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ON GROWTH PROPERTIES OF TRANSCENDENTAL MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF HIGHER ORDER

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Abstract. In the paper, we study the growth properties of meromorphic solutions of higher order linear differential equations with entire coefficients of $[p,q]-\varphi$ order, φ being a non-decreasing unbounded function and establish some new results which are improvement and extension of some previous results due to Hamani-Belaidi, He-Zheng-Hu and others.

1. Introduction, definitions and notations

We shall assume that the readers are familiar with the fundamental results and the standard notations of the value distribution theory of entire and meromorphic functions, and the theory of complex differential equations (see [9], [15] and [20]). Here, we use some notations for $r \in [0,\infty)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. For all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{p+1} r = \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Also $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$ and $\log_{-1} r = \exp_1 r$. Moreover, we denote the linear measure for a set $E \subset [0,\infty)$, by $mE = \int_E dt$ and logarithmic measure for a set $E \subset (1,\infty)$, by $m_l E = \int_E \frac{dt}{t}$.

The upper density of a set $E \subset [0,\infty)$ is defined as

$$\overline{dens}\; E = \overline{\lim_{r\to\infty}} \frac{m\left(E \cap [0,r]\right)}{r},$$

and the upper logarithmic density of a set $E \subset (1,\infty)$ is defined as

$$\overline{\log dens}\; E = \overline{\lim_{r\to\infty}} \frac{m_l\left(E \cap [1,r]\right)}{\log r} \;.$$

For $k \geq 2$, consider the complex linear differential equations

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{1.1}$$

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and

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{1.2}$$

where the coefficients $A_0(z)$, $A_1(z), \ldots, A_{k-1}(z)$, $A_k(z)\,(\not\equiv 0)$ and $F(z)\,(\not\equiv 0)$ are entire functions. It is well-known that if $A_k(z) \equiv 1$, then all solutions of the differential equations (1.1) and (1.2) are entire functions but when $A_k(z)$ is a non constant entire function, then solutions of the equations can possess meromorphic. For example, the equation

$$zf''' + 3f'' - 2e^{-2z}f' + \left((z-2)e^{-3z} + (3z-2)e^{-2z} + ze^{-z}\right) f = 0$$

has a meromorphic solution $f(z) = \frac{e^{e^{-z}}}{z}$.

From the last few years, many authors have been investigated the growth properties of solutions of the complex linear differential equations and obtained many valuable results about their growth (see [1, 4, 5, 7, 8, 10, 11, 16]).

In 1976, O. P. Juneja and his coauthors ([13] and [14]) investigated some properties of entire functions of $[p, q]$-order, and obtain some results. In [17], in order to maintain accordance with general definitions of the entire function $f(z)$ of iterated p-order, Liu-Tu-Shi gave a minor modification of the original definition of the $[p, q]$-order given in [13] and [14]. With this concept of $[p, q]$-order, the solutions of complex linear differential equations are investigated (see, [2, 11, 16, 17]). Recently, X. Shen, J. Tu and H. Y. Xu [18] introduced the new concept of $[p, q] - \varphi$ order of meromorphic functions in the complex plane to study the growth and zeros of second order linear differential equations, where p, q are positive integers satisfying $p \geq q \geq 1$. In this paper, we consider this subject and investigate the complex linear differential equations (1.1) and (1.2) when the coefficients are entire functions of $[p, q] - \varphi$ order.

To express the rate of growth of meromorphic functions, we recall the following definitions:

Definition 1.1 ([18])**.** Let $\varphi : [0, \infty) \to (0, \infty)$ be a non-decreasing unbounded function. Then the $[p, q] - \varphi$ order and $[p, q] - \varphi$ lower order of a meromorphic function f are respectively defined by

$$\sigma_{[p,q]}(f, \varphi) = \varlimsup_{r \to \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)},$$

$$\mu_{[p,q]}(f, \varphi) = \varliminf_{r \to \infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)}.$$

Definition 1.2 ([3])**.** Let f be a meromorphic function satisfying

$$0 < \sigma_{[p,q]}(f, \varphi) = \sigma < \infty.$$

Then the $[p, q] - \varphi$ type of f is defined by

$$\tau_{[p,q]}(f, \varphi) = \varlimsup_{r \to \infty} \frac{\log_{p-1} T(r, f)}{\left[\log_{q-1} \varphi(r)\right]^{\sigma}}.$$

Definition 1.3 ([18]). Let f be a meromorphic function. Then the $[p,q]-\varphi$ exponent of convergence of zero-sequence (distinct zero-sequence) of f is defined by

$$\lambda_{[p,q]}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log_p n\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)},$$

$$\overline{\lambda}_{[p,q]}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log_p \overline{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.$$

Remark 1.1. If $\varphi(r)=r$ in Definitions (1.1)-(1.3), then we obtain the standard definitions of the $[p,q]$-order, $[p,q]$-type and $[p,q]$-exponent of convergence.

