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# SOME RESULTS ON ALMOST KENMOTSU MANIFOLDS WITH GENERALIZED $(k,\mu)'$-NULLITY DISTRIBUTION

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Abstract. In the present paper, we prove that if there exists a second order parallel tensor on an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution and $h' \neq 0$, then either the manifold is isometric to $H^{n+1}(-4) \times \mathbb{R}^n$, or, the second order parallel tensor is a constant multiple of the associated metric tensor of $M^{2n+1}$ under certain restriction on $k, \mu$. Besides this, we study Ricci soliton on an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution. Finally, we characterize such a manifold admitting generalized Ricci soliton.

## 1. Introduction

Geometry of Kenmotsu manifolds was originated by Kenmotsu [18] and became an interesting area of research in differential geometry. As a generalization of Kenmotsu manifolds, the notion of almost Kenmotsu manifold was first introduced by Janssens and Vanhecke [26]. In recent years, for some results regarding such manifolds we refer to ([6, 8–22, 24, 25, 36], [27–31, 33, 35]). Almost Kenmotsu manifolds satisfying $(k,\mu)$ and $(k,\mu)'$-nullity conditions were introduced by Dileo and Pastore [12], where both $k$ and $\mu$ are constants. In 2011, Pastore and Saltarelli [23] extended the above nullity conditions to the corresponding generalized nullity conditions for which both $k$ and $\mu$ are smooth functions. Recently some results on generalized $(k,\mu)$ and $(k,\mu)'$-almost Kenmotsu manifolds satisfying some curvature conditions have been obtained by Wang et al. ([34, 35]), De et al. ([10, 15]) and many others.

In [13], Einsenhart proved that if a Riemannian manifold admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. Latter in [19], Levy proved that a second order parallel symmetric non-degenerate tensor in a space form is proportional to the metric tensor. Since then, many authors investigated the Einsenhart problem of finding symmetric parallel tensor on various spaces and obtained some fruitful results. Recently Wang et al. [29] studied second order parallel

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tensors on an almost Kenmotsu manifold satisfying $(k, \mu)$ and $(k, \mu)'$-nullity condition.

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold $(M, g)$, $g$ is called a Ricci soliton [16] if

$$(\pounds_V g + 2S + 2\gamma g)(X, Y) = 0, \tag{1}$$

where $\pounds$ is the Lie derivative, $S$ is the Ricci tensor, $V$ is a complete vector field on $M$ and $\gamma$ is a constant. The Ricci soliton is said to be shrinking, steady and expanding according as $\gamma$ is negative, zero and positive respectively. For more details about Ricci solitons we refer to ([1, 2, 6, 7, 24, 25, 32]).

In 2016, Nurowski et al. [22] introduced the notion of generalized Ricci soliton. Given a smooth function $f$ on $M$, the gradient of $f$ is defined by

$$g(grad\ f, X) = X(f), \tag{2}$$

the Hessian of $f$ is defined by

$$(Hess\ f)(X, Y) = g(\nabla_X grad\ f, Y) \tag{3}$$

for all smooth vector fields $X, Y$. For a smooth vector field $X$, its dual 1-form is $X^b$ ([21], [22]) given by

$$X^b(Y) = g(X, Y). \tag{4}$$

The generalized Ricci soliton equation in a Riemannian manifold $(M, g)$ is defined by [22]

$$\pounds_X g = -2c_1 X^b \odot X^b + 2c_2 S + 2\beta g, \tag{5}$$

where $\pounds_X g$ is the Lie derivative of $g$ along $X$ for all smooth vector fields $X$, $(X^b \odot X^b)(X, Y) = X^b(X)X^b(Y)$ for all smooth vector fields $X, Y$ and $c_1, c_2, \beta \in \mathbb{R}$. For different values of $c_1, c_2$ and $\beta$, equation (5) is a generalization of Killings equation ($c_1 = c_2 = \beta = 0$), equation for homotheties ($c_1 = c_2 = 0$), Ricci soliton ($c_1 = 0, c_2 = -1$), Vacuum near-horizon geometry equation ($c_1 = 1, c_2 = \frac{1}{2}$) etc. For more details we refer to ([8, 17, 22]). The authors in [21] studied generalized Ricci solitons in Sasakian manifolds. Recently we have studied generalized Ricci solitons on contact manifolds [14].

