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SOME RESULTS IN η-RICCI SOLITON AND GRADIENT ρ-EINSTEIN SOLITON IN A COMPLETE RIEMANNIAN MANIFOLD

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Abstract. The main purpose of the paper is to prove that if a compact Riemannian manifold admits a gradient ρ-Einstein soliton such that the gradient Einstein potential is a non-trivial conformal vector field, then the manifold is isometric to the Euclidean sphere. We have showed that a Riemannian manifold satisfying gradient ρ-Einstein soliton with convex Einstein potential possesses non-negative scalar curvature. We have also deduced a sufficient condition for a Riemannian manifold to be compact which satisfies almost η-Ricci soliton.

1. Introduction

In 1982, Hamilton [14] introduced the notion of Ricci flow in a Riemannian manifold (M, g_0) to find the various geometric and topological structures of Riemannian manifolds. The Ricci flow is defined by an evolution equation for metrics on (M, g_0):

$$\frac{\partial}{\partial_t} g(t) = -2Ric, \quad g(0) = g_0.$$

A Ricci soliton on a Riemannian manifold (M, g) is a generalization of Einstein metric and is defined as

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, e, \tag{1}$$

where X is a smooth vector field on M, $\mathcal{L}$ denotes the Lie-derivative operator and $\lambda \in \mathbb{R}$. Ricci almost solitons, which were introduced by Pigola et al. [17], correspond to self-similar solutions of the Ricci-Bourguignon flow, as it was showed by M. Brozos-Vazquez et al. in [6]. Moreover, it is well known that they can be seen as conformal solution of the Ricci flow. Some examples and rigidity results were obtained in several papers in the last 7 years, as for instance, by

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Pigola et al. [17], Barros et al. in [3,4], Calvino-Louzao et al. in [7], R. Sharma in [18], Catino et al. [10] and many other works. Ricci soliton is called shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. The vector field X is called the potential vector field of the Ricci soliton. If X is either Killing or vanishing vector field, then Ricci soliton is called trivial Ricci soliton and (1) reduces to an Einstein metric. If X becomes the gradient of a smooth function $f \in C^\infty(M)$, the ring of smooth functions on M, then the Ricci soliton is called gradient Ricci soliton and (1) reduces to the form

$$Ric + \nabla^2 f = \lambda g, \tag{2}$$

where $\nabla^2 f$ is the Hessian of f. Perelman [15] showed that Ricci soliton on any complete manifold is always a gradient Ricci soliton. If we replace the constant λ in (1) with a smooth function $\lambda \in C^\infty(M)$, called soliton function, then we say that (M, g) is an almost Ricci soliton, see ([3, 4, 17]).

Almost gradient Ricci soliton motivated Catino [8] to introduce a new class of Riemannian metrics which is a natural generalization of Einstein metrics. In particular, a Riemannian manifold (M^n, g), $n \geq 2$, is called a generalized quasi-Einstein manifold if there are smooth functions f, λ and μ on M such that

$$Ric + \nabla^2 f = \lambda g + \mu df \otimes df.$$

Cho and Kimura [12] further generalized the notion of Ricci soliton and developed the concept of η-Ricci soliton. If a Riemannian manifold M satisfies

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g + \mu \eta \otimes \eta$$

for some constant λ and μ, then M is said to admit an η-Ricci soliton with soliton vector field X. A further generalization is the notion of almost η-Ricci soliton defined by Blaga [5].

Definition ([5]). A complete Riemannian manifold (M, g) is said to satisfy almost η-Ricci soliton if there exists a smooth vector field $X \in \mathfrak{X}(M)$, the algebra of smooth vector fields on M, such that

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g + \mu \eta \otimes \eta, \tag{3}$$

where λ and μ are smooth functions on M and η is an 1-form on M.

If X is the gradient of $f \in C^\infty(M)$, then (M, g) is called a gradient almost η-Ricci soliton. Hence (3) reduces to the form

$$Ric + \nabla^2 f = \lambda g + \mu \eta \otimes \eta. \tag{4}$$

Instead of Ricci flow, Catino and Mazzieri [11] considered the following gradient flow

$$\frac{\partial}{\partial_t} g(t) = -2(Ric - \frac{1}{2}Rg), \tag{5}$$

and introduced the concept of gradient Einstein soliton in a Riemannian manifold, where R is the scalar curvature of the manifold.