Remark 1.2 ([18]). Throughout this paper, we assume that $\varphi:[0,\infty)\to(0,\infty)$ is a non-decreasing unbounded function and always satisfies the following two conditions:

(i) $\lim_{r\to+\infty}\frac{\log_{p+1} r}{\log_q \varphi(r)}=0$ and

(ii) $\lim_{r\to+\infty}\frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)}=1$ for some $\alpha>1$.

Recently, K. Hamani and B. Belaidi [8] have proved the following theorems:

Theorem A ([8]). *Let $p\geq 1$ be an integer and let $A_0(z),A_1(z),\ldots,A_{k-1}(z)$, $A_k(z)$ with $A_0(z)\not\equiv 0$ and $A_k(z)\not\equiv 0$ be entire functions such that $i_\lambda(A_k)\leq 1$, $i(A_j)=p\,(j=0,1,2,\ldots,k)$ and $\max\{\sigma_p(A_j):j=1,2,\ldots,k\}<\sigma_p(A_0)=\sigma$. Suppose that for real constants α,β,θ_1 and θ_2 satisfying $0\leq\beta<\alpha$ and $\theta_1<\theta_2$ and for $\varepsilon>0$ sufficiently small, we have*

$$|A_0(z)|\geq \exp_p\left\{\alpha|z|^{\sigma-\varepsilon}\right\},$$

and

$$|A_j(z)|\leq \exp_p\left\{\beta|z|^{\sigma-\varepsilon}\right\}\qquad(j=1,2,\ldots,k)$$

as $z\to\infty$ in $\theta_1\leq\arg z\leq\theta_2$. Then every meromorphic solution $f\not\equiv 0$ whose poles are of uniformly bounded multiplicity of the equation (1.1) has an infinite iterated p-order and satisfies $i(f)=p+1$, $\sigma_{p+1}(f)=\sigma$.

Theorem B ([8]). *Let $A_0(z)$, $A_1(z),\ldots,A_{k-1}(z)$, $A_k(z)$ with $A_0(z)\not\equiv 0$ and $A_k(z)\not\equiv 0$ be entire functions satisfying the hypotheses of Theorem B and let $F\not\equiv 0$ be an entire function of iterated order with $i(F)=q$.*

1) If $q<p+1$ or $q=p+1$ and $\sigma_{p+1}(F)<\sigma_p(A_0)=\sigma$, then every meromorphic solution f whose poles are of uniformly bounded multiplicity of the equation (1.2) satisfies $i_{\overline{\lambda}}(f)=i_\lambda(f)=i(f)=p+1$ and $\overline{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\sigma_{p+1}(f)=\sigma$ with at most one exceptional solution f_0 satisfying $i(f_0)<p+1$ or $\sigma_{p+1}(f_0)<\sigma$.

2) If $q>p+1$ or $q=p+1$ and $\sigma_p(A_0)<\sigma_{p+1}(F)<\infty$, then every meromorphic solution f whose poles are of uniformly bounded multiplicity of the equation (1.2) satisfies $i(f)=q$ and $\sigma_q(f)=\sigma_q(F)$.

In this paper, our aim is to investigate the growth properties of meromorphic solutions of the linear differential equations (1.1) and (1.2) by using the concept of $[p,q]-\varphi$ order of entire functions and obtained some results which improve and extend some previous results due to Hamani-Belaidi [8], and He-Zheng-Hu[10].

2. Main results

In this section we state the main results of the paper.

Theorem 2.1. *Let H be a set of complex numbers satisfying*

$$\overline{\log dens}\,\{|z| : z \in H\} > 0,$$

and let $A_0(z)$, $A_1(z), \ldots, A_{k-1}(z)$, $A_k(z)$ with $A_0(z)(\not\equiv 0)$ and $A_k(z)(\not\equiv 0)$ be entire functions satisfying

$$\max\left\{\sigma_{[p,q]}(A_j,\varphi) : j = 1,2,\ldots,k\right\} < \sigma_{[p,q]}(A_0,\varphi) \quad (p \geq q \geq 1 \text{ are integers}).$$

Suppose that there exist real constants α and β satisfying $0 \leq \beta < \alpha$, and for $\varepsilon > 0$ sufficiently small, we have

$$|A_0(z)| \geq \exp_p\left\{\alpha\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\},$$

and

$$|A_j(z)| \leq \exp_p\left\{\beta\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\} \quad (j = 1,2,\ldots,k)$$

as $z \to \infty$ for $z \in H$. Then every transcendental meromorphic solution $f(\not\equiv 0)$ of equation (1.1) with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ satisfies $\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$.