If $X = grad\ f$, then the generalized Ricci soliton equation is given by

$$Hess\ f = -c_1 df \odot df + c_2 S + \beta g. \tag{6}$$

In this paper we study second order parallel tensors in an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution. Moreover, we consider Ricci soliton and generalized Ricci soliton in an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution.

The paper is organized as follows:

In Section 2, we give a brief description on almost Kenmotsu manifolds. Section 3 contains some results of generalized $(k, \mu)'$-nullity distribution. Section 4 deals with second order parallel symmetric tensors on an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution, while Section 5 is devoted to the study of Ricci solitons on an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution. Finally, we characterize an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution admitting generalized Ricci solitons.

## 2. Almost Kenmotsu manifolds

A differentiable $(2n+1)$-dimensional manifold $M$ is said to be an almost contact structure if it admits a $(1,1)$-tensor field $\phi$, a characteristic vector field $\xi$ and a 1-form $\eta$ satisfying ([3,4])

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1, \tag{7}$$

where $I$ denote the identity endomorphism.

From the above equations we have $\phi\xi = 0$ and $\eta \circ \phi = 0$.
If a manifold $M$ with a $(\phi,\xi,\eta)$-structure admits a Riemannian metric $g$ such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields $X$, $Y$, then $M$ is said to have an almost contact metric structure $(\phi,\xi,\eta,g)$. The fundamental 2-form $\Phi$ on an almost contact metric manifold is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any vector fields $X$, $Y$. The condition for an almost contact metric manifold to be normal is equivalent to vanishing of the $(1,2)$-type tensor $N_\phi$, defined by $N_\phi = [\phi,\phi] + 2d\eta \otimes \xi$, where $[\phi,\phi]$ is the Nijenhuis torsion of $\phi$. An almost contact metric manifold is called an almost Kenmotsu manifold if $\eta$ is closed and $d\Phi = 2\eta \wedge \Phi$. Let us denote by $\mathcal{D}$ the distribution orthogonal to $\xi$ which is defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since $\eta$ is closed, $\mathcal{D}$ is an integrable distribution.

Let $M^{2n+1}$ be an almost Kenmotsu manifold. We denote $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot,\xi)\xi$ on $M^{2n+1}$. The tensor fields $l$ and $h$ are symmetric operators and satisfy the following relations ([5,12]):

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{8}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{9}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{10}$$

$$\begin{aligned} R(X,Y)\xi = {} & \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) \\ & + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y \end{aligned} \tag{11}$$

for any vector fields $X, Y$. The $(1,1)$-type symmetric tensor field $h' = h\phi$ is anti-commuting with $\phi$ and $h'\xi = 0$. Also it is clear that [12]

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{12}$$

## 3. $\xi$ belongs to the generalized $(k, \mu)'$-nullity distribution

In this section we give a brief description of almost Kenmotsu manifolds with $\xi$ belonging to the generalized $(k, \mu)'$-nullity distribution. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $\xi$ belonging to the generalized $(k, \mu)'$-nullity distribution. Then according to Pastore and Saltarelli [23] we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{13}$$

where $k$, $\mu$ are smooth functions on $M^{2n+1}$ and $h' = h\phi$. Let $X \in \mathcal{D}$ be the eigenvector of $h'$ corresponding to the eigenvalue $\lambda$. Then from (10) it is clear that $\lambda^2 = -(k+1)$. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigenvalue $\lambda$ and $-\lambda$ of $h'$, respectively. Before presenting our main theorems we recall some results:

**Lemma 3.1** (Theorem 5.1 of [23])**.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a generalized $(k, \mu)'$-almost Kenmotsu manifold such that $h' \neq 0$. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies*

$$\begin{aligned}
R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0,\\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0,\\
R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda},\\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,\\
R(X_\lambda, Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],\\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{aligned}$$

