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INEQUALITIES AND COMPLETE MONOTONICITY FOR THE GAMMA AND RELATED FUNCTIONS

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Abstract. It is well-known that if $\phi'' > 0$ for all x, $\phi(0) = 0$, and ϕ/x is interpreted as $\phi'(0)$ for $x = 0$, then ϕ/x increases for all x. This has been extended in [Complete monotonicity and logarithmically complete monotonicity properties for the gamma and psi functions, J. Math. Anal. Appl. **336** (2007), 812–822]. In this paper, we extend the above result to the very general cases, and then use it to prove some (logarithmically) completely monotonic functions related to the gamma function. We also establish some inequalities for the gamma function and generalize some known results.

1. Introduction

A function f is said to be completely monotonic on an open interval $(a,\, b)$ $(-\infty \leq a < b \leq \infty)$ if

$$(1)\qquad (-1)^n f^{(n)}(x) \geq 0 \quad \big(a < x < b;\ n \in \mathbb{N}_0\big).$$

Here and throughout, we denote $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ by sets of complex numbers, real numbers, positive integers, and non-positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If, in addition, f is continuous at $x = a$, then it is called completely monotonic on $[a,\, b)$, with similar definitions for $(a,\, b]$ and $[a,\, b]$.

Dubourdieu [20, p. 98] pointed out that if a non-constant function f is completely monotonic over (a, ∞), then the strict inequality in (1) holds true. It is known (Bernstein's Theorem) that f is completely monotonic on $[0,\, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where μ is a bounded and non-decreasing measure and the integral converges for $0 \leq x < \infty$ (see [54, pp. 160–163]). This means that a completely monotonic function f on $[0,\, \infty)$ is a Laplace transform with respect to the measure μ. The

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main properties of completely monotonic functions are given in [54, Chapter IV]. For an extensive list of references on completely monotonic functions, we also refer to [11].

Recall [27] that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(x)]^{(k)} \geq 0 \quad (x \in I;\ k \in \mathbb{N}). \tag{2}$$

A logarithmically completely monotonic function f on I must be completely monotonic on I (see, e.g., [13–15, 43, 44]). This result, in fact, can be derived (see, e.g., [19]) from the following Faá di Bruno's formula (see, e.g., [41, p. 5]):

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[g(h(x))] = \sum_{\substack{1\leq i\leq n, i_k\geq 0 \\ \left(\sum_{k=1}^{n} i_k=i;\ \sum_{k=1}^{n} k i_k=n\right)}} \left(\frac{n!}{\prod_{k=1}^{n} i_k!}\right) g^{(i)}(h(x)) \prod_{k=1}^{n}\left(\frac{h^{(k)}(x)}{k!}\right)^{i_k}. \tag{3}$$

Recall that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and satisfies the following inequality:

$$f^{(k)}(x) \geq 0 \qquad \big(x \in I;\ k \in \mathbb{N}_0\big).$$

By Faá di Bruno's formula (3), Chen and Srivastava [19, Theorem 1] proved that if the function f is absolutely monotonic on $\mathbb{R}$ and the function g is completely monotonic on I, then their composite function $(f \circ g)(x) = f(g(x))$ is completely monotonic on I. For example, let $f(x) = e^x$ and the function g is completely monotonic on I, then $e^{g(x)}$ is also completely monotonic on I. If we let here $f(x) = e^x$ and $g(x) = \ln F(x)$, we see that the complete monotonicity of $\ln F(x)$ implies the complete monotonicity of $e^{\ln F(x)} = F(x)$.

Recall that a function f is said to be a Bernstein function on an interval I if $f > 0$ and f' is completely monotonic on I. By Faá di Bruno's Formula (3), Chen *et al.* [18, Theorem 3] proved that if f is a Bernstein function on an interval I, then $1/f$ is logarithmically completely monotonic on I.

In [14, Theorem 1.1] and [26, 45] it is pointed out that the logarithmically completely monotonic functions on $(0, \infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [30, Theorem 4.4] and that the set of all Stieltjes transforms is a subset of the set of logarithmically completely monotonic functions on $(0, \infty)$.

The Euler gamma function Γ is usually defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,\mathrm{d}t \quad (x > 0).$$

The psi (or digamma) function $\psi(x)$ is defined by $\psi(x) := \frac{\mathrm{d}}{\mathrm{d}x} \ln \Gamma(x)$ and the polygamma functions $\psi^{(m)}(x)$ are defined by $\psi^{(m)}(x) := \frac{\mathrm{d}^m}{\mathrm{d}x^m}\,\psi(x)$. Among diverse integral representations of $\psi(x)$ and $\psi^{(m)}(x)$ (see, e.g., [36, p. 16]), we choose to recall the following representations:

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\,\mathrm{d}t \quad (x > 0) \tag{4}$$

and

$$\psi^{(m)}(x) = (-1)^{m+1} \int_0^\infty \frac{t^m}{1 - e^{-t}} e^{-xt}\,\mathrm{d}t \qquad (x > 0;\ m \in \mathbb{N}), \tag{5}$$

where γ denotes the Euler-Mascheroni constant (see, e.g., [52, Section 1.2]) defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.57721\,56649\,01532\,86060\,6512\ldots. \tag{6}$$

We recall the well-known *Gauss's summation theorem* (see, e.g., [52, Section 1.5]):

$$\begin{gathered} {}_2F_1\,(a,\, b;\, c;\, 1) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{n!\,(c)_n} = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \\ \big(\Re(c-a-b) > 0;\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-\big), \end{gathered} \tag{7}$$

where $(\alpha)_n$ denotes the Pochhammer symbol defined (for $\alpha \in \mathbb{C}$) by

$$\begin{aligned} (\alpha)_n : &= \begin{cases} 1 & (n = 0) \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (n \in \mathbb{N}) \end{cases} \\ &= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad \big(\alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-\big). \end{aligned} \tag{8}$$