Corollary 2.1. *Let $A_j(z)$ $(j = 0,1,\ldots,k)$, H satisfy all of the hypotheses of Theorem 2.1, and let $g(z)(\not\equiv 0)$ be a meromorphic function satisfying*

$$\sigma_{[p+1,q]}(g,\varphi) < \sigma_{[p,q]}(A_0,\varphi).$$

Then every transcendental meromorphic solution $f(z)(\not\equiv 0)$ with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ of equation (1.1) satisfies

$$\overline{\lambda}_{[p+1,q]}(f-g,\varphi) = \lambda_{[p+1,q]}(f-g,\varphi) = \sigma_{[p+1,q]}(f-g,\varphi) = \sigma_{[p,q]}(A_0,\varphi).$$

Theorem 2.2. *Let H be a set of complex numbers satisfying $\overline{dens}\,\{|z| : z \in H\} > 0$, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z), A_k(z)$ with $A_0(z)(\not\equiv 0)$ and $A_k(z)(\not\equiv 0)$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}(A_j,\varphi) : j = 1,2,\ldots,k\right\} < \sigma_{[p,q]}(A_0,\varphi) \quad (p \geq q \geq 1 \text{ are integers}).$$

Suppose that there exists a sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} z_n = \infty$ and two real numbers α and β satisfying $0 \leq \beta < \alpha$ such that for all $\varepsilon > 0$ sufficiently small, we have

$$|A_0(z_n)| \geq \exp_p\left\{\alpha\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\},$$

and

$$|A_j(z_n)| \leq \exp_p\left\{\beta\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\} \quad (j=1,2,\ldots,k)$$

as $n\to\infty$, $z_n\in H$. *Then every transcendental meromorphic solution* $f\,(\not\equiv 0)$ *of equation* (1.1) *with* $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ *satisfies* $\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$.

Corollary 2.2. *Let* $A_j(z)$ $(j=0,1,\ldots,k)$, H *satisfy all of the hypotheses of Theorem 2.2, and let* $g(z)\,(\not\equiv 0)$ *be a meromorphic function satisfying*

$$\sigma_{[p+1,q]}(g,\varphi) < \sigma_{[p,q]}(A_0,\varphi).$$

Then every transcendental meromorphic solution $f(z)\,(\not\equiv 0)$ *with* $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ *of equation* (1.1) *satisfies*

$$\overline{\lambda}_{[p+1,q]}(f-g,\varphi) = \lambda_{[p+1,q]}(f-g,\varphi) = \sigma_{[p+1,q]}(f-g,\varphi) = \sigma_{[p,q]}(A_0,\varphi).$$

Now, for the non-homogeneous linear differential equation (1.2), we obtain the following results:

Theorem 2.3. *Let* $A_0(z)$, $A_1(z),\ldots,A_{k-1}(z)$, $A_k(z)$ *and* $F\,(\not\equiv 0)$ *with* $A_0(z)\,(\not\equiv 0)$ *and* $A_k(z)\,(\not\equiv 0)$ *be entire functions. Suppose that* H *and* $A_0(z), A_1(z), \ldots, A_{k-1}(z), A_k(z)$ *satisfy the hypotheses of Theorem 2.1, then the following holds*:

1) *If* $p\geq q\geq 1$ *and* $\sigma_{[p+1,q]}(F,\varphi) < \sigma_{[p,q]}(A_0,\varphi)$, *then every transcendental meromorphic solution* f *of the equation* (1.2) *with* $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ *satisfies* $\overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$ *with at most one exceptional solution* f_0 *satisfying* $\sigma_{[p+1,q]}(f_0,\varphi) < \sigma_{[p,q]}(A_0,\varphi)$.

2) *If* $p\geq q\geq 1$ *and* $\sigma_{[p+1,q]}(F,\varphi) > \sigma_{[p,q]}(A_0,\varphi)$, *then every transcendental meromorphic solution* $f\,(\not\equiv 0)$ *of the equation* (1.2) *with* $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ *satisfies* $\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p+1,q]}(F,\varphi)$.

Theorem 2.4. *Let* $A_0(z), A_1(z),\ldots,A_{k-1}(z)$, $A_k(z)$ *and* $F\,(\not\equiv 0)$ *with* $A_0(z)\,(\not\equiv 0)$ *and* $A_k(z)\,(\not\equiv 0)$ *be entire functions. Suppose that* H *and* $A_0(z), A_1(z), \ldots, A_{k-1}(z), A_k(z)$ *satisfy the hypotheses of Theorem 2.2, then the following holds*:

1) *If* $p\geq q\geq 1$ *and* $\sigma_{[p+1,q]}(F,\varphi) < \sigma_{[p,q]}(A_0,\varphi)$, *then every transcendental meromorphic solution* $f\,(\not\equiv 0)$ *of the equation* (1.2) *with* $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ *satisfies*

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$$

with at most one exceptional solution f_0 *satisfying*

$$\sigma_{[p+1,q]}(f_0,\varphi) < \sigma_{[p,q]}(A_0,\varphi).$$

2) *If* $p \geq q \geq 1$ *and* $\sigma_{[p+1,q]}(F,\varphi) > \sigma_{[p,q]}(A_0,\varphi)$, *then every transcendental meromorphic solution* $f\,(\not\equiv 0)$ *of the equation* (1.2) *with* $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ *satisfies* $\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p+1,q]}(F,\varphi)$.

3. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 3.1 (see [15])**.** *Let* $g : [0,\infty) \to \mathbb{R}$ *and* $h : [0,\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $g(r) \leq h(r)$ *outside of an exceptional set* E_5 *of finite linear measure. Then for any* $\alpha > 1$, *there exists* $r_0 > 0$ *such that* $g(r) \leq h(\alpha r)$ *for all* $r > r_0$.