**Lemma 3.2.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution and $h \neq 0$. Then*

$$\xi(\lambda) = -\lambda(\mu+2),\ \xi(k) = -2(k+1)(\mu+2).$$

*Moreover, if $2n+1 \geq 5$, then we have*

$$X(\lambda) = 0,\ X(k) = 0,\ X(\mu) = 0$$

*for any $X \in \mathcal{D}$.*

**Lemma 3.3** ([34])**.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ $(n \geq 2)$ be an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution and $h \neq 0$. Then the Ricci operator $Q$ of $M^{2n+1}$ defined by $g(QX, Y) = S(X, Y)$ is given by*

$$Q = -2nI + 2n(k+1)\eta \otimes \xi + [\mu - 2(n-1)]h'. \tag{14}$$

*If both $k$ and $\mu$ are constant, then*

$$Q = -2nI + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{15}$$

*In both of the cases,*

$$S(X, \xi) = 2nk\eta(X). \tag{16}$$

Moreover, an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution satisfies [23]

$$(17)\quad (\nabla_X h')Y = -g(h'X + h'^2X, Y) - \eta(Y)(h'X + h'^2X) - (\mu+2)\eta(X)h'Y$$

for all smooth vectors fields $X, Y$.

From (13) it follows that

$$(18)\qquad R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X].$$

## 4. Second order parallel tensors on an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution

A covariant tensor $\alpha$ of second order is said to be a parallel tensor if $\nabla\alpha = 0$, where $\nabla$ denotes the operator of the covariant differentiation with respect to the metric tensor $g$. Let $\alpha$ be a $(0,2)$-type symmetric tensor field on an almost Kenmotsu manifold such that $\nabla\alpha = 0$, which implies

$$(19)\qquad \alpha(R(X,Y)Z,W) + \alpha(Z,R(X,Y)W) = 0$$

for all smooth vector fields $X, Y, Z, W$. Putting $X = Z = W = \xi$ in (19) implies

$$(20)\qquad \alpha(R(\xi,Y)\xi,\xi) + \alpha(\xi,R(\xi,Y)\xi) = 0.$$

Since $\alpha$ is symmetric, so

$$(21)\qquad \alpha(R(\xi,Y)\xi,\xi) = 0.$$

Using (18) in (21) yields

$$(22)\qquad k\eta(Y)\alpha(\xi,\xi) - k\alpha(Y,\xi) - \mu\alpha(hY,\xi) = 0.$$

Differentiating (22) along the arbitrary vector field $X$ we obtain

$$\begin{aligned} &kg(\nabla_X Y,\xi)\alpha(\xi,\xi) + kg(Y,\nabla_X\xi) + X(k)g(Y,\xi)\alpha(\xi,\xi)\\ &+ 2kg(Y,\xi)\nabla_X\xi - X(k)\alpha(Y,\xi) - k\alpha(\nabla_X Y,\xi) - k\alpha(Y,\nabla_X\xi)\\ (23)\qquad &- X(\mu)\alpha(hY,\xi) - \mu\alpha((\nabla_X h)Y,\xi) - \mu\alpha(hY,\nabla_X\xi). \end{aligned}$$

Replacing $Y$ by $hY$ in (23) and using the fact $h^2 = (k+1)\phi^2$ we have

$$(24)\qquad [k^2 + \mu^2(k+1)][\alpha(Y,\xi) - \eta(Y)\alpha(\xi,\xi)] = 0.$$

Therefore either

$$k^2 + \mu^2(k+1) = 0,$$

or,

$$\alpha(Y,\xi) - \eta(Y)\alpha(\xi,\xi) = 0.$$

Suppose

$$(25)\qquad \alpha(Y,\xi) - \eta(Y)\alpha(\xi,\xi) = 0$$

for all smooth vector fields $Y$. Noticing that $\alpha$ is parallel, then, by differentiating (25) along an arbitrary vector field $X$, we obtain

$$(26)\qquad \alpha(\nabla_X Y,\xi) + \alpha(X,\nabla_Y\xi) = g(\nabla_X Y,\xi)\alpha(\xi,\xi) + g(Y,\nabla_X\xi)\alpha(\xi,\xi)$$

$$+ 2g(Y,\xi)\alpha(\nabla_X \xi, \xi).$$

On the other hand, replacing $Y$ by $\nabla_X Y$ in (25) implies

$$\alpha(\nabla_X Y, \xi) - g(\nabla_X Y, \xi)\alpha(\xi, \xi) = 0 \tag{27}$$

for all smooth vector fields $Y$.

