# $k^{t h}$-ORDER ESSENTIALLY SLANT WEIGHTED TOEPLITZ OPERATORS 

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#### Abstract

The notion of $k^{t h}$-order essentially slant weighted Toeplitz operator on the weighted Lebesgue space $L^{2}(\beta)$ is introduced and its algebraic properties are investigated. In addition, the compression of $k^{t h}$-order essentially slant weighted Toeplitz operators on the weighted Hardy space $H^{2}(\beta)$ is also studied


## 1. Introduction

Let $\beta=\left(\beta_{n}\right)_{n \in \mathbb{Z}}$ be a sequence of positive numbers with $\beta_{0}=1$ and $0<\frac{\beta_{n}}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $0<\frac{\beta_{n}}{\beta_{n-1}} \leq 1$ for $n<0$. Consider the space $L^{2}(\beta)=\left\{f(z)=\sum_{n=-\infty}^{\infty} a_{n} z^{n} ; a_{n} \in \mathbb{C},\|f\|_{\beta}^{2}=\sum_{n=-\infty}^{\infty}\left|a_{n}\right|^{2} \beta_{n}^{2}<\infty\right\}$. The space $L^{2}(\beta)$ is a Hilbert space with the norm $\|\cdot\|_{\beta}$ induced by the inner product

$$
\langle f, g\rangle=\sum_{n=-\infty}^{\infty} a_{n} \overline{b_{n}} \beta_{n}^{2}
$$

for $f(z)=\sum_{n=-\infty}^{\infty} a_{n} z^{n}$ and $g(z)=\sum_{n=-\infty}^{\infty} b_{n} z^{n}$ in $L^{2}(\beta)$. The collection $\left\{e_{k}(z)=\frac{z^{k}}{\beta_{k}}\right\}_{k \in \mathbb{Z}}$ forms an orthonormal basis for $L^{2}(\beta)$. The space $H^{2}(\beta)$ is the collection of all those power series in $L^{2}(\beta)$ which are analytic, that is, $H^{2}(\beta)=\left\{f(z)=\sum_{n=0}^{\infty} a_{n} z^{n} ; a_{n} \in \mathbb{C},\|f\|_{\beta}^{2}=\sum_{n=0}^{\infty}\left|a_{n}\right|^{2} \beta_{n}^{2}<\infty\right\}$. The space $H^{2}(\beta)$ is a subspace of $L^{2}(\beta)$ and the collection $\left\{e_{k}(z)=\frac{z^{k}}{\beta_{k}}\right\}_{k \geq 0}$ forms an orthonormal basis for $H^{2}(\beta)$.

Let $L^{\infty}(\beta)=\left\{\phi(z)=\sum_{n=-\infty}^{\infty} a_{n} z^{n}\right.$ such that $\phi L^{2}(\beta) \subseteq L^{2}(\beta)$ satisfying $\|\phi f\|_{\beta} \leq c\|f\|_{\beta}$ for some $c>0$ and for all $\left.f \in L^{2}(\beta)\right\}$. Then $L^{\infty}(\beta)$ is a Banach space with respect to norm defined by

$$
\|\phi\|_{\infty}=\inf \left\{c:\|\phi f\|_{\beta} \leq c\|f\|_{\beta} \text { for all } f \in L^{2}(\beta)\right\} .
$$

[^0]Let $H^{\infty}(\beta)=\left\{\phi(z)=\sum_{n=0}^{\infty} a_{n} z^{n}\right.$ such that $\phi H^{2}(\beta) \subseteq H^{2}(\beta)$ satisfying $\|\phi f\|_{\beta} \leq c\|f\|_{\beta}$ for some $c>0$ and for all $\left.f \in H^{2}(\beta)\right\}$. One can refer [15] and the references therein for the details of the spaces $L^{2}(\beta), H^{2}(\beta), L^{\infty}(\beta)$ and $H^{\infty}(\beta)$. If for each $n, \beta_{n}=1$ and if we consider the functions as the complex valued measurable functions over the unit circle $\mathbb{T}$, then these weighted spaces are nothing but the spaces $L^{2}(\mathbb{T}), H^{2}(\mathbb{T}), L^{\infty}(\mathbb{T})$ and $H^{\infty}(\mathbb{T})$. For the detailed information of these spaces one can see [9]. Around the year 1966, the notions of the weighted sequence spaces $H^{2}(\beta)$ and $L^{2}(\beta)$ were brought forth by R. L. Kelley [13] during his systematic study of weighted shift operators. Further Shields [15] extended the study of multiplication operators over the space $L^{2}(\beta)$ and that made the study interesting as the space $L^{2}(\beta)$ cover spaces such as Bergman spaces, Hardy spaces and Dirichlet spaces. Barria and Halmos in [6] studied the class of essentially Toeplitz operators which are basically the essential commutant of the unilateral shift. In 1996, M. C. Ho. [11, 12] introduced and studied the class of slant Toeplitz operators having the property that the matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Laurent operators. Later on Arora and his research associates [3,5] introduced and studied the notion of slant weighted Toeplitz operators on the spaces $L^{2}(\beta)$ and $H^{2}(\beta)$. Lauric [14] in the year 2004, studied particular cases of weighted Toeplitz operators on the space $H^{2}(\beta)$ and obtain the description of its commutant. Many other invariants of Toeplitz and weighted Toeplitz operators are further studied by many mathematicians. Recently, Datt and his research associate [7] studied the notion of essentially generalized $\lambda$-slant Toeplitz operators on the Hilbert space $L^{2}$ for general complex number $\lambda$. In [10], the notion of slant H-Toeplitz operator on the Hardy space is introduced, which connects with the study of slant Toeplitz and slant Hankel operators.

