariant and proper slant distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively such that $\xi \in \Gamma(\mathcal{D}^\theta)$. Then*

$$g(\nabla_X Y, Z) = \sec^2\theta [g(h(X, PY), \phi Z) + g(h(X,Z), QPY)] \tag{4.1}$$

for any $X,\ Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(D^\perp)$.

Proof. For any $X,\ Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\overline{\nabla}_X \phi Y, \phi Z) - g((\overline{\nabla}_X \phi)Y, \phi Z) \\ &= g(\overline{\nabla}_X PY, \phi Z) + g(\overline{\nabla}_X QY, \phi Z) \\ &= g(\overline{\nabla}_X PY, \phi Z) + g(\overline{\nabla}_X \phi QY, Z) - g((\overline{\nabla}_X \phi)QY, Z) \\ &= g(\overline{\nabla}_X PY, \phi Z) + g(\overline{\nabla}_X bQY, Z) + g(\overline{\nabla}_X cQY, Z). \end{aligned}$$

Using (2.12) in the above relation, we get

$$g(\nabla_X Y, Z) = g(\overline{\nabla}_X PY, \phi Z) + \sin^2\theta g(\overline{\nabla}_X Y, Z) - g(\overline{\nabla}_X QPY, Z). \tag{4.2}$$

Using (2.8) and (2.9) in (4.2) we get (4.1). □

Corollary 4.1. *Let M be a proper pseudo-slant submanifold $\overline{M}$ with anti-invariant and proper slant distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively such that $\xi \in \Gamma(\mathcal{D}^\theta)$. Then the distribution $\mathcal{D}^\theta$ defines a totally geodesic foliation if and only if*

$$g(h(X, PY), \phi Z) + g(h(X, Z), QPY) = 0$$

for every $X,\ Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

Lemma 4.2. *Let $M = N_\theta \times_f N_\perp$ be a warped product pseudo-slant submanifolds of $\overline{M}$, where $N_\perp$ and N_θ are ant-invariant and proper slant submanifold of $\overline{M}$ such that $\xi \in \Gamma(TN_\theta)$. Then*

$$g(h(X, Y), \phi Z) + g(h(X, Z), QY) = 0, \tag{4.3}$$

$$g(h(Z, W), QX) + g(h(X, Z), QW) = (\phi X \ln f)g(Z, W), \tag{4.4}$$

$$g(h(Z, W), QPX) + g(h(PX, Z), QW) = \cos^2\theta[(X \ln f) + \alpha\eta(X)]g(Z, W). \tag{4.5}$$

Proof. For any $X,\ Y \in \Gamma(TN_\theta)$ and $Z \in \Gamma(TN_\perp)$, we have from (2.8), (2.6) and (2.10) that

$$g(h(X, Y), \phi Z) = g(\overline{\nabla}_X PY, Z) + g(\overline{\nabla}_X QY, Z). \tag{4.6}$$

Using (2.9) and (2.15) in (4.6), we obtain

$$g(h(X, Y), \phi Z) = (X \ln f)g(Z, PY) - g(h(X, Z), QY). \tag{4.7}$$

Thus (4.3) follows from (4.7). Also, for any $X \in \Gamma(TN_\theta)$ and $Z,\ W \in \Gamma(TN_\perp)$, we have

$$\begin{aligned} g(h(Z, W), QX) &= g(\overline{\nabla}_Z W, \phi X) - g(\overline{\nabla}_Z W, PX) \\ &= g(\overline{\nabla}_Z \phi W, X) - g((\overline{\nabla}_Z \phi)W, X) - g(\overline{\nabla}_Z W, PX) \\ &= -g(h(X, Z), \phi W) + g(W, \overline{\nabla}_Z PX). \end{aligned}$$

Using (2.15) in the above relation, we get (4.4) and (4.5) is obtained by interchanging X by PX in (4.4). □

Now, we prove the following:

Theorem 4.1. *Let M be a proper pseudo-slant submanifold of $\overline{M}$ with anti-invariant distribution $\mathcal{D}^{\perp}$ and proper pseudo-slant distribution $\mathcal{D}^{\theta}$, respectively such that $\xi \in \Gamma(\mathcal{D}^{\theta})$. Then M is locally a mixed-geodesic warped product submanifold of the form $N_{\theta} \times_f N_{\perp}$ if and only if*

$$A_{\phi Z}X = 0 \quad \text{and} \quad A_{QPX}Z = \cos^2\theta[(X\mu) + \alpha\eta(X)]Z \tag{4.8}$$

for any $X \in \Gamma(\mathcal{D}^{\theta})$, $Z \in \Gamma(\mathcal{D}^{\perp})$ and for some function μ on M satisfying $(Z\mu) = 0$, for any $Z \in \Gamma(\mathcal{D}^{\perp})$.