Substracting (27) from (26) we have

$$\alpha(X, \nabla_Y \xi) = g(Y, \nabla_X \xi)\alpha(\xi, \xi) + 2g(Y, \xi)\alpha(\nabla_X \xi, \xi). \tag{28}$$

Using (9) in (28) implies

$$\alpha(X, Y) - \alpha(Y, \phi hX) = g(X, Y)\alpha(\xi, \xi) - g(Y, \phi hX)\alpha(\xi, \xi). \tag{29}$$

Replacing $X$ by $\phi X$ in (29) yields

$$\alpha(\phi X, Y) - \alpha(Y, hX) = g(\phi X, Y)\alpha(\xi, \xi) - g(Y, hX)\alpha(\xi, \xi). \tag{30}$$

Again replacing $X$ by $hX$ and using the fact $h^2 = (k+1)\phi^2$ in (30) we obtain

$$(k+2)[\alpha(X, Y) - g(X, Y)\alpha(\xi, \xi)] = 0.$$

Therefore, either $k = -2$, or,

$$\alpha(X, Y) - g(X, Y)\alpha(\xi, \xi) = 0.$$

If $k = -2$, from Lemma 3.1, the curvature tensor satisfies the following:

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$
$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0$$

for any vector fields $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Making use of $k = -2$ in Proposition 3.2 of [23] we have

$$\mu + 2 = 0,$$

which implies $\mu = -2$. Also noticing $\mu = -2$, it follows from Lemma 3.1 that $K(X, \xi) = -4$ for any $X \in [\lambda]'$ and $K(X, \xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.1, we see that $K(X, Y) = -4$ for any $X, Y \in [\lambda]'$, $K(X, Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X, Y) = 0$ for any $X \in [\lambda]'$, $Y \in [-\lambda]'$. Also the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where $H$ is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in $M^{2n+1}$. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in $M^{2n+1}$. Then we can say that $M^{2n+1}$ is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.

On the other hand, if

$$\alpha(X, Y) - g(X, Y)\alpha(\xi, \xi) = 0,$$

that is,

$$\alpha(X, Y) = g(X, Y)\alpha(\xi, \xi),$$

then the second order parallel tensor is a constant multiple of the associate metric tensor of $M^{2n+1}$.

Hence we can state the following:

**Theorem 4.1.** *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ $(2n+1 \geq 5)$ *be an almost Kenmotsu manifold with generalized* $(k, \mu)'$*-nullity distribution and* $h' \neq 0$*. If the manifold admits a second order parallel symmetric tensor, then either the manifold is locally isometric to* $H^{n+1}(-4) \times \mathbb{R}^n$*, or the second order parallel tensor is a constant multiple of the associated metric tensor of* $M^{2n+1}$*, provided* $k^2+\mu^2(k+1) \neq 0$.

**Corollary 4.2.** *The above theorem is the generalization of Theorem* 4.2 *of Wang et al.* [29].

## 5. Ricci soliton on an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution

In this section we consider almost Kenmotsu manifolds with $\xi$ belonging to the generalized $(k, \mu)'$-nullity distribution. Suppose that the manifold $M^{2n+1}$ admits a Ricci soliton. Then we have

$$(\pounds_V g + 2S + 2\gamma g)(X, Y) = 0,$$

which implies

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2S(X, Y) + 2\gamma g(X, Y) = 0 \tag{31}$$

for all smooth vector fields $X, Y$.