Let $P^{\beta}: L^{2}(\beta) \rightarrow H^{2}(\beta)$ be the orthogonal projection of $L^{2}(\beta)$ onto $H^{2}(\beta)$. The operator $M_{z}^{\beta}: L^{2}(\beta) \rightarrow L^{2}(\beta)$ defined by $M_{z}^{\beta} e_{n}(z)=w_{n} e_{n+1}(z)$ where $w_{n}=\frac{\beta_{n+1}}{\beta_{n}}$ for all $n \in \mathbb{Z}$, is known as weighted shift operator. The sequence $\left(w_{n}\right)$ is usually called weight sequence. For a given $\phi \in L^{\infty}(\beta)$, the induced weighted multiplication operator (also known as the weighted Laurent operator) is denoted by $M_{\phi}^{\beta}$ and given by $M_{\phi}^{\beta}: L^{2}(\beta) \rightarrow L^{2}(\beta)$ such that $M_{\phi}^{\beta} f=$ $\phi f$ for all $f$ in $L^{2}(\beta)$. A weighted Toeplitz operator $T_{\phi}^{\beta}$ on the space $L^{2}(\beta)$ is nothing but the operator that is in the commutant of weighted multiplication operators, that is, an operator $A$ on $L^{2}(\beta)$ is a weighted Toeplitz operator if it satisfy $A M_{z}^{\beta}=M_{z}^{\beta} A$. For detailed study of weighted Toeplitz operator one can refer [4] and the reference therein. An operator $A$ on the space $L^{2}(\beta)$ is an essentially weighted Toeplitz operator if the operator $M_{z}^{\beta} A-A M_{z}^{\beta}$ is compact on $L^{2}(\beta)$.

Through out the paper we assume that $k \geq 2$ is an integer and the sequence $\beta_{n}$ satisfies an extra condition $\sup _{n}\left|\frac{\beta_{n+1}}{\beta_{n}}\right|<\infty$. The $k^{\text {th }}$-order slant weighted

Toeplitz operators $A_{\phi}^{k^{\beta}}$, with the symbol $\phi$, on the space $L^{2}(\beta)$ is defined as $A_{\phi}^{k^{\beta}}=W_{k} M_{\phi}^{\beta}$, where $W_{k}$ is the operator on $L^{2}(\beta)$ given by

$$
W_{k} e_{n}(z)= \begin{cases}\frac{\beta_{n / k}}{\beta_{n}} e_{n / k} & \text { if } n \text { is divisible by } k \\ 0 & \text { otherwise }\end{cases}
$$

for $n \in \mathbb{Z}$. The adjoint of the operator $W_{k}$ is given by $W_{k}^{*}\left(e_{n}\right)=\frac{\beta_{k n}}{\beta_{n}} e_{k n}$. In particular, for $k=2$, the operator $A_{\phi}^{k^{\beta}}$ becomes slant weighted Toeplitz operator on the space $L^{2}(\beta)$ (see [3]). A bounded linear operator $T$ on a Hilbert space is said to be a Fredholm operator (see [1]) if Range $(T)$ is closed and $\operatorname{Ker}(T)$ and $\operatorname{Ker}\left(T^{*}\right)$ are finite dimensional. For a Fredholm operator $T$, its index, denoted as $\operatorname{ind}(T)$, is given by $\operatorname{dim}(\operatorname{ker}(T))-\operatorname{dim}\left(\operatorname{ker}\left(T^{*}\right)\right)$. The essential spectrum of the operator $T$, denoted by $\sigma_{e}(T)$, is defined as $\sigma_{e}(T)=\{\lambda \in \mathbb{C}$ : $T-\lambda I$ is not a Fredholm operator $\}$.

Motivated by the recent progress of the spaces $L^{2}(\beta)$ and $H^{2}(\beta)$ and the multi-directional applications of the Toeplitz operators, we further extended the study of Toeplitz operators by introducing the notion of $k^{t h}$-order essentially slant weighted Toeplitz operator and studied its algebraic properties. In particular, the conditions under which the set of essentially $k^{t h}$-order slant weighted Toeplitz operator on $L^{2}(\beta)$ is closed with respect to multiplication are obtained. We have also studied the compression of $k^{t h}$-order essentially slant weighted Toeplitz operators on the weighted Hardy space $H^{2}(\beta)$. Moreover, it has been shown that the Fredholm operators can never be a $k^{t h}$-order essentially slant weighted Toeplitz operator on $H^{2}(\beta)$.

## 2. $k^{t h}$-order essentially slant weighted Toeplitz operator on $L^{2}(\beta)$

In 2009, Arora and his research associates [2] introduced the notion of essentially slant Toeplitz operators and studied its properties. Arora in [5] gave the following characterization for the class of $k^{t h}$-order slant weighted Toeplitz operators:
Theorem 2.1. A bounded operator $A$ on $L^{2}(\beta)$ is a $k^{\text {th }}$-order slant weighted Toeplitz operator on $L^{2}(\beta)$ if and only if $A M_{z^{k}}^{\beta}=M_{z}^{\beta} A$, where $M_{z}^{\beta}$ and $M_{z^{k}}^{\beta}$ are weighted multiplication operators induced by $z$ and $z^{k}$ respectively.