An interesting special case of (7) when the numerator parameter a or b is a nonpositive integer $-n$ gives

$$ {}_2F_1\,(-n,\, b;\, c;\, 1) = \frac{(c-b)_n}{(c)_n} \quad (n \in \mathbb{N}_0;\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{9}$$

which is, in fact, equivalent to *Vandermonde's convolution theorem*:

$$\begin{gathered} \sum_{k=0}^{n} \binom{\lambda}{k} \binom{\mu}{n-k} = \binom{\lambda+\mu}{n} = \sum_{k=0}^{n} \binom{\lambda}{n-k} \binom{\mu}{k} \\ (n \in \mathbb{N}_0;\ \lambda,\, \mu \in \mathbb{C}). \end{gathered} \tag{10}$$

There exists a very extensive literature on the gamma function, the psi and polygamma functions. In particular, a variety of inequalities, monotonicity and complete monotonicity properties for these functions and their related functions have been investigated by many authors. For example, see the works [3,5,6,42] and the references therein.

In this paper, we will extend Lemma 1 in [17] to a general case and establish Theorem 2.1 in the following section. We will show usefulness of our main result, Theorem 2.1, by applying it to prove some (logarithmically) complete monotonicity of functions related to the gamma function. We will also establish some inequalities for the gamma function and generalize some known results.

2. Main results

In [29, p. 99] it was stated that if $\phi'' > 0$ for all x, $\phi(0) = 0$, and ϕ/x is interpreted as $\phi'(0)$ for $x = 0$, then ϕ/x increases for all x. The first author [17, Lemma 1] provided an extension of this result and proved the case $m = 1$ in (i) of Theorem 2.1. We give further extensions of the cited results asserted in the following theorem.

Theorem 2.1. *Let the function ϕ have derivatives of all orders on $(-\infty, \infty)$ and $\phi^{(k-1)}(0) = 0$ for $1 \le k \le m$ $(m \in \mathbb{N})$. Define the function f_m by*

$$f_m(x) = \begin{cases} \dfrac{\phi(x)}{x^m} & (x \neq 0) \\ \dfrac{\phi^{(m)}(0)}{m!} & (x = 0). \end{cases}$$

Then the following properties hold true:

(i)

$$f_m^{(n)}(x) = \begin{cases} \dfrac{1}{x^{n+m}} \Phi_{m,n}(x) & (x \neq 0) \\ \dfrac{n!}{(m+n)!} \phi^{(m+n)}(0) & (x = 0), \end{cases} \tag{11}$$

where

$$\Phi_{m,n}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k (k+m-1)!}{(m-1)!} x^{n-k} \phi^{(n-k)}(x).$$

Moreover,

$$\Phi_{m,n}^{(m)}(x) = x^n \phi^{(n+m)}(x). \tag{12}$$

(ii) *For all $x \in (-\infty, \infty)$ and a given $n \in \mathbb{N}_0$,*

$$\phi^{(n+m)}(x) \ge 0 \Longrightarrow f_m^{(n)}(x) \ge 0 \tag{13}$$

and

$$\phi^{(n+m)}(x) \le 0 \Longrightarrow f_m^{(n)}(x) \le 0. \tag{14}$$

Proof. By applying Leibniz rule to the definition of $f_m(x)$, it is easy to see the first identity in (11).

By recalling the following known identity (see, e.g., [28, p. 26])

$$\sum_{k=0}^{n} \frac{(-1)^k}{m+k} \binom{n}{k} = \frac{n!}{m(m+1)\cdots(m+n)}$$

and using L'Hôspital's rule, we have

$$f_m^{(n)}(0) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k (k+m-1)!}{(m-1)!} \lim_{x\to 0} \frac{\phi^{(n-k)}(x)}{x^{m+k}}$$
$$= \frac{\phi^{(n+m)}(0)}{(m-1)!} \sum_{k=0}^{n} \frac{(-1)^k}{m+k} \binom{n}{k} = \frac{n!}{(m+n)!} \phi^{(m+n)}(0).$$

This completes the proof of the second identity in (11).

Now we will prove (12). Use (8) to rewrite the form $\Phi_{m,n}(x)$ as follows:

$$\Phi_{m,n}(x) = \sum_{k=0}^{n} \frac{n!\,(m)_k}{k!} (-1)^k \frac{x^{n-k}}{(n-k)!} \phi^{(n-k)}(x).$$

Differentiate the relation just obtained m times by applying Leibniz rule and use $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^j/j!\right) = x^{j-1}/(j-1)!$, we obtain

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} \Phi_{m,n}(x) = \sum_{k=0}^{n} \frac{n!\,(m)_k}{k!} (-1)^k \sum_{j=0}^{m} \binom{m}{j} \frac{x^{n-k-j}}{(n-k-j)!} \phi^{(n-k+m-j)}(x)$$

(15)
$$= \sum_{k=0}^{n} \frac{n!\,(m)_k}{k!} (-1)^k \sum_{j=0}^{n-k} \frac{m!}{j!\,(m-j)!} \frac{x^{n-k-j}}{(n-k-j)!} \phi^{(n-k+m-j)}(x).$$

Let $s = j + k$ at the last expression in (15). Then we have

(16)
$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} \Phi_{m,n}(x) = \sum_{k=0}^{n} \frac{n!\,(m)_k\,(-1)^k}{k!} \sum_{s=k}^{n} \frac{m!}{(s-k)!\,(m-s+k)!} \frac{x^{n-s}}{(n-s)!} \phi^{(n+m-s)}(x).$$