Proof. Let $M = N_{\theta} \times_f N_{\perp}$ be a mixed geodesic warped product submanifold of $\overline{M}$ such that (4.8) holds. Then for any $X,\ Y \in \Gamma(TN_{\theta})$ and $Z \in \Gamma(TN_{\perp})$, from (4.3) and (4.5), we get (4.8).

Conversely, let M is a proper pseudo-slant submanifold of $\overline{M}$ such that (4.8) holds. Then for any for any $X,\ Y \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, from (4.1) and (4.8), we get $g(\nabla_X Y, Z) = 0$ and hence the leaves of $\mathcal{D}^{\theta}$ are totally geodesic in M.

Also, for any $X \in \Gamma(\mathcal{D}^{\theta})$ and $Z,\ W \in \Gamma(\mathcal{D}^{\perp})$, we have

$$\begin{aligned} g([Z,W],X) &= g(\overline{\nabla}_Z W, X) - g(\overline{\nabla}_W Z, X) \\ &= g(\phi\overline{\nabla}_Z W, \phi X) - g(\phi\overline{\nabla}_W Z, \phi X) \\ &= g(\overline{\nabla}_Z \phi W, \phi X) - g(\overline{\nabla}_W \phi Z, \phi X) \\ &= -g(\phi W, \overline{\nabla}_Z PX) - g(\phi W, \overline{\nabla}_Z QX) + g(\phi Z, \overline{\nabla}_W PX) \\ &\quad + g(\phi Z, \overline{\nabla}_W QX) \\ &= -g(\phi W, \overline{\nabla}_Z PX) - g(W, \overline{\nabla}_Z bQX) - g(W, \overline{\nabla}_Z cQX) \\ &\quad + g(\phi Z, \overline{\nabla}_W PX) + g(Z, \overline{\nabla}_W bQX) + g(Z, \overline{\nabla}_W cQX). \end{aligned}$$

Using (2.12), (2.8) and (2.9) in the above relation, we find

$$\begin{aligned} g([Z,W],X) = &-g(A_{\phi W}PX, Z) + g(A_{\phi Z}PX, W) + \sin^2\theta g([Z,W],X) \\ &+ g(A_{QPX}Z, W) - g(A_{QPX}W, Z). \end{aligned} \tag{4.9}$$

Using (4.8) in (4.9), we get

$$\cos^2\theta g([Z,W],X) = 0. \tag{4.10}$$

Since $\mathcal{D}^{\theta}$ is proper pseudo-slant so, $\theta \neq 0, \frac{\pi}{2}$. Therefore, $g([Z,W],X) = 0$ and hence $\mathcal{D}^{\perp}$ is integrable.

Now, let $h^{\perp}$ be the second fundamental form of a leaf $N_{\perp}$ of $\mathcal{D}^{\perp}$ in M. Then for any $Z,\ W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D}^{\theta})$, we have

$$\begin{aligned} g(h^{\perp}(Z,W),X) &= g(\nabla_Z W, X) \\ &= g(\overline{\nabla}_Z \phi W, \phi X) - g((\overline{\nabla}_Z \phi)W, \phi X) \\ &= g(\overline{\nabla}_Z \phi W, PX) - g(W, \overline{\nabla}_Z \phi QX) \\ &= g(\overline{\nabla}_Z \phi W, PX) - g(W, \overline{\nabla}_Z bQX) - g(W, \overline{\nabla}_Z cQX). \end{aligned}$$

Using (2.14) in the above relation we obtain
(4.11)
$$g(h^{\perp}(Z,W),X) = -g(\overline{\nabla}_Z\phi W, PX) + \sin^2\theta g(\overline{\nabla}_Z W, X) + g(W, \overline{\nabla}_Z QPX).$$

By virtue of (2.9), (4.11) yields

$$\cos^2\theta g(h^{\perp}(Z,W),X) = -g(A_{\phi W}PX, Z) - g(A_{QPX}W, Z). \tag{4.12}$$

Using (4.8) in (4.12), we get

$$\cos^2\theta g(h^{\perp}(Z,W),X) = -\cos^2\theta[(X\mu) + \alpha\eta(X)]g(Z,W). \tag{4.13}$$

Thus, we find

$$h^{\perp}(Z,W) = -[\boldsymbol{\nabla}\mu + \alpha\xi]g(Z,W),$$

where $\boldsymbol{\nabla}\mu$ is gradient of the function μ. Therefore $N_{\perp}$ is totally umbilical in M with the mean curvature $H^{\perp} = -(\boldsymbol{\nabla}\mu + \alpha\xi)$.