Motivated by the characterization of the class of $k^{\text {th }}$-order slant weighted Toeplitz operators, in this section we introduce the notion of $k^{t h}$-order essentially slant weighted Toeplitz operators. Let $\mathcal{K}\left(L^{2}(\beta)\right)$ denote the set of all compact operators on the weighted space $L^{2}(\beta)$.
Definition. A bounded linear operator $A$ on $L^{2}(\beta)$ is a $k^{t h}$-order essentially slant weighted Toeplitz operator if $M_{z}^{\beta} A-A M_{z^{k}}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)$.

We denote the set of all $k^{\text {th }}$-order essentially slant weighted Toeplitz operator on $L^{2}(\beta)$ by $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$.

Proposition 2.2. Let $A$ be a bounded linear operator on $L^{2}(\beta)$. Then the following are equivalent:
(i) $A \in W-E S T O_{k}\left(L^{2}(\beta)\right)$.
(ii) $A-M_{z^{-1}}^{\beta} A M_{z^{k}}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)$.
(iii) $A-M_{z}^{\beta} A M_{z^{k-1}}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)$.

Proposition 2.3. The identity operator can not be a $k^{\text {th }}$-order essentially slant weighted Toeplitz operator on $L^{2}(\beta)$.

Proof. Let $I$ be the identity operator on $L^{2}(\beta)$. Then for each integer $n$, we have

$$
\left(M_{z}^{\beta} I-I M_{z^{k}}^{\beta}\right)\left(e_{n}(z)\right)=\frac{\beta_{n+1}}{\beta_{n}} e_{n+1}(z)-\frac{\beta_{n+k}}{\beta_{n}} e_{n+k}(z) .
$$

Hence, the operator $M_{z}^{\beta} I-I M_{z^{k}}^{\beta}$ is not compact on $L^{2}(\beta)$ and therefore identity operator $I$ is not a $k^{t h}$-order essentially slant weighted Toeplitz operator on $L^{2}(\beta)$.

We denote the set of all $k^{\text {th }}$-order slant weighted Toeplitz operators on $L^{2}(\beta)$ by $\mathrm{W}-\mathrm{STO}_{k}\left(L^{2}(\beta)\right)$. Since the zero operator on $L^{2}(\beta)$ is a compact operator, therefore definitions of the classes W-STO $k k$ ( $\left.L^{2}(\beta)\right)$ and W-ESTO ${ }_{k}\left(L^{2}(\beta)\right)$ gives that $\mathrm{W}-\mathrm{STO}_{k}\left(L^{2}(\beta)\right) \subset \mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. That is, every $k^{\text {th }}$-order slant weighted Toeplitz operator is trivially in the set W-ESTO $k\left(L^{2}(\beta)\right)$. In fact, if $A$ is any compact perturbation of a $k^{t h}$-order slant weighted Toeplitz operator on $L^{2}(\beta)$, then $A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. It is known that the only compact $k^{t h}$-order slant weighted Toeplitz operator is the zero operator. Therefore, it follows that

$$
{\mathrm{W}-\mathrm{STO}_{k}\left(L^{2}(\beta)\right) \cap \mathcal{K}\left(L^{2}(\beta)\right)=\{0\} . ~}_{\text {. }}
$$

Also, from the definition of $k^{t h}$-order essentially slant weighted Toeplitz operator on $L^{2}(\beta)$, it follows that every compact operator on $L^{2}(\beta)$ is in W$\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$ and hence

$$
\mathrm{W-ESTO}_{k}\left(L^{2}(\beta)\right) \cap \mathcal{K}\left(L^{2}(\beta)\right)=\mathcal{K}\left(L^{2}(\beta)\right) .
$$

Thus, every non-zero compact operator on $L^{2}(\beta)$ is in the class

$$
{\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)}^{2}
$$

but not in $\mathrm{W}-\mathrm{STO}_{k}\left(L^{2}(\beta)\right)$. In the following example, we construct a noncompact operator in the class $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ but not in the class

$$
{\mathrm{W}-\mathrm{STO}_{k}\left(L^{2}(\beta)\right) .}
$$

Example 2.4. Consider an operator $A$ on $L^{2}(\beta)$ defined by

$$
A\left(e_{n}\right)= \begin{cases}e_{0}+e_{1} & \text { if } n=0 \\ \frac{\beta_{n} / k}{\beta_{n}} e_{n / k} & \text { if } n \neq 0 \text { and } n \text { is a multiple of } k, \\ 0 & \text { otherwise },\end{cases}
$$

where $e_{n}(z)=\frac{z^{n}}{\beta_{n}}$ for all $n \in \mathbb{Z}$.
The matrix representation of $A$ with respect to orthonormal basis $\left\{e_{n}\right\}$ of $L^{2}(\beta)$ is given by

$$
\left[\begin{array}{cccccc|ccccccc} 
& \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
\vdots & \\
& \frac{\beta-2}{\beta-2 k} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & \cdots & 0 & \frac{\beta_{-1}}{\beta_{-k}} \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\
\cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\
\hline \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & & \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \frac{\beta_{1}}{\beta_{k}} & 0 & \cdots \\
0 & \cdots \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
\frac{\beta_{2}}{\beta_{2 k}} & \cdots \\
& \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots
\end{array}\right] .
$$