Using (9) in (31) yields

$$(1+\gamma)g(X, Y) - \eta(X)\eta(Y) - g(X, \phi hY) + S(X, Y) = 0. \tag{32}$$

Putting $X = Y = \xi$ in (32) we have

$$\gamma = -2nk. \tag{33}$$

Since $\gamma$ is constant, so from (33) we have $k$ is also constant. Therefore, $\xi(k) = 0$. Making use of $\xi(k) = 0$ in Proposition 3.2 of [23] we obtain

$$-2(k+1)(\mu+2) = 0,$$

which implies either $k+1 = 0$, or $\mu + 2 = 0$. If $k+1 = 0$, then $h' = 0$, which is a contradiction. Hence $\mu = -2$, a constant.

Thus we have the following:

**Theorem 5.1.** *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost Kenmotsu manifold with* $\xi$ *belonging to the generalized* $(k, \mu)'$*-nullity distribution. If the manifold admits a Ricci soliton, then the manifold reduces to an almost Kenmotsu manifold with* $\xi$ *belonging to the* $(k, \mu)'$*-nullity distribution.*

Since we know that for an almost Kenmotsu manifold $k < -1$, so from (33) it is clear that $\gamma$ is always positive.

Thus we can state the following:

**Theorem 5.2.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $\xi$ belonging to the generalized $(k, \mu)'$-nullity distribution. If the manifold admits a Ricci soliton, then the Ricci soliton is expanding.*

## 6. Generalized Ricci solitons

In this section we characterize an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution and $(k, \mu)'$-nullity distribution admitting a generalized Ricci soliton. First we prove the following lemma:

**Lemma 6.1.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution or $(k, \mu)'$-nullity distribution. Then*

$$\nabla_\xi grad\ f = (\beta + 2c_2 nk)\xi - c_1 \xi(f) grad\ f.$$

*Proof.* From (6) we get,

$$(Hess\ f)(\xi, Y) = -c_1 \xi(f) g(grad\ f, Y) + \beta\eta(Y) + c_2 S(\xi, Y). \tag{34}$$

Making use of (16) in (34) yields

$$(Hess\ f)(\xi, Y) = -c_1 \xi(f) g(grad\ f, Y) + (2nkc_2 + \beta)\eta(Y). \tag{35}$$

Hence the Lemma follows from (35) and the definition (3) of the Hessian. □

If $X = grad\ f$, then the generalized Ricci soliton equation is given by

$$Hess\ f = -c_1 df \odot df + c_2 S + \beta g. \tag{36}$$

The above equation can be written as

$$\nabla_Y grad\ f = -c_1 Y(f) grad\ f + c_2 QY + \beta Y. \tag{37}$$

Using (37) we obtain

$$\begin{aligned} R(X, Y) grad\ f = &- c_1 Y(f) \nabla_X grad\ f + c_1 X(f) \nabla_Y grad\ f \\ &+ c_2[(\nabla_X Q)Y - (\nabla_Y Q)X] = 0. \end{aligned} \tag{38}$$

Substituting $X = \xi$ in (38) and then taking inner product with $Y$ yields

$$\begin{aligned} g(R(\xi, Y) grad\ f, \xi) = &- c_1 Y(f) g(\nabla_\xi grad\ f, \xi) + c_1 \xi(f) g(\nabla_Y grad\ f, \xi) \\ &+ c_2[g((\nabla_\xi Q)Y, \xi) - g((\nabla_Y Q)\xi, \xi)] = 0. \end{aligned} \tag{39}$$

From (18), it follows that

$$\begin{aligned} R(\xi, Y) grad\ f = k[&g(Y, grad\ f)\xi - g(grad\ f, \xi)Y] \\ &+ \mu g(h'Y, grad\ f)\xi. \end{aligned} \tag{40}$$

Taking the inner product with $\xi$ of (40) yields

$$\begin{aligned} g(R(\xi, Y) grad\ f, \xi) = k[&g(Y, grad\ f) - g(grad\ f, \xi)\eta(Y)] \\ &+ \mu g(h'Y, grad\ f). \end{aligned} \tag{41}$$