Definition ([11])**.** A Riemannian manifold (M, g) of dimension n is said to be the gradient Einstein Ricci soliton if

$$Ric - \frac{1}{2}Rg + \nabla^2 f = \lambda g$$

for some function $f \in C^\infty(M)$ and some constant $\lambda \in \mathbb{R}$.

A more general type gradient Einstein soliton has been deduced by considering the following Ricci-Bourguignon flows [9]:

$$\frac{\partial}{\partial_t} g(t) = -2(Ric - \rho Rg),$$

where ρ is a real non-zero constant.

Definition ([11])**.** A Riemannian manifold (M, g) of dimension $n \geq 3$ is said to be the gradient ρ-Einstein Ricci soliton if

$$Ric + \nabla^2 f = \lambda g + \rho Rg, \quad \rho \in \mathbb{R},\ \rho \neq 0$$

for some function $f \in C^\infty(M)$ and some constant $\lambda \in \mathbb{R}$. The function f is called Einstein potential. The gradient ρ-Einstein soliton is called expanding if $\lambda < 0$, steady if $\lambda = 0$ and shrinking if $\lambda > 0$.

The paper is arranged as follows: Section 2 discusses some basic concepts of Riemannian manifold and some definitions, which are needed for the rest of the paper. Section 3 deals with the study of almost η-Ricci soliton in a complete Riemannian manifold and provides a proof of the statement saying that in a compact manifold the potential of such soliton turns into the Hodge-de Rham potential, up to a constant. In this section, we have also deduced a sufficient condition for a Riemannian manifold admitting an almost η-Ricci soliton structure to be compact. In the last section, as the main result of the paper, we will prove that a compact Riemannian manifold satisfying a gradient ρ-Einstein soliton with gradient of Einstein potential as a conformal vector field, is isometric to the Euclidean sphere. In this section, we have also studied some properties of gradient ρ-Einstein soliton in a complete Riemannian manifold. Among others it will be proved that if (M, g) is a compact gradient ρ-Einstein soliton with ρ as non-positive real number and gradient of the Einstein potential is a conformal vector field, then such soliton can not be expanding.

2. Preliminaries

Let M be a complete Riemannian manifold of dimension n endowed with some positive definite metric g unless otherwise stated. In this section, we have discussed some rudimentary facts of M (for reference see [16]). The tangent space at the point $p \in M$ is denoted by T_pM. The geodesic with initial point p and final point q is denoted by γ_{pq}. A smooth section of the tangent bundle TM

is called a smooth vector field. The gradient of a smooth function $u : M \to \mathbb{R}$ at the point $p \in M$ is defined by $\nabla u(p) = g^{ij} \frac{\partial u}{\partial x^j} \frac{\partial}{\partial x^i} \mid_p$. It is the unique vector field such that any smooth vector field X in M satisfies $g(\nabla u, X) = X(u)$. The Hessian $Hess(u)$ is the symmetric $(0,2)$-tensor field and is defined by $\nabla^2 u(X,Y) = Hess(u)(X,Y) = g(\nabla_X \nabla u, Y)$ for all smooth vector fields X, Y of M. In local coordinates this can be written as

$$(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u,$$

where Γ_{ij}^k is the Christoffel symbol of g. For any vector field $X \in \mathfrak{X}(M)$ and a covariant tensor field ω of order r on M, the Lie derivative of ω with respect to X is defined by

$$(\mathcal{L}_X \omega)(X_1, \ldots, X_r) = X(\omega(X_1, \ldots, X_r)) - \sum_{i=1}^{r} \omega(X_1, \ldots, [X, X_i], \ldots, X_n),$$

where $X_i \in \chi(M)$ for $i = 1, \ldots, r$. In particular, when $\omega = g$, then

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \text{ for } Y, Z \in \mathfrak{X}(M).$$

Given a vector field X, the divergence of X is defined by

$$div(X) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \sqrt{g} X^j,$$

where $g = \det(g_{ij})$ and $X = X^j \frac{\partial}{\partial x^j}$. The Laplacian of u is defined by $\Delta u = div(\nabla u)$.