From the above matrix, it follows that the operator $A$ is not in the class $\mathrm{W}-\mathrm{STO}_{k}\left(L^{2}(\beta)\right)$. In order to show that the operator $A$ is in the class W $\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$, we define the operators $K$ and $S$ on $L^{2}(\beta)$ as follow:

$$
K\left(e_{n}\right)= \begin{cases}e_{1} & \text { if } n=0 \\ 0 & \text { otherwise }\end{cases}
$$

and

$$
S\left(e_{n}\right)= \begin{cases}e_{1} & \text { if } n \text { is not a multiple of } k \\ e_{n} & \text { if } n \text { is a multiple of } k\end{cases}
$$

Recall that the operator $W_{k}$ on $L^{2}(\beta)$, is defined by

$$
W_{k} e_{n}(z)= \begin{cases}\left(\frac{\beta_{n / k}}{\beta_{n}}\right) e_{n / k} & \text { if } k \text { divides } n \\ 0 & \text { otherwise }\end{cases}
$$

is a bounded linear operator with $\left\|W_{k}\right\|_{\beta}=\sup _{n}\left|\frac{\beta_{n}}{\beta_{k n}}\right|$. Thus, we can write the operator $A=W_{k} S+K$. Moreover,

$$
\begin{aligned}
M_{z}^{\beta} A-A M_{z^{k}}^{\beta} & =M_{z}^{\beta}\left(W_{k} S+K\right)-\left(W_{k} S+K\right) M_{z^{k}}^{\beta} \\
& =M_{z}^{\beta} W_{k} S+M_{z}^{\beta} K-W_{k} S M_{z^{k}}^{\beta}-K M_{z^{k}}^{\beta} \\
& =\left(M_{z}^{\beta} W_{k} S-W_{k} S M_{z^{k}}^{\beta}\right)+K^{\prime} \text { for some } K^{\prime} \in \mathcal{K}\left(L^{2}(\beta)\right) .
\end{aligned}
$$

We consider the following two cases:
Case 1. If $n$ is not a multiple of $k$, then

$$
\begin{aligned}
\left(M_{z}^{\beta} W_{k} S-W_{k} S M_{z^{k}}^{\beta}\right)\left(e_{n}(z)\right) & =M_{z}^{\beta} W_{k} S\left(e_{n}(z)\right)-W_{k} S M_{z^{k}}^{\beta}\left(e_{n}(z)\right) \\
& =M_{z}^{\beta} W_{k}\left(e_{1}(z)\right)-\frac{\beta_{n+k}}{\beta_{n}} W_{k} S\left(e_{n+k}(z)\right) \\
& =0\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right)
\end{aligned}
$$

Case 2. If $n$ is a multiple of $k$, then

$$
\begin{aligned}
\left(M_{z}^{\beta} W_{k} S-W_{k} S M_{z^{k}}^{\beta}\right)\left(e_{n}(z)\right) & =M_{z}^{\beta} W_{k}\left(e_{n}(z)\right)-\frac{\beta_{n+k}}{\beta_{n}} W_{k} S\left(e_{n+k}(z)\right) \\
& =\frac{\beta_{n / k}}{\beta_{n}} M_{z}^{\beta}\left(e_{n / k}(z)\right)-\frac{\beta_{n+k}}{\beta_{n}} W_{k}\left(e_{n+k}(z)\right) \\
& =\frac{1}{\beta_{n}} z^{(n+k) / k}-\frac{\beta_{(n+k) / k}}{\beta_{n}}\left(e_{(n+k) / k}(z)\right) \\
& =\frac{1}{\beta_{n}}\left(z^{\frac{n+k}{k}}-z^{\frac{n+k}{k}}\right)=0\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) .
\end{aligned}
$$

Thus, it follows that, $\left(M_{z}^{\beta} W S-W S M_{z^{k}}^{\beta}\right)\left(e_{n}(z)\right)=0\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right)$ for all $n \in \mathbb{Z}$ and from the equation (1), we get that $M_{z}^{\beta} A-A M_{z^{k}}^{\beta}$ is in $\mathcal{K}\left(L^{2}(\beta)\right)$. Hence, the operator $A$ is in W-ESTO ${ }_{k}\left(L^{2}(\beta)\right)$ but not in W-STO ${ }_{k}\left(L^{2}(\beta)\right)$.
Proposition 2.5. If $k_{1}$ and $k_{2}$ are integers such that $k_{1} \neq k_{2}$ and $k_{1}, k_{2} \geq 2$, then $W$-ESTO $\mathcal{k}_{1}\left(L^{2}(\beta)\right) \cap W-\operatorname{ESTO}_{k_{2}}\left(L^{2}(\beta)\right)=\mathcal{K}\left(L^{2}(\beta)\right)$.
Proof. Let $k_{1}, k_{2} \geq 2$ be distinct integers and assume that the operator $A \in \mathrm{~W}$ $\operatorname{ESTO}_{k_{1}}\left(L^{2}(\beta)\right) \cap \mathrm{W}^{-E S T O} k_{k_{2}}\left(L^{2}(\beta)\right)$. Then, it follows that $M_{z}^{\beta} A-A M_{z^{k_{1}}}^{\beta} \in$ $\mathcal{K}\left(L^{2}(\beta)\right)$ and $M_{z}^{\beta} A-A M_{z^{k_{2}}}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)$. This further implies that $A\left(M_{z^{k_{2}}}^{\beta}-\right.$ $\left.M_{z^{k_{1}}}^{\beta}\right) \in \mathcal{K}\left(L^{2}(\beta)\right)$ and hence we must have $A \in \mathcal{K}\left(L^{2}(\beta)\right)$. This shows that $\mathrm{W}-\mathrm{ESTO}_{k_{1}}\left(L^{2}(\beta)\right) \cap{\mathrm{W}-\mathrm{ESTO}_{k_{2}}\left(L^{2}(\beta)\right) \subset \mathcal{K}\left(L^{2}(\beta)\right) \text {. Since the reverse con- }}_{\text {( }}$ tainment always holds, therefore we get the required result.