Use (8) to write $(m-s+k)!$ in the following form:

$$(m-s+k)! = \Gamma(m-s+k+1) = \lim_{\epsilon\to 0+} \Gamma(m-s+k+1+\epsilon)$$
$$= \lim_{\epsilon\to 0+} \Gamma(m-s+1+\epsilon)\,(m-s+1+\epsilon)_k.$$

Using the last expression for the term $(m-s+k)!$ in (16), we obtain

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} \Phi_{m,n}(x) = \lim_{\epsilon\to 0+} \sum_{k=0}^{n} \sum_{s=k}^{n} \frac{n!\,(m)_k\,(-1)^k}{k!} \frac{m!}{(s-k)!}$$
$$\cdot \frac{1}{\Gamma(m-s+1+\epsilon)\,(m-s+1+\epsilon)_k} \frac{x^{n-s}}{(n-s)!} \phi^{(n+m-s)}(x).$$

Using a manipulation for double series:

$$\sum_{k=0}^{n} \sum_{s=k}^{n} A_{s,k} = \sum_{s=0}^{n} \sum_{k=0}^{s} A_{s,k}$$

and the following well-known identity:

$$(s-k)! = \frac{(-1)^k\, s!}{(-s)_k} \quad \bigl(0 \leq k \leq s;\ k,\, s \in \mathbb{N}_0\bigr),$$

we find

$$\begin{aligned}
&\frac{\mathrm{d}^m}{\mathrm{d}x^m}\,\Phi_{m,n}(x)\\
&= \lim_{\epsilon\to 0+} \sum_{s=0}^{n} \frac{n!\,m!}{s!\,\Gamma(m-s+1+\epsilon)}\,\frac{x^{n-s}}{(n-s)!}\,\phi^{(n+m-s)}(x) \sum_{k=0}^{s} \frac{(m)_k\,(-s)_k}{k!\,(m-s+1+\epsilon)_k}\\
&= \lim_{\epsilon\to 0+} \sum_{s=0}^{n} \binom{n}{s}\,\phi^{(n+m-s)}(x)\,\frac{m!\,x^{n-s}}{\Gamma(m-s+1+\epsilon)}\,{}_2F_1\,(-s,\, m\,;\, m-s+1+\epsilon\,;\, 1).
\end{aligned}$$

Applying the Chu-Vandermonde formula (9) to ${}_2F_1\,(1)$, we get

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\,\Phi_{m,n}(x) = \sum_{s=0}^{n} \binom{n}{s}\,\phi^{(n+m-s)}(x)\,x^{n-s}\,(1-s)_s.$$

Considering

$$(1-s)_s = \begin{cases} 0 & (s \in \mathbb{N}) \\ 1 & (s = 0) \end{cases}$$

in the last identity, we obtain

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\,\Phi_{m,n}(x) = x^n\,\phi^{(n+m)}(x).$$

This completes the proof of (12).

It remains to prove (13) and (14). Here we only prove (13). A similar argument will establish (14). Since (see (11))

$$f_m^{(n)}(0) = \frac{n!}{(m+n)!}\phi^{(m+n)}(0),$$

(13) holds true for $x = 0$.

Since $\phi^{(k-1)}(0) = 0$ for $1 \leq k \leq m$ ($m \in \mathbb{N}$), it is easy to see that $\Phi_{m,n}^{(k-1)}(0) = 0$ for $1 \leq k \leq m$, and $\Phi_{0,0}$ is interpreted as ϕ.

Assume that $\phi^{(n+m)}(x) \geq 0$ for $x \neq 0$ and a given $n \in \mathbb{N}_0$. Here we prove (13) case by case.

Case 1. $x > 0$. We find from (12) that

$$\Phi_{m,n}^{(m)}(x) = x^n \phi^{(n+m)}(x) \geq 0 \quad (x \in [0,\, \infty)). \tag{17}$$

We see from (17) that $\Phi_{m,n}^{(m-1)}(x)$ is increasing on $[0,\, \infty)$ and so

$$\Phi_{m,n}^{(m-1)}(x) \geq \Phi_{m,n}^{(m-1)}(0) = 0 \quad (x \in [0,\, \infty)).$$

We also see from the last inequality that $\Phi_{m,n}^{(m-2)}(x)$ is increasing on $[0,\, \infty)$ and so

$$\Phi_{m,n}^{(m-2)}(x) \geq \Phi_{m,n}^{(m-2)}(0) = 0 \quad (x \in [0,\, \infty)).$$

Continuing this process, we are finally led to see the following inequality:

$$\Phi_{m,n}(x) \geq \Phi_{m,n}(0) = 0 \quad (x \in [0, \infty)).$$

Hence it follows from (11) that $f_m^{(n)}(x) \geq 0$ on $[0, \infty)$.

Case 2. $x < 0$ and n and m are even. We find from (12) that

$$\Phi_{2m,n}^{(2m)}(x) = x^n \phi^{(n+2m)}(x) \geq 0 \quad (x \in (-\infty, 0]). \tag{18}$$

We see from (18) that $\Phi_{2m,n}^{(2m-1)}(x)$ is increasing on $(-\infty, 0]$ and so

$$\Phi_{2m,n}^{(2m-1)}(x) \leq \Phi_{2m,n}^{(2m-1)}(0) = 0 \quad (x \in (-\infty, 0]).$$

We also see from the last inequality that $\Phi_{2m,n}^{(2m-2)}(x)$ is decreasing on $(-\infty, 0]$ and so

$$\Phi_{2m,n}^{(2m-2)}(x) \geq \Phi_{2m,n}^{(2m-2)}(0) = 0 \quad (x \in (-\infty, 0]).$$

Continuing this process, we are finally led to find the following inequality:

$$\Phi_{2m,n}(x) \geq \Phi_{2m,n}(0) = 0 \quad (x \in (-\infty, 0]).$$

Hence it follows from (11) that

$$f_{2m}^{(n)}(x) = \frac{1}{x^{n+2m}} \Phi_{2m,n}(x) \geq 0 \quad (x \in (-\infty, 0]).$$

The remaining cases (**Case 3.** $x < 0$ and n is even and m is odd; **Case 4.** $x < 0$ and n is odd and m is even; **Case 5.** $x < 0$ and n is odd and m is odd) can be established by using the similar argument in **Case 2** and are left to an interested reader. Thus the proof of the assertion (13) is complete.