3. Some results of almost η-Ricci soliton in a compact Riemannian manifold

Let M be a compact orientable Riemannian manifold and $X \in \mathfrak{X}(M)$ a vector field on M. Then Hodge-de Rham decomposition theorem [2] implies that X can be expressed as

$$X = \nabla h + Y,$$

where $h \in C^\infty(M)$ and $div(Y) = 0$. In particular, the function h is called the Hodge-de Rham potential [4]. Now, we may state our first result as follows.

Theorem 3.1. *Let (M^n, g, X, λ) be a compact gradient almost η-Ricci soliton. If M is also a gradient almost η-Ricci soliton with potential function f, then, up to a constant, f is equal to the Hodge-de Rham potential.*

Proof. Since (M, g, X, λ) is a compact almost η-Ricci soliton, so taking trace of (3), we get

$$R + div(X) = \lambda n + tr(\mu\eta \otimes \eta).$$

Now Hodge-de Rham decomposition implies that $div(X) = \Delta h$, hence from the above equation, we obtain

$$R = \lambda n - \Delta h + tr(\mu\eta \otimes \eta).$$

Again since M is gradient almost η-Ricci soliton with Perelman potential f, taking trace of (4), we have

$$R = \lambda n - \Delta f + tr(\mu\eta \otimes \eta).$$

Combining the last two equations we get $\Delta(f - h) = 0$. Hence $f - h$ is a harmonic function in M. Since M is compact, we have $f = h + c$ for some constant c. This finishes the proof of the theorem. □

Theorem 3.2. *Let (M, g) be a complete Riemannian manifold satisfying*

$$Ric + \frac{1}{2}\mathcal{L}_g \geq \lambda g + \mu\eta \otimes \eta, \tag{6}$$

where X is a smooth vector field, μ and λ are smooth functions and η is an 1-*form. Then M is compact if $\|X\|$ is bounded and one of the following conditions holds*:

(i) $\lambda \geq 0$ *and* $\mu > c > 0$,
(ii) $\lambda > c > 0$ *and* $\mu \geq 0$

for some constant c.

Proof. Let $p \in M$ be a fixed point and $\gamma : [0, \infty) \to M$ be a geodesic ray such that $\gamma(0) = p$. Then along γ we calculate

$$\mathcal{L}_X g(\gamma', \gamma') = 2g(\nabla_{\gamma'} X, \gamma') = 2\frac{d}{dt}[g(X, \gamma')].$$

This data jointly with (6) yields

$$\begin{aligned}
\int_0^T Ric(\gamma', \gamma')dt \geq& \int_0^T \lambda(\gamma(t))g(\gamma', \gamma')dt - \int_0^T \frac{d}{dt}[g(X, \gamma')]dt \\
&+ \int_0^T \mu(\gamma(t))(\eta \otimes \eta)(\gamma', \gamma')dt \\
=& \int_0^T \lambda(\gamma(t))dt + g(X_p, \gamma'(0)) - g(X_{\gamma(T)}, \gamma'(T)) \\
&+ \int_0^T \mu(\gamma(t))\eta^2(\gamma')dt \\
\geq& \int_0^T \lambda(\gamma(t))dt + g(X_p, \gamma'(0)) - \|X_{\gamma(T)}\| \\
&+ \int_0^T \mu(\gamma(t))\eta^2(\gamma')dt.
\end{aligned}$$

The last inequality follows by Cauchy-Schwarz inequality. If any one of the conditions (i) or (ii) holds, then above inequality implies

$$\int_0^\infty Ric(\gamma', \gamma')dt = \infty.$$

Hence Ambrose's compactness theorem [1] implies that M is compact, which finishes the proof of the theorem. $\square$

4. Gradient ρ-Einstein soliton in a compact Riemannian manifold

We start this section recalling a sphere theorem obtained by Yano in [19], which is going to use the proof of our next result. More preciously, Yano proved the following result. Throughout this section M is a complete Riemannian manifold with dimension $n \geq 2$.