Next, we study the basic properties of the set $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. Let the operator $A^{*}$ denotes the adjoint of the operator $A$.
Proposition 2.6. The space $W$ - $E S T O_{k}\left(L^{2}(\beta)\right)$ is a closed vector subspace of $\mathcal{B}\left(L^{2}(\beta)\right)$ under the $*$-strong operator topology, where $\mathcal{B}\left(L^{2}(\beta)\right)$ denote the set of all bounded linear operators on $L^{2}(\beta)$.
Proof. Let $A_{1}, A_{2} \in \mathrm{~W}-\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$ and $\alpha_{1}, \alpha_{2} \in \mathbb{C}$. Then,

$$
\begin{aligned}
& M_{z}^{\beta}\left(\alpha_{1} A_{1}+\alpha_{2} A_{2}\right)-\left(\alpha_{1} A_{1}+\alpha_{2} A_{2}\right) M_{z^{k}}^{\beta} \\
= & \alpha_{1}\left(M_{z}^{\beta} A_{1}-A_{1} M_{z^{k}}^{\beta}\right)+\alpha_{2}\left(M_{z}^{\beta} A_{2}-A_{2} M_{z^{k}}^{\beta}\right) \in \mathcal{K}\left(L^{2}(\beta)\right) .
\end{aligned}
$$

Hence, $\alpha_{1} A_{1}+\alpha_{2} A_{2} \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ and the set $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ forms a subspace. Now for each $n$, consider the sequences of operators $A_{n}$ and $A_{n}^{*}$ in $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ such that $A_{n} \rightarrow A$ and $A_{n}^{*} \rightarrow A^{*}$ uniformly in $\mathcal{B}\left(L^{2}(\beta)\right)$ as $n \rightarrow \infty$. For each integer $n$, we have

$$
M_{z}^{\beta} A_{n}-A_{n} M_{z^{k}}^{\beta}=K_{n} \text { and } M_{z}^{\beta} A_{n}^{*}-A_{n}^{*} M_{z^{k}}^{\beta}=C_{n},
$$

where $K_{n}$ and $C_{n}$ are compact operators on $L^{2}(\beta)$. On using this we compute

$$
\begin{aligned}
\left\|M_{z}^{\beta} A-A M_{z^{k}}^{\beta}-K_{n}\right\|_{\beta} & =\left\|M_{z}^{\beta} A-A M_{z^{k}}^{\beta}-M_{z}^{\beta} A_{n}+A_{n} M_{z^{k}}^{\beta}\right\|_{\beta} \\
& \leq\left(\left\|M_{z}^{\beta}\right\|_{\beta}+\left\|M_{z^{k}}^{\beta}\right\|_{\beta}\right)\left\|A_{n}-A\right\|_{\beta}
\end{aligned}
$$

$$
\leq\left(\sup _{n}\left|\frac{\beta_{n+1}}{\beta_{n}}\right|+\sup _{n}\left|\frac{\beta_{k+n}}{\beta_{n}}\right|\right)\left\|A_{n}-A\right\|_{\beta}
$$

which goes to 0 as $n \rightarrow \infty$. Therefore, as $n$ tends to $\infty$, the sequence of compact operators $\left\langle K_{n}\right\rangle \rightarrow M_{z}^{\beta} A-A M_{z^{k}}^{\beta}$ uniformly. Since $\left\langle K_{n}\right\rangle$ is uniformly closed, therefore $M_{z}^{\beta} A-A M_{z^{k}}^{\beta}$ is compact and hence $A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. Similarly, using the same technique, we have

$$
\left\|M_{z}^{\beta} A^{*}-A^{*} M_{z^{k}}^{\beta}-C_{n}\right\|_{\beta} \rightarrow 0 \text { as } n \rightarrow \infty
$$

So, it follows that $A^{*} \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. Thus the set $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ is a closed subspace of $\mathcal{B}\left(L^{2}(\beta)\right)$ in $*$-strong operator topology.

In general the product of two $k^{t h}$-order essentially slant weighted Toeplitz operators need not be in the class $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ which can be seen by the following example:

Example 2.7. Let $A=B=W_{k} S+K$ on $L^{2}(\beta)$, where the operators $S$ and $K$ are defined as in Example 2.4. Then, $A$ and $B$ are in the class W-ESTO ${ }_{k}\left(L^{2}(\beta)\right)$ and take $C=A B$. We have

$$
\begin{aligned}
M_{z}^{\beta} C-C M_{z^{k}}^{\beta} & =M_{z}^{\beta} A B-A B M_{z^{k}}^{\beta} \\
& =\left(M_{z}^{\beta}\left(W_{k} S\right)^{2}-\left(W_{k} S\right)^{2} M_{z^{k}}^{\beta}\right)\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) .
\end{aligned}
$$