Lemma 3.2 (see [8])**.** *Let* $f(z)$ *be a meromorphic function. Let* $\alpha > 1$ *and* $\varepsilon > 0$ *be given constants. Then there exist a constant* $B > 0$ *and a set* $E_2 \subset [0,\infty)$ *having finite linear measure such that for all* z *satisfying* $|z| = r \notin E_2$, *we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(\alpha r,f)\, r^{\varepsilon}\log T(\alpha r,f)\right]^{j} \quad (j \in \mathbb{N}).$$

By using the similar proof of Lemma 3.5 in [19], *we easily obtain the following lemma when* $\sigma_{[p,q]}(g,\varphi) = \sigma_{[p,q]}(f,\varphi) = \infty$.

Lemma 3.3. *Let* $f(z) = \frac{g(z)}{d(z)}$ *be a meromorphic function, where* $g(z)$ *and* $d(z)$ *are entire functions satisfying* $\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \leq \sigma_{[p,q]}(g,\varphi) = \sigma_{[p,q]}(f,\varphi) \leq \infty$ *and* $\lambda_{[p,q]}(d,\varphi) = \sigma_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu$. *Then there exists a set* $E_6 \subset (1,\infty)$ *of finite logarithmic measure such that for all* $|z| = r \notin [0,1] \cup E_6$ *and* $|g(z)| = M(r,g)$ *we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^{n}(1+o(1)) \quad (n \in \mathbb{N}),$$

where $\nu_g(r)$ *is the central index of* $g(z)$ *where* φ *satisfies the conditions* (i) *and* (ii) *of Remark 1.2.*

Lemma 3.4 (see [18])**.** *Let* $f(z)$ *be an entire function of* $[p,q]-\varphi$ *order, and let* $\nu_f(r)$ *be the central index of* $f(z)$. *Then*

$$\varlimsup_{r\to\infty}\frac{\log_p \nu_f(r)}{\log_q \varphi(r)} = \sigma_{[p,q]}(f,\varphi).$$

By using the similar proof of Lemma 2.8 in [6]; we can easily obtain the following lemma.

Lemma 3.5. *Let* $f(z)$ *be a meromorphic function such that* $\sigma_{[p,q]}(f,\varphi) = \sigma < \infty$, *then there exists a set* $E_4 \subset (1,\infty)$ *of* r *of finite linear measure such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_4$, $r \to \infty$. *Then for any given* $\varepsilon > 0$, *we have*

$$|f(z)| \leq \exp_p\left(\left(\log_{q-1}\varphi(r)\right)^{\sigma+\varepsilon}\right).$$

By inequalities in [12, Chapter 6] and in [15, Corollary 2.3.5], we obtain the following lemma.

Lemma 3.6 (see [16])**.** *Let $f(z)$ be an meromorphic function of $[p,q]-\varphi$ order. Then*

$$\sigma_{[p,q]}(f,\varphi)=\sigma_{[p,q]}(f',\varphi).$$

Lemma 3.7 (see [3])**.** *Let f_1, f_2 be meromorphic functions of $[p.q]-\varphi$ order satisfying $\sigma_{[p,q]}(f_1,\varphi)>\sigma_{[p,q]}(f_2,\varphi)$, where $\varphi(r)$ only satisfies $\lim\limits_{r\to+\infty}\frac{\log_q\varphi(\alpha r)}{\log_q\varphi(r)}$ $=1$ for some $\alpha>1$. Then there exists a set $E_3\subset(1,+\infty)$ having infinite logarithmic measure such that for all $r\in E_3$, we have*

$$\lim_{r\to+\infty}\frac{T(r,f_2)}{T(r,f_1)}=0.$$

Lemma 3.8. *Let p, q be integers such that $p\geq q\geq 1$, and let $F(z)\,(\not\equiv 0)$, $A_j(z)$ $(j=0,1,\dots,k)$ entire functions, let $f(z)$ be a meromorphic solution of the equation* (1.2) *satisfying*

$$\max\left\{\sigma_{[p,q]}(A_j,\varphi),\ \sigma_{[p,q]}(F,\varphi):j=0,1,\dots,k\right\}<\sigma_{[p,q]}(f,\varphi).$$

Then

$$\overline{\lambda}_{[p,q]}(f,\varphi)=\lambda_{[p,q]}(f,\varphi)=\sigma_{[p,q]}(f,\varphi),$$

where φ satisfies the conditions (i) *and* (ii) *of Remark 1.2.*

Proof. By (1.2) we have

$$\frac{1}{f}=\frac{1}{F}\left(A_k(z)\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\cdots+A_1(z)\frac{f'}{f}+A_0\right). \tag{3.1}$$

It is easy to see that if f has a zero at z_0 of order β $(\beta>k)$ and if $A_0,A_1,\dots,A_k$ are all analytic at z_0, then F has a zero at z_0 of order $\beta-k$. Hence

$$n\left(r,\frac{1}{f}\right)\leq k\overline{n}\left(r,\frac{1}{f}\right)+n\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}n(r,A_j) \tag{3.2}$$

and

$$N\left(r,\frac{1}{f}\right)\leq k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}N(r,A_j). \tag{3.3}$$

By Lemma of the logarithmic derivative and the equation (3.1), we have

$$m\left(r,\frac{1}{f}\right)\leq m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}m(r,A_j)+O(\log T(r,f)+\log r), \tag{3.4}$$

holds for all z satisfying $|z|=r\notin E_3$, where E_3 is a set of finite linear measure.