This completes the proof of Theorem 2.1. □

Remark 2.2. Let $I = (a, b)$ with $a \leq 0$ and $0 < b \leq \infty$. By replacing $(-\infty, \infty)$ by I, we see that in fact Theorem 2.1 also holds on I.

The following properties hold from (13) and (14).

(i) Let $\phi(0) = 0$. It is easy to see that, if ϕ' is completely monotonic on $(-\infty, \infty)$, then f_1 is completely monotonic on $(-\infty, \infty)$; if ϕ'' is completely monotonic on $(-\infty, \infty)$, then the function f_1' is completely monotonic on $(-\infty, \infty)$. In general, if the function $\phi^{(k)}$ for some integer $k \in \mathbb{N} \setminus \{1\}$ is completely monotonic on $(-\infty, \infty)$, then the function $f_1^{(k-1)}$ is completely monotonic on $(-\infty, \infty)$.

(ii) Let $\phi(0) = \phi'(0) = 0$. It is clear that, if ϕ'' is completely monotonic on $(-\infty, \infty)$, then f_2 and f_1' are completely monotonic on $(-\infty, \infty)$; if ϕ''' is completely monotonic on $(-\infty, \infty)$, then the functions f_2' and f_1'' are completely monotonic on $(-\infty, \infty)$. In general, if $\phi^{(k)}$ for some $k \in \mathbb{N} \setminus \{1\}$ is completely monotonic on $(-\infty, \infty)$, then $f_2^{(k-2)}$ and $f_1^{(k-1)}$ are completely monotonic on $(-\infty, \infty)$.

(iii) Let $\phi(0) = \phi'(0) = \phi''(0) = 0$. It follows that, if ϕ''' is completely monotonic on $(-\infty, \infty)$, then f_3, f_2' and f_1'' are completely monotonic on $(-\infty, \infty)$; if $\phi^{(4)}$ is completely monotonic on $(-\infty, \infty)$, then the functions f_3', f_2'' and f_1''' are completely monotonic on $(-\infty, \infty)$. In general, if $\phi^{(k)}$ for some integer $k \geq 3$ is completely monotonic on $(-\infty, \infty)$, then $f_3^{(k-3)}$, $f_2^{(k-2)}$ and $f_1^{(k-1)}$ are completely monotonic on $(-\infty, \infty)$.

3. Applications of main results

As applications of Theorem 2.1, we present some (logarithmically) completely monotonic functions related to the gamma function, and we establish some inequalities for the gamma function and generalize some known results.

3.1. Anderson *et al.* [12, Lemma 2.39] proved that the function

$$x \mapsto \frac{\ln \Gamma(1 + x/2)}{x} \tag{19}$$

is strictly increasing from $[2, \infty)$ onto $[0, \infty)$ and

$$\lim_{x \to \infty} \frac{\ln \Gamma(1 + x/2)}{x \ln x} = \frac{1}{2}.$$

From this, Anderson *et al.* [12, Lemma 2.40] derived the following conclusions: The sequence $\Omega_n^{1/n}$ decreases strictly to 0 as $n \to \infty$, the series $\sum_{n=2}^{\infty} \Omega_n^{1/\ln n}$ is convergent, and

$$\lim_{n \to \infty} \Omega_n^{1/(n \ln n)} = e^{-1/2},$$

where

$$\Omega_n := \frac{\pi^{n/2}}{\Gamma(1 + n/2)}$$

denotes the volume of the unit ball in $\mathbb{R}^n$ (see, e.g., [53, Theorem 12.69]).

Clearly, the function (19) is strictly increasing on $[2, \infty)$, i.e., the function

$$f_1(x) = \frac{\ln \Gamma(1 + x)}{x} \tag{20}$$

is strictly increasing on $[1, \infty)$. Kershaw and Laforgia [32] proved that the function $[\Gamma(1 + 1/x)]^x$ decreases with $x > 0$, i.e., the function $[\Gamma(1 + x)]^{1/x}$ increases with $x > 0$.

Grabner *et al.* [25] proved that the function $f_1 : (-1, \infty) \to \mathbb{R}$, defined by $f_1(x) = \frac{\ln \Gamma(x+1)}{x}$, is concave, strictly increasing and satisfies an analogue of the famous Bohr-Mollerup theorem (see, e.g., [52, p. 12]).

Sándor [48] proved that the function

$$f_2(x) = \frac{x\psi(x + 1) - \ln \Gamma(x + 1)}{x^2} \tag{21}$$

is strictly decreasing and strictly convex for $x \geq 6$.

Let the functions f_1 and f_2 be defined by (20) and (21), respectively. We prove that the functions f_1' and f_2 are completely monotonic on $(-1,\infty)$. Moreover, we present a very short proof.

It is easy to see that $f_1'(x)=f_2(x)$.