Therefore, $M_{z}^{\beta} C-C M_{z^{k}}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)$ if and only if $M_{z}^{\beta}\left(W_{k} S\right)^{2}-\left(W_{k} S\right)^{2} M_{z^{k}}^{\beta} \in$ $\mathcal{K}\left(L^{2}(\beta)\right)$. But, from the definitions of the operators $W_{k}, S$ and $K$, we have

$$
\left(M_{z}^{\beta}\left(W_{k} S\right)^{2}-\left(W_{k} S\right)^{2} M_{z^{k}}^{\beta}\right)\left(e_{n}\right)= \begin{cases}0 & \text { if } k \text { does not divide } n, \\ \frac{-\beta_{1}}{\beta_{k^{2}}-k} e_{1} & \text { if } n=k^{2}-k, \\ \frac{-k_{k}}{\beta_{k^{3}}-k} e_{k} & \text { if } n=k^{3}-k, \\ \frac{-\beta_{k^{2}}}{\beta_{k^{4}-k}} e_{k^{2}} & \text { if } n=k^{4}-k, \\ \frac{-\beta_{k} 3}{\beta_{k^{5}-k}} e_{k^{3}} & \text { if } n=k^{5}-k, \\ \vdots & \end{cases}
$$

and therefore $M_{z}^{\beta}(W S)^{2}-(W S)^{2} M_{z^{k}}^{\beta} \notin \mathcal{K}\left(L^{2}(\beta)\right)$. Hence, it follows that $C \notin$ $\mathrm{W}^{-E S T O}{ }_{k}\left(L^{2}(\beta)\right)$.

From the above example, it is clear that the set of $k^{t h}$-order essentially slant weighted Toeplitz operator does not form an algebra. In the following theorem, we determine the condition under which the set W-ESTO${ }_{k}\left(L^{2}(\beta)\right)$ is closed with respect to multiplication.

Theorem 2.8. Let $T_{1}, T_{2} \in W-E S T O_{k}\left(L^{2}(\beta)\right)$. Then the operator $T_{1} T_{2} \in$ $W-\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$ if and only if $T_{1} M_{z^{k}-z}^{\beta} T_{2} \in \mathcal{K}\left(L^{2}(\beta)\right)$.

Proof. Let $T_{1}, T_{2} \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. Then

$$
M_{z}^{\beta} T_{1}-T_{1} M_{z^{k}}^{\beta}=K_{1} \text { and } M_{z}^{\beta} T_{2}-T_{2} M_{z^{k}}^{\beta}=K_{2}
$$

where $K_{1}$ and $K_{2}$ are in $\mathcal{K}\left(L^{2}(\beta)\right)$. Also,

$$
\begin{aligned}
M_{z}^{\beta} T_{1} T_{2}-T_{1} T_{2} M_{z^{k}}^{\beta} & =\left(T_{1} M_{z^{k}}^{\beta}+K_{1}\right) T_{2}-T_{1}\left(M_{z}^{\beta} T_{2}-K_{2}\right) \\
& =T_{1} M_{z^{k}}^{\beta} T_{2}-T_{1} M_{z}^{\beta} T_{2}+K_{1} T_{2}+T_{1} K_{2} \\
& =T_{1}\left(M_{z^{k}-z}^{\beta}\right) T_{2}+K^{\prime}
\end{aligned}
$$

where $K^{\prime}=K_{1} T_{2}+T_{1} K_{2} \in \mathcal{K}\left(L^{2}(\beta)\right)$. Thus, $M_{z}^{\beta} T_{1} T_{2}-T_{1} T_{2} M_{z^{k}}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)$ if and only if $T_{1} M_{z^{k}-z} T_{2}$ is in $\mathcal{K}\left(L^{2}(\beta)\right)$ and hence the result follows.

For a natural number $s \geq 2$, let $n(s)$ denotes the number of partitions of $s$ as sum of two natural numbers. Then for each $i ; 1 \leq i \leq n(s)$, we can write $s=p_{i}+q_{i}$ such that $p_{i}$ and $q_{i}$ are natural numbers. The following result is a direct consequence of Theorem 2.8.
Corollary 2.9. Let $A \in W-\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$ and $s \in \mathbb{N}$, $s>1$. If $s=p_{i}+q_{i}$ where $p_{i}, q_{i} \in \mathbb{N}$ for $i=1,2, \ldots, n(s)$ and $A^{p_{i}}, A^{q_{i}} \in W-E S T O_{k}\left(L^{2}(\beta)\right)$, then the following are equivalent:
(1) $A^{s} \in W-E S T O_{k}\left(L^{2}(\beta)\right)$.
(2) $A^{p_{i}}\left(M_{z^{k}-z}\right) A^{q_{i}} \in \mathcal{K}, i=1,2, \ldots, n(s)$.
(3) $A^{q_{i}}\left(M_{z^{k}-z}\right) A^{p_{i}} \in \mathcal{K}, i=1,2, \ldots, n(s)$.

In the following theorem, we provide necessary condition for the class W $\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ to be closed under multiplication.
Theorem 2.10. If $T_{1}, T_{2} \in W-E S T O_{k}\left(L^{2}(\beta)\right)$ such that either $T_{1}$ commutes essentially with $M_{z}^{\beta}$ or $T_{2}$ commutes essentially with $M_{z^{k}}^{\beta}$, then $T_{1} T_{2} \in W$ $\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$.
Proof. Let $T_{1}, T_{2} \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ and consider the following two cases:
Case 1. If $T_{1} M_{z}^{\beta}=M_{z}^{\beta} T_{1}\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right)$, then

$$
\begin{aligned}
M_{z}^{\beta} T_{1} T_{2}-T_{1} T_{2} M_{z^{k}}^{\beta} & =\left(M_{z}^{\beta} T_{1} T_{2}-T_{1} M_{z}^{\beta} T_{2}\right)\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =\left(T_{1} M_{z}^{\beta} T_{2}-T_{1} M_{z}^{\beta} T_{2}\right)\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =0\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) .
\end{aligned}
$$

Therefore, it follows that $T_{1} T_{2} \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$.
Case 2. If $T_{2} M_{z^{k}}^{\beta}=M_{z^{k}}^{\beta} T_{2}\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right)$, then

$$
\begin{aligned}
M_{z}^{\beta} T_{1} T_{2}-T_{1} T_{2} M_{z^{k}}^{\beta} & =\left(T_{1} M_{z^{k}}^{\beta} T_{2}-T_{1} T_{2} M_{z^{k}}^{\beta}\right)\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =\left(T_{1} M_{z^{k}}^{\beta} T_{2}-T_{1} M_{z^{k}}^{\beta} T_{2}\right)\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =0\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) .
\end{aligned}
$$

Therefore, it follows that $T_{1} T_{2} \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$.