From (3.1), (3.3) and (3.4) for $|z|=r\notin E_3$, we get that

$$T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)=N\left(r,\frac{1}{f}\right)+m\left(r,\frac{1}{f}\right)+O(1)$$

$$\leq k\overline{N}\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k} T\left(r, A_j\right) + O\left(\log rT\left(r, f\right)\right). \tag{3.5}$$

Since

$$\max\left\{\sigma_{[p,q]}\left(A_j, \varphi\right),\ \sigma_{[p,q]}\left(F, \varphi\right) : j = 0, 1, \ldots, k\right\} < \sigma_{[p,q]}\left(f, \varphi\right).$$

Then using Lemma 3.7, we get that

$$\max\left\{\frac{T\left(r, F\right)}{T\left(r, f\right)}, \frac{T\left(r, A_j\right)}{T\left(r, f\right)} : j = 0, 1, \ldots, k\right\} \to 0 \text{ as } r \to \infty. \tag{3.6}$$

For sufficiently large r, we have

$$O\left(\log r_n T\left(r_n, f\right)\right) = o\left(T\left(r_n, f\right)\right).$$

By (3.5) and (3.6), we obtain that for sufficiently large $r \notin E_3$, there holds

$$\left(1 - o\left(1\right) T\left(r_n, f\right)\right) \leq k\overline{N}\left(r_n, \frac{1}{f}\right).$$

Then by Definition we get that

$$\sigma_{[p,q]}\left(f, \varphi\right) \leq \overline{\lambda}_{[p,q]}\left(f, \varphi\right).$$

Again, by Definition we have

$$\overline{\lambda}_{[p,q]}\left(f, \varphi\right) \leq \lambda_{[p,q]}\left(f, \varphi\right) \leq \sigma_{[p,q]}\left(f, \varphi\right).$$

Hence

$$\overline{\lambda}_{[p,q]}\left(f, \varphi\right) = \lambda_{[p,q]}\left(f, \varphi\right) = \sigma_{[p,q]}\left(f, \varphi\right).$$

This proves the lemma. □

4. Proof of Theorems and Corollaries

Proof of Theorem 2.1. Let $f\,(\not\equiv 0)$ be a transcendental meromorphic solution of the equation (1.1) with $\lambda_{[p,q]}\left(\frac{1}{f}, \varphi\right) < \mu_{[p,q]}\left(f, \varphi\right)$. From equation (1.1), we have

$$\left|A_0\left(z\right)\right| \leq \sum_{j=1}^{k} \left|A_j\left(z\right)\right| \left|\frac{f^{(j)}\left(z\right)}{f}\right|. \tag{4.1}$$

By Lemma 3.2, there exist a constant $B > 0$ and a set $E_2 \subset [0, \infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_2$, we have

$$\left|\frac{f^{(j)}\left(z\right)}{f\left(z\right)}\right| \leq Br\left[T\left(2r, f\right)\right]^{k+1} \qquad \left(j = 1, 2, \ldots, k\right). \tag{4.2}$$

By the hypotheses of theorem there exists a set H with $\overline{\log dens}\left\{|z| : z \in H\right\} > 0$ such that for all $z \in H$ as $z \to \infty$, we have

$$\left|A_0\left(z\right)\right| \geq \exp_p\left\{\alpha\left(\log_{q-1}\varphi\left(r\right)\right)^{\sigma_{[p,q]}\left(A_0, \varphi\right) - \varepsilon}\right\}, \tag{4.3}$$

and

$$|A_j(z)| \leq \exp_p \left\{ \beta \left(\log_{q-1} \varphi(r) \right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon} \right\} \quad (j = 1, 2, \ldots, k). \tag{4.4}$$

Let $H_1 = \{|z| : z \in H\} \backslash E_2$, so that $\overline{\log dens}\{|z| : z \in H_1\} > 0$. It follows from (4.1), (4.2), (4.3) and (4.4)

$$\begin{aligned}\exp_p \left\{ \alpha \left(\log_{q-1} \varphi(r) \right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon} \right\} &\leq Brk \left[T(2r, f) \right]^{k+1} \\ &\cdot \exp_p \left\{ \beta \left(\log_{q-1} \varphi(r) \right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon} \right\}\end{aligned} \tag{4.5}$$

or

$$\exp \left\{ (1 - o(1)) \exp_{p-1} \left\{ \alpha \left(\log_{q-1} \varphi(r) \right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon} \right\} \right\} \leq Bkr \left[T(2r, f) \right]^{k+1}$$

as $|z| \to \infty$, $|z| = r \in H_1$.