Theorem 4.1 ([19, Yano]). *Let (M^n, g) be a compact Riemannian manifold with constant scalar curvature. Suppose that M admits a non-trivial conformal vector field X. If $\mathcal{L}_X Ric = \alpha g$ for some $\alpha \in C^\infty(M)$, then M is isometric to the Euclidean sphere $\mathbb{S}^n$.*

Let (M, g) be a gradient ρ-Einstein soliton. Then

$$Ric + \nabla^2 f = \rho R g + \lambda g.$$

If ∇f is a conformal vector field, then $\nabla^2 f = \psi g$ for some $\psi \in C^\infty(M)$. Therefore above equation reduces to the form

$$Ric = (\rho R + \lambda - \psi) g. \tag{7}$$

Hence Ricci curvature depends only on the points of M. Then it follows from Schur's lemma that R is constant. Again by taking $X = \nabla f$, we have

$$\mathcal{L}_X Ric = (\rho R + \lambda - \psi)\mathcal{L}_X g = (\rho R + \lambda - \psi)\psi g.$$

Therefore, it follows by Theorem 4.1 the following result.

Theorem 4.2. *Let (M, g) be a compact gradient ρ-Einstein soliton with Einstein potential f. If ∇f is a non-trivial conformal vector field, then M is isometric to the Euclidean sphere $\mathbb{S}^n$.*

Theorem 4.3 ([19]). *If M is compact with constant scalar curvature and admits a non-trivial conformal vector field X: $\mathcal{L}_X g = 2\psi g$, $\psi \neq 0$, then*

$$\int_M \psi dV = 0.$$

Taking the trace in (7), we get

$$R = n(\rho R + \lambda - \psi),$$

which implies that

$$\int_M (1 - n\rho) R = n \int_M (\lambda - \psi).$$

If X is a conformal vector field and M has constant scalar curvature, then applying Theorem 4.3 we get

$$R \int_M (1 - n\rho) = n \int_M \lambda. \tag{8}$$

Now if $\lambda < 0$, then the above equation becomes

$$R \int_M (1 - n\rho) < 0.$$

If M is compact, then Theorem 4.2 implies that M is isometric to $\mathbb{S}^n$. Since isometry preserves scalar curvature, so $R > 0$. Therefore, the above equation guarantees

$$Vol(M) < n \int_M \rho. \tag{9}$$

This computations allow us to infer the following result.

Theorem 4.4. *Let (M, g) be a compact gradient ρ-Einstein soliton with Einstein potential f and $\rho \leq 0$. If ∇f is a conformal vector field, then M is either shrinking or steady gradient ρ-Einstein soliton.*

Lemma 4.5 ([11])**.** *Let (M, g) be a gradient ρ-Einstein Ricci soliton with Einstein potential f. Then we have*

$$\Delta f = -(1 - n\rho)R + n\lambda. \tag{10}$$

The following results are about the effect of scalar curvature on Einstein potential function in ρ-Einstein Ricci soliton.

Proposition 4.6. *Suppose (M, g) is an expanding or steady gradient ρ-Einstein Ricci soliton with Einstein potential f and $n\rho > 1$. If f is a convex function, then M has non-negative scalar curvature.*

Proof. The convexity of f implies that f is subharmonic [13], i.e., $\Delta f \geq 0$. Therefore (10) implies that

$$(1 - n\rho)R - n\lambda \leq 0.$$

Now take $1 - n\rho = -h$, where $h > 0$ is a real constant. Then we obtain

$$R \geq -\frac{n\lambda}{h}. \tag{11}$$

Since M is expanding or steady, so $\lambda \leq 0$. Thus we can conclude from (11) that $R \geq 0$. □

The following can be easily derived from (10):

Proposition 4.7. *Suppose (M, g) is a steady gradient ρ-Einstein Ricci soliton with Einstein potential f and $n\rho > 1$. If f is a harmonic function, then the scalar curvature of M vanishes.*

Integrating (8) on M, we get

$$R(1 - n\rho)Vol(M) = n\lambda Vol(M),$$

which yields

$$R = \frac{n\lambda}{1 - n\rho}.$$

If $R > 0$, then $n\lambda > 1 - n\rho$, i.e., $\rho > \frac{1}{n}(1 - n\lambda)$. Thus Theorem 4.2 implies the following:

Proposition 4.8. *Let (M, g) be a compact gradient ρ-Einstein soliton with Einstein potential f. If ∇f is a non-trivial conformal vector field, then ρ satisfies*

$$\rho > \frac{1}{n}(1 - n\lambda).$$

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