Corollary 2.11. If the operator $T_{1}$ (or $T_{2}$ ) commutes essentially with $M_{z}^{\beta}$ and $T_{2}\left(\right.$ or $\left.T_{1}\right) \in W-E S T O_{k}\left(L^{2}(\beta)\right)$, then $T_{1} T_{2} \in W-E S T O_{k}\left(L^{2}(\beta)\right)$.

From the above result, it follows that if $M_{\phi}^{\beta}$ is a weighted multiplication operator on $L^{2}(\beta)$ induced by $\phi \in L^{\infty}(\beta)$ and $A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$, then both the operators $A M_{\phi}^{\beta}$ and $M_{\phi}^{\beta} A$ are in $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. In the next result we show that the product of $k^{t h}$-order essentially slant weighted Toeplitz operators with essentially weighted Toeplitz operator is in W-ESTO ${ }_{k}\left(L^{2}(\beta)\right)$.
Theorem 2.12. If $A \in W$-ESTO$O_{k}\left(L^{2}(\beta)\right)$ and $X$ is an essentially weighted Toeplitz operator, then $A X$ and $X A$ are in $W-E S T O_{k}\left(L^{2}(\beta)\right)$.

Proof. Let $A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ and $X$ is an essentially weighted Toeplitz operator on $L^{2}(\beta)$. Then, by definition it follows that $M_{z}^{\beta} A-A M_{z^{k}}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)$ and $M_{z}^{\beta} X-X M_{z}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)$. On using this we have

$$
\begin{aligned}
M_{z}^{\beta} A X-A X M_{z^{k}}^{\beta} & =A M_{z^{k}}^{\beta} X-A X M_{z^{k}}^{\beta}\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =A M_{z^{k}}^{\beta} X-A M_{z^{k}}^{\beta} X\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =0\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right),
\end{aligned}
$$

therefore, the operator $A X \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. Again,

$$
\begin{aligned}
M_{z}^{\beta} X A-X A M_{z^{k}}^{\beta} & =X M_{z}^{\beta} A-X A M_{z^{k}}^{\beta}\left(\bmod \left(\mathcal{K}\left(L^{2}(\beta)\right)\right)\right. \\
& =X M_{z}^{\beta} A-X M_{z}^{\beta}\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =0\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right)
\end{aligned}
$$

and so it gives that the operator $X A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$.
Remark 2.13. It can be noted that if $X$ is an essentially weighted Toeplitz operator which is also essentially invertible, then the converse of Theorem 2.12 holds. Indeed, whenever the operator $A X$ or $X A$ is in the class W $\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ for some bounded operator $A$ on $L^{2}(\beta)$, then

$$
A \in \mathrm{~W}_{-\mathrm{ESTO}_{k}}\left(L^{2}(\beta)\right)
$$

The class $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ is not closed with respect to adjoint, which is shown in the following example:

Example 2.14. For $k \geq 2$, the operator $A=W_{k}$ defined on $L^{2}(\beta)$ is in the class W-ESTO ${ }_{k}\left(L^{2}(\beta)\right)$. Indeed, when the integer $n$ is a multiple of $k$, then we have

$$
\begin{aligned}
\left(M_{z}^{\beta} A-A M_{z^{k}}^{\beta}\right)\left(e_{n}\right) & =M_{z}^{\beta} W_{k}\left(e_{n}\right)-W_{k} M_{z^{k}}^{\beta}\left(e_{n}\right) \\
& =\frac{\beta_{n / k}}{\beta_{n}} M_{z}^{\beta} e_{n / k}-\frac{\beta_{n+k}}{n} e_{n+k} \\
& =\frac{\beta_{(n+k) / k}}{\beta_{n}} e_{(n+k) / k}-\frac{\beta_{(n+k) / k}}{\beta_{n}} e_{(n+k) / k}=0
\end{aligned}
$$

If $n$ is not a multiple of $k$, then again we have

$$
\left(M_{z}^{\beta} A-A M_{z^{k}}^{\beta}\right)\left(e_{n}\right)=0
$$

Therefore, for each integer $n,\left(M_{z}^{\beta} A-A M_{z^{k}}^{\beta}\right)\left(e_{n}\right)=0$ and hence the operator $W_{k} \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. Since the adjoint $W_{k}^{*}$ of the operator $W_{k}$ satisfies that $W_{k}^{*}\left(e_{n}\right)=\frac{\beta_{n k}}{\beta_{n}} e_{n k}$ and therefore using this we compute

$$
\begin{aligned}
\left(M_{z}^{\beta} W_{k}^{*}-W_{k}^{*} M_{z^{k}}^{\beta}\right)\left(e_{n}\right) & =\frac{\beta_{n k}}{\beta_{n}} M_{z}^{\beta} e_{n k}-\frac{\beta_{n+k}}{\beta_{n}} W_{k}^{*}\left(e_{n+k}\right) \\
& =\frac{\beta_{n k+1}}{\beta_{n}} e_{n k+1}-\frac{\beta_{(n+k) k}}{\beta_{n}} e_{(n+k) k}
\end{aligned}
$$

which does not belong to $\mathcal{K}\left(L^{2}(\beta)\right)$. Hence, the operator $A^{*}=W_{k}^{*}$ is not in the class W-ESTO $k\left(L^{2}(\beta)\right)$.