By help of Lemma 3.1 and (4.5) we obtained

$$\sigma_{[p+1,q]}(f, \varphi) \geq \sigma_{[p,q]}(A_0, \varphi). \tag{I}$$

We can rewrite equation (1.1) as

$$f^{(k)} + \frac{A_{k-1}(z)}{A_k(z)} f^{(k-1)} + \cdots + \frac{A_1(z)}{A_k(z)} f' + \frac{A_0(z)}{A_k(z)} f = 0, \quad A_k(z) (\not\equiv 0).$$

Obviously, poles of f can only occur at the zeros of $A_k(z)$. Note that the multiplicity of the poles of f is uniformly bounded, and thus we have $\lambda_{[p,q]}\left(\frac{1}{f}, \varphi\right) < \sigma_{[p,q]}(A_0, \varphi) < \infty$.

Again, by the Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu_{[p,q]}(g, \varphi) = \mu_{[p,q]}(f, \varphi) = \mu \leq \sigma_{[p,q]}(g, \varphi) = \sigma_{[p,q]}(f, \varphi) \leq \infty$$

and

$$\lambda_{[p,q]}(d, \varphi) = \sigma_{[p,q]}(d, \varphi) = \lambda_{[p,q]}\left(\frac{1}{f}, \varphi\right) < \mu.$$

By Lemma 3.3 there exists a set $E_6 \subset (1, \infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_6$ and $|g(z)| = M(r, g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z} \right)^j (1 + o(1)) \quad (j = 1, 2, \ldots, k). \tag{4.6}$$

Again by Definition 1.1, for any given $\varepsilon > 0$ and for sufficiently large r, we have

$$|A_j(z)| \leq \exp_p \left\{ \left(\log_{q-1} \varphi(r) \right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon} \right\} \quad (j = 0, 1, \ldots, k-1). \tag{4.7}$$

And by Lemma 3.5, for the above $\varepsilon > 0$, there exists a set $E_4 \subset [1, \infty)$ of r of finite linear measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4$, $r \to \infty$, we have

$$\left| \frac{1}{A_k(z)} \right| \leq \exp_p \left(\left(\log_{q-1} \varphi(r) \right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon} \right). \tag{4.8}$$

Also, we can rewrite equation (1.1) as

$$-A_k(z)\frac{f^{(k)}}{f}=A_{k-1}(z)\frac{f^{(k-1)}}{f}+\cdots+A_1(z)\frac{f'}{f}+A_0(z)\,. \tag{4.9}$$

Substituting (4.6), (4.7) and (4.8) into (4.9), we obtain

$$\left(\frac{1}{\exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\}}\right)\left|\frac{\nu_g(r)}{z}\right|^k|1+o(1)|$$
$$\leq k\exp_p\left\{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)+\varepsilon}\right\}\left|\frac{\nu_g(r)}{z}\right|^{k-1}|1+o(1)|, \tag{4.10}$$

where $|z|=r\notin[0,1]\cup E_6\cup E_4$, $r\to\infty$ and $|g(z)|=M(r,g)$.

By Lemma 3.1 and (4.10), we obtain

$$\limsup_{r\to\infty}\frac{\log_{p+1}\nu_g(r)}{\log_q\varphi(r)}\leq\sigma_{[p,q]}(A_0,\varphi)\,. \tag{4.11}$$

By Lemma 3.4, and (4.11) we obtain

$$\sigma_{[p+1,q]}(f,\varphi)\leq\sigma_{[p,q]}(A_0,\varphi)\,. \tag{II}$$

Hence from (I) and (II), we have

$$\sigma_{[p+1,q]}(f,\varphi)=\sigma_{[p,q]}(A_0,\varphi)\,.$$

This proves the theorem. □

Proof of Corollary 2.1. Suppose $h=f-g$ such that

$$\sigma_{[p+1,q]}(g,\varphi)<\sigma_{[p,q]}(A_0,\varphi)\,.$$

By Theorem 2.1, we have $\sigma_{[p+1,q]}(f,\varphi)=\sigma_{[p,q]}(A_0,\varphi)$. Using the properties of $[p,q]-\varphi$ order, we have

$$\sigma_{[p+1,q]}(h,\varphi)=\sigma_{[p+1,q]}(f,\varphi)=\sigma_{[p,q]}(A_0,\varphi)\,.$$

Substituting $f=h+g$ into equation (1.1), we obtain

$$\begin{aligned}&A_k(z)h^{(k)}+A_{k-1}(z)h^{(k-1)}+\cdots+A_1(z)h'+A_0(z)h\\=&-\left[A_k(z)g^{(k)}+A_{k-1}(z)g^{(k-1)}+\cdots+A_1(z)g'+A_0(z)g\right].\end{aligned}$$

Let

$$F(z)=-\left[A_k(z)g^{(k)}+A_{k-1}(z)g^{(k-1)}+\cdots+A_1(z)g'+A_0(z)g\right].$$

If $F(z)\equiv 0$, by the first part of Theorem 2.1, we get

$$\sigma_{[p+1,q]}(g,\varphi)\geq\sigma_{[p,q]}(A_0,\varphi)$$

but it contradict with $\sigma_{[p+1,q]}(g,\varphi)<\sigma_{[p,q]}(A_0,\varphi)$.