Now from Lemma 6.1 it follows that

$$g(\nabla_\xi grad\ f, \xi) = (\beta + 2c_2 nk) - c_1 \xi(f) g(grad\ f, \xi). \tag{42}$$

Taking the covariant differentiation of (14) we get

$$(\nabla_X Q)Y = 2n(k+1)[((\nabla_X\eta)Y)\xi + \eta(Y)\nabla_X\xi] + [\mu - 2(n-1)](\nabla_X h')Y + 2nX(k)\eta(Y)\xi + X(\mu)h'Y. \tag{43}$$

Making use of (7), (9) and (17) in (43) implies

$$\begin{aligned}(\nabla_X Q)Y = {}& 2n(k+1)[g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + g(X,h'Y)\xi \\ & + \eta(Y)X + \eta(Y)h'X] + [\mu - 2(n-1)][g(h'Y,X)\xi + g(h'^2X,Y)\xi \\ & + \eta(Y)h'^2Y(\mu+2)\eta(X)h'Y] + 2nX(k)\eta(Y)\xi + X(\mu)h'Y.\end{aligned} \tag{44}$$

Putting $X = \xi$ in (44) and then using (8) yields

$$(\nabla_\xi Q)Y = [\mu - 2(n-1)](\mu+2)h'X + 2n\eta(Y)\xi(k)\xi + \xi(\mu)h'Y. \tag{45}$$

Taking the inner product with $\xi$ of the above equation yields

$$g((\nabla_\xi Q)Y, \xi) = 2n\eta(Y)\xi(k). \tag{46}$$

Similarly, from (44) we obtain

$$g((\nabla_Y Q)\xi, \xi) = 2nY(k). \tag{47}$$

Finally using (41), (42), (46) and (47) in (39) we infer

$$\begin{aligned}& k[g(Y, grad\ f) - g(grad\ f, \xi)\eta(Y)] + \mu g(h'Y, grad\ f) \\ = {}& -c_1Y(f)[c_1\xi(f)g(grad\ f,\xi) + 2nkc_2 + \beta] \\ & + c_1\xi(f)[-c_1Y(f)g(grad\ f,\xi) \\ & + (2nkc_2+\beta)\eta(Y)] + c_2[2n\xi(k)\eta(Y) - 2nY(k)] \\ = {}& c_1(2nkc_2+\beta)\eta(Y)\xi(f) - c_1(2nkc_2+\beta)Y(f) \\ & + 2nc_2[\eta(Y)\xi(k) - Y(k)].\end{aligned} \tag{48}$$

Let $Y \in [\lambda]'$. Then from (48) we have

$$[k + \mu\lambda + c_1(2nkc_2+\beta)]g(Y, grad\ f) = -2nc_2Y(k). \tag{49}$$

Let $Y \in [-\lambda]'$. Then from (48), we get

$$[k - \mu\lambda + c_1(2nkc_2+\beta)]g(Y, grad\ f) = -2nc_2Y(k). \tag{50}$$

Applying Lemma 3.1 in (49) yields

$$[k + \mu\lambda + c_1(2nkc_2+\beta)]g(Y, grad\ f) = 0. \tag{51}$$

Similarly, applying Lemma 3.1 in (50) implies

$$[k - \mu\lambda + c_1(2nkc_2+\beta)]g(Y, grad\ f) = 0. \tag{52}$$

Substracting (52) from (51) we have

$$\lambda\mu\ grad\ f = 0, \tag{53}$$

which implies either $grad\ f = 0$, or $\mu\lambda = 0$.

If $grad\ f = 0$, then $f$ is constant.

On the other hand, if $\mu\lambda = 0$, then either $\lambda = 0$ or $\mu = 0$. If $\lambda = 0$, then $k = -1$, which is a contradiction as $h' \neq 0$. Therefore $\mu = 0$, that is, the manifolds turns out to be a generalized $(k, 0)'$-almost Kenmotsu manifolds.

Thus we have the following:

**Theorem 6.2.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution with $h' \neq 0$. If $M$ satisfies the generalized Ricci soliton equation, then either $f$ is constant, or the manifold turns out to be a generalized $(k, 0)'$-almost Kenmotsu manifold.*

Now we characterize almost Kenmotsu manifolds with $(k, \mu)'$-nullity distribution with $h' \neq 0$ admitting generalized Ricci soliton equation.