Theorem 3.1. *For $x>-1$, let*

$$f_1(x)=\begin{cases}\dfrac{\ln\Gamma(x+1)}{x} & (x\neq 0)\\ -\gamma & (x=0),\end{cases}$$

where γ is the Euler-Mascheroni constant given in (6)*. Then the function f_1' is completely monotonic on $(-1,\infty)$, that is*

$$(-1)^n f_1^{(n+1)}(x)\ge 0 \quad \bigl(x>-1;\ n\in\mathbb{N}_0\bigr). \tag{22}$$

Proof. Let $\phi(x)=\ln\Gamma(x+1)$. Then

$$f_1(x)=\begin{cases}\dfrac{\phi(x)}{x} & (x\neq 0)\\ -\gamma & (x=0).\end{cases}$$

Clearly, $\phi(0)=0$, and

$$(-1)^n\phi^{(n+2)}(x)=(-1)^n\psi^{(n+1)}(x+1)=\int_0^\infty \frac{t^{n+1}}{1-e^{-t}}e^{-(x+1)t}\,dt>0$$

for $x>-1$ and $n\in\mathbb{N}_0$, and therefore, the function ϕ'' is completely monotonic on $(-1,\infty)$. By Remark 2.2, the function f_1' is completely monotonic on $(-1,\infty)$. □

Remark 3.2. It is known [21, Lemma 2.2] that if $f(x)$ is completely monotonic on some interval (a,b), then so is $f(x)-f(x+c)$ on $(a,b)\cap(a-c,b-c)$ for any $c>0$. Consequently, if $f(x)$ is logarithmically completely monotonic on some interval (a,b), then so is $f(x)/f(x+c)$ on $(a,b)\cap(a-c,b-c)$ for any $c>0$. Write (22) as

$$(-1)^n\left(\ln\frac{1}{[\Gamma(1+x)]^{1/x}}\right)^n\ge 0 \quad (x>-1;\ n\in\mathbb{N}). \tag{23}$$

We see that the function $x\mapsto \frac{1}{[\Gamma(1+x)]^{1/x}}$ is logarithmically completely monotonic on $(-1,\infty)$. This implies that the function $x\mapsto\frac{\sqrt{\pi}}{[\Gamma(1+x/2)]^{1/x}}$ is logarithmically completely monotonic on $(-2,\infty)$, and then, the function

$$x\mapsto\frac{\left[\Gamma\left(1+\frac{x+1}{2}\right)\right]^{1/(x+1)}}{\left[\Gamma\left(1+\frac{x}{2}\right)\right]^{1/x}}$$

is logarithmically completely monotonic on $(-1,\infty)$. Because

$$\Omega_n^{1/n}=\frac{\sqrt{\pi}}{[\Gamma(1+n/2)]^{1/n}},$$

we see that the sequences $\Omega_n^{1/n}$ and $\Omega_n^{1/n}/\Omega_{n+1}^{1/(n+1)}$ are both decreasing for $n \in \mathbb{N}$.

Some inequalities for the volume of the unit ball in $\mathbb{R}^n$ can be found in [8, 10, 47].

3.2. In 1974, Gautschi [22] proved the inequality conjectured by Rao Uppuluri

$$\frac{2}{1/\Gamma(x) + 1/\Gamma(1/x)} \geq 1 \quad (x > 0) \tag{24}$$

which states that the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is greater than or equal to 1. In 1997, Alzer [4] deduced that the harmonic mean of $[\Gamma(x)]^2$ and $[\Gamma(1/x)]^2$ is greater than or equal to 1, i.e.,

$$\frac{2}{1/[\Gamma(x)]^2 + 1/[\Gamma(1/x)]^2} \geq 1 \quad (x > 0). \tag{25}$$

Let $M_r(a, b)$ be the rth power mean of two positive real numbers a and b. The inequalities (24) and (25) can be written as

$$M_1\left(\frac{1}{\Gamma(x)}, \frac{1}{\Gamma(1/x)}\right) \leq 1 \quad \text{and} \quad M_2\left(\frac{1}{\Gamma(x)}, \frac{1}{\Gamma(1/x)}\right) \leq 1 \quad (x > 0).$$

Since the function $r \mapsto M_r(a, b)$ is increasing in $r \in (-\infty, \infty)$, we see that inequality (25) is stronger than (24). In 2000, Alzer [7] proved that the inequality

$$M_r(\Gamma(x), \Gamma(1/x)) \geq 1$$

holds for all $x \in (0, \infty)$ if and only if $r \geq 1/\gamma - \pi^2/(6\gamma^2)$, where γ denotes Euler-Mascheroni constant.

In the second paper on this subject, Gautschi [23] showed that the conjecture

$$\frac{n}{\sum_{k=1}^n 1/\Gamma(x_k)} \geq 1 \quad \big(x_k > 0;\ k = 1, \ldots, n;\ x_1 x_2 \cdots x_n = 1\big) \tag{26}$$

is evident for $n = 1, 2, \ldots, 8$, but false for $n = 9$. He also showed that, for all $n \in \mathbb{N}$,

$$\left[\prod_{k=1}^n \Gamma(x_k)\right]^{1/n} \geq 1 \quad \big(x_k > 0;\ k = 1, \ldots, n;\ x_1 x_2 \cdots x_n = 1\big).$$

In 2003, Alzer [9] proved that the harmonic mean inequality (26) holds for all positive real numbers $x_1, x_2, \ldots, x_n$ with $x_1 x_2 \cdots x_n = 1$ if and only if $n \leq 8$.