Now, let D^N be the normal connection of $N_{\perp}$ in M. Then for any $Y \in \Gamma(\mathcal{D}^{\theta})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, we have

$$g(D^N_Z\boldsymbol{\nabla}\mu + \alpha\xi, Y) = g(\nabla_Z\boldsymbol{\nabla}\mu, Y) + \alpha g(\nabla_Z\xi, Y).$$

Also, from (2.1) and (2.8) we get $\nabla_Z\xi = 0$.

Therefore, $g(D^N_Z\boldsymbol{\nabla}\mu + \alpha\xi, Y) = g(\nabla_Z\boldsymbol{\nabla}\mu, Y) = 0$, since $(Z\mu) = 0$ for every $Z \in \Gamma(\mathcal{D}^{\perp})$ and hence the mean curvature of $N_{\perp}$ is parallel. Thus the leaves of the distribution $\mathcal{D}^{\perp}$ are totally umbilical in M with non-vanishing parallel mean curvature vector $H^{\perp}$, i.e., $N_{\perp}$ is an extrinsic sphere in M. Therefore by Hiepko's Theorem (see [5]), M is a warped product submanifold. □

5. Warped product bi-slant submanifolds

Definition 5.1 ([22])**.** The submanifold M of $\overline{M}$ is said to be a bi-slant submanifold if ∃ a pair of orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ of M such that

(i) $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$,
(ii) $\phi\mathcal{D}_1 \perp \mathcal{D}_2$ and $\phi\mathcal{D}_2 \perp \mathcal{D}_1$,
(iii) $\mathcal{D}_1, \mathcal{D}_2$ are slant submanifolds with slant angles θ_1 and θ_2, respectively.

If $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then M is a CR-submanifold and if $\theta_1 = 0$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then M is a semi-slant submanifold. Also, if $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then M is a pseudo-slant submanifold. A bi-slant submanifold M of $\overline{M}$ is said to be proper if $\theta_1,\ \theta_2 \neq 0, \frac{\pi}{2}$.

If M is bi-slant submanifold of $\overline{M}$, then any $X \in \Gamma(TM)$ can be expressed as

$$X = T_1X + T_2X. \tag{5.1}$$

Here T_1, T_2 represent the projections from TM onto $\mathcal{D}_1, \mathcal{D}_2$ respectively.

Taking $P_1 = T_1 \circ P$ and $P_2 = T_2 \circ P$ from (5.1), we obtain

$$\phi X = P_1X + P_2X + QX. \tag{5.2}$$

Then the normal bundle of M can be written as $T^{\perp}M = Q\mathcal{D}_1 \oplus Q\mathcal{D}_2 \oplus \nu$.
Now we construct a bi-slant submanifold of $\overline{M}$.

Example 5.1. Consider the semi-Euclidean space $\mathbb{R}^7$ with the cartesian coordinates $(x_1, y_1 \ldots, x_3, y_3,\, t)$ and the paracontact structure

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\ \ \phi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j},\ \phi\left(\frac{\partial}{\partial t}\right) = 0,\ \ 1 \le i, j \le 3.$$

It is clear that $\mathbb{R}^7$ is a Lorentzian metric manifold with usual semi-Euclidean metric tensor. For any $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$, let M be a submanifold of $\mathbb{R}^7$ defined by

$$\chi(u, v, w, s,\, t) = (w + u\cos\theta_1,\ u\sin\theta_1,\ s + v\cos\theta_2,\ v\sin\theta_2,\ 0,\ 0,\ t).$$

Then the tangent space of M is spanned by the following vectors

$$Z_1 = \cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial x_2}, \qquad Z_2 = \cos\theta_2\frac{\partial}{\partial y_1} + \sin\theta_2\frac{\partial}{\partial y_2},$$
$$Z_3 = \frac{\partial}{\partial x_1},\ Z_4 = \frac{\partial}{\partial y_1},\ \text{and} \qquad Z_5 = \frac{\partial}{\partial t}.$$

Then we have

$$\phi Z_1 = \cos\theta_1\frac{\partial}{\partial y_1} + \sin\theta_1\frac{\partial}{\partial y_2}, \qquad \phi Z_2 = \cos\theta_2\frac{\partial}{\partial x_1} + \sin\theta_2\frac{\partial}{\partial x_2},$$
$$\phi Z_3 = \frac{\partial}{\partial y_1},\ \phi Z_4 = \frac{\partial}{\partial x_1},\ \text{and} \qquad \phi Z_5 = 0.$$

We take $\mathcal{D}_1 = \mathrm{Span}\{Z_1,\ Z_4\}$ and $\mathcal{D}_2 = \mathrm{Span}\{Z_2,\ Z_3\}$, then $g(Z_1, \phi Z_4) = \cos\theta_1$ and $g(Z_2, \phi Z_3) = \cos\theta_2$. Thus the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are slant with slant angles θ_1 and θ_2 respectively and hence M is a bi-slant submanifold.