Thus, $F(z)\not\equiv 0$ and

$$\sigma_{[p+1,q]}(F,\varphi)\leq\sigma_{[p+1,q]}(g,\varphi)<\sigma_{[p,q]}(A_0,\varphi)=\sigma_{[p+1,q]}(f,\varphi)\,.$$

Therefore, we have

$$\max\left\{\sigma_{[p+1,q]}\left(A_j,\varphi\right),\sigma_{[p+1,q]}\left(F,\varphi\right):j=0,1,\ldots,k\right\}<\sigma_{[p+1,q]}\left(f,\varphi\right).$$

By Lemma 3.8, we obtain

$$\overline{\lambda}_{[p+1,q]}\left(h,\varphi\right)=\lambda_{[p+1,q]}\left(h,\varphi\right)=\sigma_{[p+1,q]}\left(h,\varphi\right).$$

Hence,

$$\overline{\lambda}_{[p+1,q]}\left(f-g,\varphi\right)=\lambda_{[p+1,q]}\left(f-g,\varphi\right)=\sigma_{[p+1,q]}\left(f-g,\varphi\right)=\sigma_{[p,q]}\left(A_0,\varphi\right).$$

This completes the proof of the corollary. □

Proof of Theorem 2.2. Suppose that $f\,(\not\equiv 0)$ be a transcendental meromorphic solution of equation (1.1) with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right)<\mu_{[p,q]}\left(f,\varphi\right)$. From the equation (1.1), it follows that

$$|A_0(z)|\leq\sum_{j=1}^{k}|A_j(z)|\left|\frac{f^{(j)}(z)}{f}\right|. \tag{4.12}$$

By Lemma 3.2, there exist a constant $B>0$ and a set $E_2\subset[0,\infty)$ having finite linear measure such that for all z satisfying $|z_n|=r_n\notin E_2$, we have

$$\left|\frac{f^{(j)}(z_n)}{f(z_n)}\right|\leq Br_n\left[T\left(2r_n,f\right)\right]^{k+1}\qquad(j=1,2,\ldots,k). \tag{4.13}$$

By the hypotheses of Theorem there exists a set H with $\overline{\log dens}\left\{|z_n|:z_n\in H\right\}>0$ such that for all $z_n\in H$ as $n\to\infty$, we have

$$|A_0(z_n)|\geq\exp_p\left\{\alpha\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\}, \tag{4.14}$$

and

$$|A_j(z_n)|\leq\exp_p\left\{\beta\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\}\quad(j=1,2,\ldots,k). \tag{4.15}$$

Set $H_1=\left\{|z_n|:z_n\in H\right\}\backslash E_2$, so that $\overline{\log dens}\left\{|z_n|:z_n\in H_1\right\}>0$. It follows from (4.13), (4.14), (4.15) and (4.12), we have

$$\begin{aligned}\exp_p\left\{\alpha\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\}&\leq Br_nk\left[T\left(2r_n,f\right)\right]^{k+1}\\ &\quad\cdot\exp_p\left\{\beta\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\},\end{aligned} \tag{4.16}$$

$$\exp\left\{(1-o(1))\exp_{p-1}\left\{\alpha\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(A_0,\varphi)-\varepsilon}\right\}\right\}\leq Bkr_n\left[T\left(2r_n,f\right)\right]^{k+1}$$

for all z_n satisfying $|z_n|=r_n\in H_1, r_n\to\infty$. By Lemma 3.1 and (4.16) we obtained

$$\sigma_{[p+1,q]}\left(f,\varphi\right)\geq\sigma_{[p,q]}\left(A_0,\varphi\right).$$

On the other hand we replace z by sequence $\{z_n\}$ in the same arguments as in proof of the last part of Theorem 2.1, as the similar way, we can get $\sigma_{[p+1,q]}\left(f,\varphi\right)\leq\sigma_{[p,q]}\left(A_0,\varphi\right)$.

Hence

$$\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi).$$

This proves the theorem. □

Proof of Corollary 2.2. Similarly as the proof of Corollary 2.1, one can easily prove Corollary 2.2 in the line Theorem 2.2. □

Proof of Theorem 2.3. 1) Suppose that $f\,(\not\equiv 0)$ is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicities of equation (1.2) and $f_1, f_2, \ldots, f_k$ are solution base of the corresponding homogeneous equation (1.1) of equation (1.2).

By Theorem 2.1, we have $\sigma_{[p+1,q]}(f_j,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$ $(j = 1, 2, \ldots, k)$. Then f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z) + \cdots + B_k(z) f_k(z), \tag{4.17}$$

where $B_1(z), B_2(z), \ldots, B_k(z)$ are suitable meromorphic functions satisfying

$$B_j'(z) = F \cdot G_j(f_1, f_2, \ldots, f_k) \cdot W(f_1, f_2, \ldots, f_k)^{-1} \quad (j = 1, 2, \ldots, k), \tag{4.18}$$

where $G_j(f_1, f_2, \ldots, f_k)$ are differential polynomials in $f_1, f_2, \ldots, f_k$ with constant coefficients, and the Wronskian $W(f_1, f_2, \ldots, f_k)$ is also a differential polynomial in $f_1, f_2, \ldots, f_k$ with constant coefficients. Thus by using Theorem 2.1, we deduce that

$$\sigma_{[p+1,q]}(W,\varphi) \le \max\left\{\sigma_{[p+1,q]}(f_j,\varphi) : j = 1, 2, \ldots, k\right\} = \sigma_{[p,q]}(A_0,\varphi) \tag{4.19}$$

and

$$\begin{aligned} \sigma_{[p+1,q]}(G_j,\varphi) &\le \max\left\{\sigma_{[p+1,q]}(f_j,\varphi) : j = 1, 2, \ldots, k\right\} \\ &= \sigma_{[p,q]}(A_0,\varphi) \;\; (j = 1, 2, \ldots, k). \end{aligned}$$