Because of the well-known inequalities among the harmonic, geometric and arithmetic means, inequality (24) implies, for example,

$$\Gamma(x)\Gamma(1/x) \geq 1 \quad (x > 0). \tag{27}$$

An alternative proof of (27) was given by Kairies [31] and an extension of (27) was presented by Laforgia and Sismondi [35]:

$$\left[\frac{\Gamma(x+1)}{\Gamma(x+\lambda)} \frac{\Gamma(1/x+1)}{\Gamma(1/x+\lambda)}\right]^{1/2} \geq \frac{1}{\Gamma(\lambda+1)} \tag{28}$$

for $x > 0$ and $0 < \lambda < 1$. In the case $\lambda > 1$ the inequality (28) must be reversed. If we define

$$f(x) = \frac{\Gamma(x+1)}{\Gamma(x+\lambda)},$$

the previous inequality shows that the geometric mean inequality

$$G\left(f(x), f\left(\frac{1}{x}\right)\right) \geq \frac{1}{\Gamma(\lambda+1)}$$

holds. The lower bound in (27) is pessimistic. For example, $\Gamma(5)\Gamma(0.2) = 110.179$. Giordano and Laforgia [24] proved more accurate inequalities than (27) for the product of gamma functions: For $x_1, x_2 > 0$ and $x_1 x_2 = 1$, then

$$\frac{1}{2}\Gamma(1+x_1+x_2) \leq \Gamma(1+x_1)\Gamma(1+x_2) < \Gamma(1+x_1+x_2). \tag{29}$$

For the lower bound in (29), the equality occurs for $x_1 = x_2 = 1$. Giordano and Laforgia [24] showed that the first inequality in (29) is only true for two variables.

Motivated by the inequality (29), we can extend the second inequality in (29) to several parameters asserted by the following theorem.

Theorem 3.3. *Let $a_i > 0$ $(i = 1, 2, \dots, n)$ be real numbers, and let*

$$f(x) = \frac{\Gamma\left(1 + \sum_{i=1}^n a_i x\right)}{\prod_{i=1}^n \Gamma(1 + a_i x)} \quad (x > 0). \tag{30}$$

Then we have

(i) *The function f is strictly increasing on $(0, \infty)$;*
(ii) *The function $x \mapsto (\ln f(x))''$ is completely monotonic on $(0, \infty)$.*

Proof. Let $a = \sum_{i=1}^n a_i$. A simple computation yields

$$(\ln f(x))' = \sum_{i=1}^n a_i \left[\psi(1+ax) - \psi(1+a_i x)\right] > 0,$$

since the function ψ is strictly increasing on $(0, \infty)$.

For $x > 0$,

$$\begin{aligned}
(\ln f(x))'' &= \sum_{i=1}^n a_i \left[a\psi'(1+ax) - a_i\psi'(1+a_i x)\right] \\
&= \sum_{i=1}^n a_i \left[a \int_0^\infty \frac{t}{e^t - 1} e^{-axt} \mathrm{d}t - a_i \int_0^\infty \frac{t}{e^t - 1} e^{-a_i x t} \mathrm{d}t\right] \\
&= \sum_{i=1}^n a_i \left[a \int_0^\infty \frac{t}{e^t - 1} e^{-axt} \mathrm{d}t - a \int_0^\infty \frac{(a/a_i) u^{m-1}}{e^{(a/a_i)t} - 1} e^{-axu} \mathrm{d}u\right] \\
&= \int_0^\infty \sum_{i=1}^n a_i a \left[g(t) - g\left(\frac{a}{a_i} t\right)\right] e^{-axt} \mathrm{d}t,
\end{aligned} \tag{31}$$

where

$$g(t) = \frac{t}{e^t - 1} \quad (t > 0).$$

Since the function g is strictly decreasing on $(0, \infty)$, (31) implies

$$(-1)^m \left(\ln f(x)\right)^{(m+2)} > 0 \quad \big(x > 0;\ m \in \mathbb{N}_0\big).$$

The proof is complete. □

Remark 3.4. Consider the function f given in (30). Since $f(0) = 1$ and f is strictly increasing on $(0, \infty)$, we find

$$1 < \frac{\Gamma\left(1 + \sum_{i=1}^{n} a_i x\right)}{\prod_{i=1}^{n} \Gamma(1 + a_i x)} \quad \big(x > 0;\ a_i > 0;\ n \in \mathbb{N} \setminus \{1\}\big).$$

By replacing $a_i x$ by x_i, we get, for all $x_i > 0$,

$$\Gamma(1 + x_1)\Gamma(1 + x_2) \cdots \Gamma(1 + x_n) < \Gamma(1 + x_1 + x_2 + \cdots + x_n). \tag{32}$$

We see that the inequality (32) generalizes the second inequality in (29) to several variables without the condition $\prod_{j=1}^{n} x_j = 1$. We can also prove the inequality (32) by using the fact that the function $x \mapsto \frac{\ln \Gamma(1+x)}{x}$ is increasing on $(0, \infty)$. Indeed, for all $x_i > 0$ $(i = 1, 2, \ldots, n)$, we have

$$\ln \Gamma(1 + x_i) \le \frac{x_i}{x_1 + x_2 + \cdots + x_n} \ln \Gamma(1 + x_1 + x_2 + \cdots + x_n),$$

summing these inequalities side by side proves the inequality in (32).

Recall that a function g is said to be *super-additive* (respectively, *sub-additive*) on an interval I if

$$g(x) + g(y) \le (\text{respectively}, \ge)\, g(x + y) \quad (x,\, y \in I;\ x + y \in I).$$

Consider the function f given in (30). Superadditive property of the function $\ln f$ will be an immediate consequence of the following Theorem 3.5.

Theorem 3.5. *Let $a_i > 0$ $(i = 1, 2, \ldots, n)$ be real numbers, and let*

$$F(x) = \left[\frac{\Gamma\left(1 + \sum_{i=1}^{n} a_i x\right)}{\prod_{i=1}^{n} \Gamma(1 + a_i x)}\right]^{1/x} \quad (x > 0). \tag{33}$$

Then the function $x \mapsto \frac{1}{F(x)}$ is logarithmically completely monotonic on $(0, \infty)$.