Lemma 5.1. *Let M be a proper bi-slant submanifold of $\overline{M}$ with the slant distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\xi \in \Gamma(\mathcal{D}_1)$. Then*

$$\cos^2\theta_2 g(\nabla_{X_1}X_2, Y_2) = g(\nabla_{X_1}P_2X_2, P_2Y_2) + g(h(X_1, P_2X_2), QY_2) + g(h(X_1, Y_2), QP_2X_2), \tag{5.3}$$

for any $X_1 \in \Gamma(\mathcal{D}_1)$ and $X_2,\ Y_2 \in \Gamma(\mathcal{D}_2)$, where θ_1 and θ_2 are the slant angles of slant distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively.

Proof. For any $X_1 \in \Gamma(\mathcal{D}_1)$ and $X_2,\ Y_2 \in \Gamma(\mathcal{D}_2)$, we have

$$\begin{aligned} g(\nabla_{X_1}X_2, Y_2) &= g(\overline{\nabla}_{X_1}\phi X_2, \phi Y_2) - g((\overline{\nabla}_{X_1}\phi)X_2, \phi Y_2) \\ &= g(\overline{\nabla}_{X_1}P_2X_2, P_2Y_2) + g(\overline{\nabla}_{X_1}P_2X_2, QY_2) + g(\overline{\nabla}_{X_1}QX_2, \phi Y_2) \\ &= g(\overline{\nabla}_{X_1}P_2X_2, P_2Y_2) + g(\overline{\nabla}_{X_1}P_2X_2, QY_2) \\ &\quad + g(\overline{\nabla}_{X_1}bQX_2, Y_2) + g(\overline{\nabla}_{X_1}cQX_2, Y_2) - g((\overline{\nabla}_{X_1}\phi)QX_2, Y_2). \end{aligned}$$

Using (2.12) in the above relation, we obtain

$$g(\nabla_{X_1}X_2, Y_2) = g(\overline{\nabla}_{X_1}P_2X_2, P_2Y_2) + g(\overline{\nabla}_{X_1}P_2X_2, QY_2) + \sin^2\theta g(\overline{\nabla}_{X_1}X_2, Y_2) - g(\overline{\nabla}_{X_1}QP_2X_2, Y_2). \tag{5.4}$$

By virtue of (2.8) and (2.9), (5.4) yields (5.3). □

Theorem 5.1. *There does not exist a proper warped product bi-slant submanifold $M = M_1 \times_f M_2$ of $\overline{M}$ such that $\xi \in \Gamma(TM_1)$.*

Proof. If possible, let $M = M_1 \times_f M_2$ be a proper warped product bi-slant submanifold of $\overline{M}$. Then for $X_1 \in \Gamma(TM_1)$ and $X_2,\ Y_2 \in \Gamma(TM_2)$, we have

$$\begin{aligned} g(h(X_1, P_2X_2), QY_2) &= g(\overline{\nabla}_{X_1} P_2X_2, \phi Y_2) - g(\overline{\nabla}_{X_1} P_2X_2, P_2Y_2) \\ &= g(\overline{\nabla}_{X_1} P_2^2 X_2, Y_2) + g(\overline{\nabla}_{X_1} QP_2X_2, Y_2) \\ &\quad - g((\overline{\nabla}_{X_1}\phi)P_2X_2, Y_2) + g(P_2X_2, \overline{\nabla}_{X_1} P_2Y_2). \end{aligned}$$

Using (2.6), (2.9), (2.11) and (2.15) in the above relation, we get

(5.5)
$$g(h(X_1, P_2X_2), QY_2) + g(h(X_1, Y_2), QP_2X_2) = 2\cos^2\theta_2(X_1 \ln f)g(X_2, Y_2).$$

Again using (2.15) in (5.3), we get

(5.6) $$g(h(X_1, P_2X_2), QY_2) + g(h(X_1, Y_2), QP_2X_2) = 0.$$

From (5.5) and (5.6), we obtain

(5.7) $$\cos^2\theta_2(X_1 \ln f) = 0.$$

Since M is a proper warped product bi-slant submanifold so, $\theta_2 \neq \frac{\pi}{2}$. Therefore $(X_1 \ln f) = 0$ for every $X_1 \in \Gamma(TM_1)$ which implies that f is a constant. This is a contradiction. Hence warped product does not exist. □

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