Therefore we get

$$\sigma_{[p+1,q]}(G_j,\varphi) \le \sigma_{[p,q]}(A_0,\varphi) \;\; (j = 1, 2, \ldots, k). \tag{4.20}$$

By Lemma 3.6, (4.18), (4.19) and (4.20) for $j = 1, 2, \ldots, k$,

$$\begin{aligned} \sigma_{[p+1,q]}(B_j,\varphi) &= \sigma_{[p+1,q]}\left(B_j',\varphi\right) \\ &\le \max\left\{\sigma_{[p+1,q]}(F,\varphi), \sigma_{[p,q]}(A_0,\varphi)\right\} \\ &= \sigma_{[p,q]}(A_0,\varphi). \end{aligned} \tag{4.21}$$

From the equations (4.17) and (4.21), we obtain

$$\begin{aligned} \sigma_{[p+1,q]}(f,\varphi) &\le \max\left\{\sigma_{[p+1,q]}(f_j,\varphi), \sigma_{[p+1,q]}(B_j,\varphi) : j = 1, 2, \ldots, k\right\} \\ &= \sigma_{[p,q]}(A_0,\varphi). \end{aligned} \tag{4.22}$$

Now, we assert that every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicity of equation (1.2) satisfies $\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$, with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0,\varphi)$

$< \sigma_{[p,q]}(A_0,\varphi)$. In fact, if f^* is another transcendental meromorphic solution with $\sigma_{[p+1,q]}(f^*,\varphi) < \sigma_{[p,q]}(A_0,\varphi)$ of the equation (1.2), then

$$\sigma_{[p+1,q]}(f_0 - f^*,\varphi) < \sigma_{[p,q]}(A_0,\varphi).$$

But $f_0 - f^*$ is a transcendental meromorphic solution of the corresponding homogeneous equation (1.1) of equation (1.2). This contradicts Theorem 2.1. Therefore $\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$ holds for all transcendental meromorphic solutions of equation (1.2) with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0,\varphi) < \sigma_{[p,q]}(A_0,\varphi)$.

So

$$\max\left\{\sigma_{[p+1,q]}(A_j,\varphi), \sigma_{[p+1,q]}(F,\varphi) : j = 0,1,2,\ldots,k\right\} < \sigma_{[p+1,q]}(f,\varphi).$$

By Lemma 3.8, we have

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \sigma_{[p+1,q]}(f,\varphi).$$

Therefore

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$$

with at most one exceptional solution f_0 satisfying

$$\sigma_{[p+1,q]}(f_0,\varphi) < \sigma_{[p,q]}(A_0,\varphi).$$

2) By Lemma 3.6, (4.18), (4.19) and (4.20) for $j = 1,2,\ldots,k$, we have

$$\begin{aligned}\sigma_{[p+1,q]}(B_j,\varphi) &= \sigma_{[p+1,q]}\left(B_j',\varphi\right)\\ &\le \max\left\{\sigma_{[p+1,q]}(F,\varphi), \sigma_{[p,q]}(f_j,\varphi) : j = 1,\ldots,k\right\}\\ &= \sigma_{[p+1,q]}(F,\varphi).\end{aligned}$$

Therefore we get

$$\sigma_{[p+1,q]}(B_j,\varphi) \le \sigma_{[p+1,q]}(F,\varphi). \tag{4.23}$$

From (4.17) and (4.23), we obtain

$$\begin{aligned}\sigma_{[p+1,q]}(f,\varphi) &\le \max\left\{\sigma_{[p+1,q]}(B_j,\varphi), \sigma_{[p,q]}(f_j,\varphi) : j = 1,2,\ldots,k\right\}\\ &= \sigma_{[p+1,q]}(F,\varphi). \end{aligned} \tag{4.24}$$

If $p \ge q \ge 1$ and $\sigma_{[p+1,q]}(F,\varphi) > \sigma_{[p,q]}(A_0,\varphi)$, it follows from the equation (1.2) that a simple consideration of $[p,q]-\varphi$ order implies $\sigma_{[p+1,q]}(f,\varphi) \ge \sigma_{[p+1,q]}(F,\varphi)$.

By this inequality and the fact (4.24) we thus obtain

$$\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p+1,q]}(F,\varphi).$$

This completes the proof of the theorem. □

Proof of Theorem 2.4. By using similar reasoning of Theorem 2.3, we can prove Theorem 2.4. □

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