Proof. Let $\phi(x) = \ln f(x)$, where f is the function in (30). Then

$$\ln F(x) = \frac{\phi(x)}{x}. \tag{34}$$

Since $\phi(0) = 0$, we find from Theorem 3.3 that the function ϕ'' is completely monotonic on $(0, \infty)$. This implies by Remark 2.2 that the function $x \mapsto (\ln F(x))'$ is completely monotonic on $(0, \infty)$. That is,

$$(-1)^n (\ln F(x))^{(n+1)} \ge 0 \quad \big(x > 0;\ n \in \mathbb{N}_0\big). \tag{35}$$

Rewrite (35) as

$$(-1)^n \left(\ln \frac{1}{F(x)}\right)^{(n)} \geq 0 \quad (x > 0;\ n \in \mathbb{N}). \tag{36}$$

We see that the function $x \mapsto \frac{1}{F(x)}$ is logarithmically completely monotonic on $(0, \infty)$. □

Remark 3.6. Clearly, the function F defined by (33) is strictly increasing on $(0, \infty)$. Using the known asymptotic expansion (see, e.g., [1, p. 257]):

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \frac{1}{12x} + O\left(\frac{1}{x^3}\right) \quad (x \to \infty), \tag{37}$$

we conclude

$$\lim_{x \to \infty} F(x) = \frac{\left(\sum_{i=1}^n a_i\right)^{\sum_{i=1}^n a_i}}{\prod_{i=1}^n a_i^{a_i}}.$$

It is easy to see that $F(0) = \lim_{x \to 0+} F(x) = 1$. Hence we find

$$1 < \left[\frac{\Gamma\left(1 + \sum_{i=1}^n a_i x\right)}{\prod_{i=1}^n \Gamma(1 + a_i x)}\right]^{1/x} < \frac{\left(\sum_{i=1}^n a_i\right)^{\sum_{i=1}^n a_i}}{\prod_{i=1}^n a_i^{a_i}} \tag{38}$$

for all $x > 0$ and $a_i > 0$ ($i = 1, 2, \ldots, n$). Taking $a_1 = a_2 = \cdots = a_n = 1$, we obtain

$$\frac{1}{n^{nx}} < \frac{\Gamma(1 + x)^n}{\Gamma(1 + nx)} < 1 \quad (x > 0). \tag{39}$$

Since the function $\ln F$ is strictly increasing on $(0, \infty)$, we obtain by (34) that the function $\phi(x) = \ln f(x)$ is super-additive on $(0, \infty)$, i.e.,

$$\frac{\Gamma\left(1 + \sum_{i=1}^n a_i x\right)}{\prod_{i=1}^n \Gamma(1 + a_i x)} \frac{\Gamma\left(1 + \sum_{i=1}^n a_i y\right)}{\prod_{i=1}^n \Gamma(1 + a_i y)} \leq \frac{\Gamma\left(1 + \sum_{i=1}^n a_i (x + y)\right)}{\prod_{i=1}^n \Gamma(1 + a_i (x + y))} \quad \big(x,\ y,\ a_i > 0\big).$$

By replacing $a_i x$ and $a_i y$ with x_i and y_i, respectively, we get for all $x_i, y_i > 0$,

$$\frac{\Gamma\left(1 + \sum_{i=1}^n x_i\right) \Gamma\left(1 + \sum_{i=1}^n y_i\right)}{\Gamma\left(1 + \sum_{i=1}^n (x_i + y_i)\right)} \leq \prod_{i=1}^n \frac{\Gamma(1 + x_i)\Gamma(1 + y_i)}{\Gamma(1 + x_i + y_i)}. \tag{40}$$

3.3. By using a geometrical method, Alsina and Tomás [2] proved the inequality:

$$\frac{1}{n!} \leq \frac{\Gamma(x + 1)^n}{\Gamma(nx + 1)} \leq 1 \quad \big(x \in [0,\ 1];\ n \in \mathbb{N}_0\big). \tag{41}$$

By certain simple analytical arguments, Sándor [49] proved the inequality

$$\psi(ax + 1) \geq \psi(x + 1) \quad (x \geq 0;\ a \geq 1). \tag{42}$$

Then he used the inequality (42) to show the following result: For all $a \geq 1$, the function

$$x \mapsto \frac{\Gamma(x + 1)^a}{\Gamma(ax + 1)}$$

is a decreasing function on $[0, \infty)$. This implies the inequality

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(x+1)^a}{\Gamma(ax+1)} \leq 1 \quad (a \geq 1;\ x \in [0,\, 1]). \tag{43}$$

We remark here in passing that, in fact, the inequality (42) holds obviously, since the function ψ is strictly increasing on $(0, \infty)$. Mercer [38] continued to create new inequalities on this subject and other special functions and obtained the following inequalities

$$\frac{\Gamma(1+x)^a}{\Gamma(1+ax)} < \frac{\Gamma(1+y)^a}{\Gamma(1+ay)} \quad (0 < a < 1)$$

and

$$\frac{\Gamma(1+x)^a}{\Gamma(1+ax)} > \frac{\Gamma(1+y)^a}{\Gamma(1+ay)} \quad (a < 0 \quad \text{or} \quad a > 1),$$

where $y > x > 0$, $1 + ax > 0$ and $1 + ay > 0$. Let f be a function defined by

$$f_{a,b}(x) = \frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b} \tag{44}$$

in which $1 + ax > 0$ and $1 + bx > 0$. Using the same method used by Sándor [49], Bougoffa [16] considered the monotonicity property of $x \mapsto f_{a,b}(x)$, and then, used his results to establish several inequalities involving the gamma function. There have been a lot of results on this subject, see, for example, [2, 16, 33, 34, 37–40, 46, 49–51]. Here we also present certain analogous results asserted by the following theorem.