Taking the covariant differentiation of (15) yields

$$(\nabla_X Q)Y = 2n(k+1)[((\nabla_X \eta)Y)\xi + \eta(Y)\nabla_X \xi] - 2n(\nabla_X h')Y. \tag{54}$$

Making use of (9) and (17) in (54) implies

$$\begin{aligned}(\nabla_X Q)Y = {} & 2n(k+1)[g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + g(X, h'Y)\xi \\ & + \eta(Y)X + \eta(Y)h'X] - 2n[g(h'Y, X)\xi + g(h'^2X, Y)\xi \\ & + \eta(Y)h'^2Y(\mu+2)\eta(X)h'Y].\end{aligned} \tag{55}$$

Putting $X = \xi$ in (55) and then using (8) yields

$$(\nabla_\xi Q)Y = -2n(\mu+2)h'X + 2n\eta(Y)\xi(k)\xi. \tag{56}$$

Taking the inner product with $\xi$ of the above equation, we have

$$g((\nabla_\xi Q)Y, \xi) = 0. \tag{57}$$

Similarly, from (55) we obtain

$$g((\nabla_Y Q)\xi, \xi) = 0. \tag{58}$$

Using (41), (42), (57) and (58) in (39) yields

$$\begin{aligned} & k[g(Y, grad\ f) - g(grad\ f, \xi)\eta(Y)] + \mu g(h'Y, grad\ f) \\ = {} & -c_1Y(f)[c_1\xi(f)g(grad\ f, \xi) + 2nkc_2 + \beta] \\ & + c_1\xi(f)[-c_1Y(f)g(grad\ f, \xi) \\ & + (2nkc_2 + \beta)\eta(Y)] + c_2[2n\xi(k)\eta(Y) - 2nY(k)] \\ = {} & c_1(2nkc_2 + \beta)\eta(Y)\xi(f) - c_1(2nkc_2 + \beta)Y(f).\end{aligned} \tag{59}$$

Let $Y \in [\lambda]'$. Then from (59) we have

$$[k + \mu\lambda + c_1(2nkc_2 + \beta)]g(Y, grad\ f) = 0. \tag{60}$$

Let $Y \in [-\lambda]'$. Then from (59), we get

$$[k - \mu\lambda + c_1(2nkc_2 + \beta)]g(Y, grad\ f) = 0. \tag{61}$$

Subtracting (61) from (60), we have

$$\lambda\mu\ grad\ f = 0, \tag{62}$$

which implies either $grad\ f = 0$, or $\mu\lambda = 0$.

If $grad\ f = 0$, then $f$ is constant.

On the other hand, if $\mu\lambda = 0$, then either $\lambda = 0$ or $\mu = 0$. If $\lambda = 0$, then $k = -1$, which is a contradiction as $h' \neq 0$. Therefore $\mu = 0$, that is, the manifold turns out to be a $(k, 0)'$-almost Kenmotsu manifold.

Thus we have the following:

**Theorem 6.3.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $(k, \mu)'$-nullity distribution with $h' \neq 0$. If $M$ satisfies the generalized Ricci soliton equation, then either $f$ is constant, or the manifold turns out to be a $(k, 0)'$-almost Kenmotsu manifold.*

If we consider $\mu \neq 0$, then both of the equations (53) and (62) hold good and we have, $grad\ f = 0$, since $\lambda = 0$ implies $k = -1$, a contradiction.

Now $grad\ f = 0$ implies $f$ is constant and from (6) it follows that the manifold is an Einstein one. This leads to the following:

**Theorem 6.4.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ $(n \geq 2)$ be an almost Kenmotsu manifold with $(k, \mu)'$ or generalized $(k, \mu)'$-nullity distribution with $h' \neq 0$. If $M$ satisfies the generalized Ricci soliton equation, then the manifold is an Einstein one, provided $c_2 \neq 0$.*

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