Theorem 3.7. *For $x, a, b > 0$, let the function $x \mapsto f_{a,b}(x)$ be given in (44), and the function $x \mapsto G_{a,b}(x)$ be defined by*

$$G_{a,b}(x) = \frac{\Gamma(1+bx)^{a/x}}{\Gamma(1+ax)^{b/x}}. \tag{45}$$

Then we have

(i) *For $b > a > 0$, the function $x \mapsto \frac{\mathrm{d}^2[\ln f_{a,b}(x)]}{\mathrm{d}x^2}$ is completely monotonic and $x \mapsto \frac{1}{G_{a,b}(x)}$ is logarithmically completely monotonic on $(0, \infty)$.*

(ii) *For $a > b > 0$, the function $x \mapsto \left(\ln \frac{1}{f_{a,b}(x)}\right)''$ is completely monotonic and $x \mapsto G_{a,b}(x)$ is logarithmically completely monotonic on $(0, \infty)$.*

Proof. For $n \geq 2$, we obtain

$$\begin{aligned}(-1)^n \left(\ln f_{a,b}(x)\right)^{(n)} &= ab^n(-1)^n\psi^{(n-1)}(1+bx) - a^n b(-1)^n\psi^{(n-1)}(1+ax) \\ &= \int_0^\infty \frac{ab^n t^{n-1}}{1-e^{-t}} e^{-(1+bx)t}\mathrm{d}t - \int_0^\infty \frac{a^n b t^{n-1}}{1-e^{-t}} e^{-(1+ax)t}\mathrm{d}t \\ &= \int_0^\infty \frac{ab^n t^{n-1}}{e^t-1} e^{-bxt}\mathrm{d}t - \int_0^\infty \frac{a^n b t^{n-1}}{e^t-1} e^{-axt}\mathrm{d}t \\ &= \int_0^\infty \frac{a^{n+1} u^{n-1}}{e^{(a/b)u}-1} e^{-axu}\mathrm{d}u - \int_0^\infty \frac{a^n b t^{n-1}}{e^t-1} e^{-axt}\mathrm{d}t\end{aligned} \tag{46}$$

$$= a^n b \int_0^\infty \left[\frac{(a/b)t}{e^{(a/b)t} - 1} - \frac{t}{e^t - 1} \right] t^{n-2} e^{-axt} \mathrm{d}t.$$

Since the function $t \mapsto \frac{t}{e^t-1}$ is strictly decreasing on $(0, \infty)$, (46) implies that

$$(47) \qquad (-1)^n (\ln f_{a,b}(x))^{(n+2)} > 0 \quad \big(x > 0;\ b > a > 0;\ n \in \mathbb{N}_0\big)$$

and

$$(48) \qquad (-1)^n \left(\ln \frac{1}{f_{a,b}(x)} \right)^{(n+2)} > 0 \quad \big(x > 0;\ a > b > 0;\ n \in \mathbb{N}_0\big).$$

Let $\phi_{a,b}(x) = \ln f_{a,b}(x)$, then

$$(49) \qquad \ln G_{a,b}(x) = \frac{\phi_{a,b}(x)}{x}.$$

Clearly, $\phi_{a,b}(0) = 0$. Since the function $\phi''_{a,b}$ is completely monotonic on $(0, \infty)$ for $b > a > 0$, this implies by Remark 2.2 that the function $x \mapsto (\ln G_{a,b}(x))'$ is completely monotonic on $(0, \infty)$ for $b > a > 0$, i.e.,

$$(-1)^n (\ln G_{a,b}(x))^{(n+1)} \geq 0 \quad \big(x > 0;\ b > a > 0;\ n \in \mathbb{N}_0\big),$$

or,

$$(50) \qquad (-1)^n \left(\ln \frac{1}{G_{a,b}(x)} \right)^{(n)} \geq 0 \quad \big(x > 0;\ b > a > 0;\ n \in \mathbb{N}\big).$$

Obviously, since $G_{a,b}(x) = \frac{1}{G_{b,a}(x)}$, (50) implies that

$$(51) \qquad (-1)^n (\ln G_{a,b}(x))^{(n)} \geq 0 \quad \big(x > 0;\ a > b > 0;\ n \in \mathbb{N}\big).$$

The proof of Theorem 3.7 is complete. □

Remark 3.8. Clearly, the function $x \mapsto G_{a,b}(x)$ given in (45) is strictly decreasing (respectively, strictly increasing) on $(0, \infty)$ if $a > b > 0$ (respectively, $b > a > 0$). Then we find from (49) that the function $x \mapsto \phi_{a,b}(x) = \ln f_{a,b}(x)$ is sub-additive (respectively, super-additive) on an interval $(0, \infty)$ if $a > b > 0$ (respectively, $b > a > 0$), where $f_{a,b}(x)$ is defined by (44). That is,

$$\phi_{a,b}(x) + \phi_{a,b}(y) \geq \text{ (respectively, } \leq) \, \phi_{a,b}(x + y) \quad (x,\, y > 0),$$

or, equivalently,

$$(52) \qquad f_{a,b}(x) f_{a,b}(y) \geq \text{ (respectively, } \leq) \, f_{a,b}(x + y) \quad (x,\, y > 0),$$

if $a > b > 0$ (respectively, $b > a > 0$).

Using the asymptotic expansion (37) we conclude

$$\lim_{x \to \infty} G_{a,b}(x) = \left(\frac{b}{a} \right)^{ab}.$$

It is easy to see that $G_{a,b}(0) = \lim_{x \to 0+} G_{a,b}(x) = 1$. Hence we find

$$(53) \qquad \left(\frac{b}{a} \right)^{ab} < \frac{\Gamma(1 + bx)^{a/x}}{\Gamma(1 + ax)^{b/x}} < 1 \quad (x > 0;\ a > b > 0).$$

The inequality (53) is seen to be reversed for $x > 0$ and $b > a > 0$.

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