In the next theorem, we have shown that whenever both the operator and its adjoint are in the class $\mathrm{W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$, then the operator $A T$ essentially equals to $T^{*} A$ on $L^{2}(\beta)$, where $T=M_{\bar{z}}^{\beta}+M_{z^{k}}^{\beta}$.

Theorem 2.15. If $A, A^{*} \in W-E S T O_{k}\left(L^{2}(\beta)\right)$, then $A T=T^{*} A(\bmod \mathcal{K})$, where $T=M_{\bar{z}}^{\beta}+M_{z^{k}}^{\beta}$.

Proof. Let $A, A^{*} \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$. Then, we have

$$
\begin{gather*}
M_{z}^{\beta} A-A M_{z^{k}}^{\beta}=K_{1}  \tag{2}\\
M_{z}^{\beta} A^{*}-A^{*} M_{z^{k}}^{\beta}=K_{2}
\end{gather*}
$$

where $K_{1}, K_{2} \in \mathcal{K}\left(L^{2}(\beta)\right)$. On taking adjoint of both sides of the equation (3) and subtracting equation (2) from this, we obtain that

$$
A M_{\bar{z}}^{\beta}-M_{\bar{z}^{k}}^{\beta} A-M_{z}^{\beta} A+A M_{z^{k}}^{\beta}=K_{2}^{*}-K_{1}
$$

which further implies that

$$
A\left(M_{\bar{z}}^{\beta}+M_{z^{k}}^{\beta}\right)-\left(M_{\bar{z}^{k}}^{\beta}+M_{z}^{\beta}\right) A=K_{3} \text { for some } K_{3} \in \mathcal{K}\left(L^{2}(\beta)\right)
$$

Therefore, $A T=T^{*} A\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right)$, where $T=M_{\bar{z}}^{\beta}+M_{z^{k}}^{\beta}$.
Hence, from Theorem 2.15, it follows that a necessary condition for any operator $A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ to be self-adjoint is that $A T=T^{*} A(\bmod \mathcal{K})$, where $T=M_{\bar{z}}^{\beta}+M_{z^{k}}^{\beta}$.
Theorem 2.16. Let $A \in W-E S T O_{k}\left(L^{2}(\beta)\right)$ be an essentially normal operator. Then the operators $A^{*} A$ and $A A^{*}$ are both in $W-E S T O_{k}\left(L^{2}(\beta)\right)$ if and only if $A^{*}$ commutes essentially with $A M_{z^{k}}^{\beta}$.

Proof. Let $A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ be such that $A A^{*}=A^{*} A\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right)$. Then, we get

$$
\begin{aligned}
M_{z}^{\beta} A^{*} A-A^{*} A M_{z^{k}}^{\beta} & =M_{z}^{\beta} A A^{*}-A^{*} A M_{z^{k}}^{\beta}\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =\left(A M_{z^{k}}^{\beta}\right) A^{*}-A^{*}\left(A M_{z^{k}}^{\beta}\right)\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right),
\end{aligned}
$$

and therefore, $A^{*} A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ if and only if $A^{*}$ commutes essentially with $A M_{z^{k}}^{\beta}$. Similarly, we have

$$
\begin{aligned}
M_{z}^{\beta} A A^{*}-A A^{*} M_{z^{k}}^{\beta} & =A M_{z^{k}}^{\beta} A^{*}-A A^{*} M_{z^{k}}^{\beta}\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =\left(A M_{z^{k}}^{\beta}\right) A^{*}-A^{*}\left(A M_{z^{k}}^{\beta}\right)\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right),
\end{aligned}
$$

which implies that $A A^{*} \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ if and only if $A^{*}$ commutes essentially with $A M_{z^{k}}^{\beta}$.

Datt et al. [8] have studied the notion of $k^{\text {th }}$-order slant weighted Hankel operator and have proved that an operator $B$ on $L^{2}(\beta)$ is a $k^{t h}$-order slant weighted Hankel operator if and only if $M_{z^{-1}}^{\beta} B=B M_{z^{k}}^{\beta}$. In the next theorem, we obtain the condition for the product of $k^{t h}$-order essentially slant weighted Toeplitz operator and $k^{t h}$-order slant weighted Hankel operator to be in the class W-ESTO ${ }_{k}\left(L^{2}(\beta)\right)$.

Theorem 2.17. Let the operator $A \in W-E S T O_{k}\left(L^{2}(\beta)\right)$ and $B$ be a $k^{t h}-$ order slant weighted Hankel operator on $L^{2}(\beta)$. Then the operator $A B \in W$ $\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$ if and only if $A\left(M_{z^{k}}^{\beta}-M_{z^{-1}}^{\beta}\right) B \in \mathcal{K}\left(L^{2}(\beta)\right)$.
Proof. Let the operator $A \in \mathrm{~W}^{-E_{S T O}^{k}}\left(L^{2}(\beta)\right)$ and $B$ be a $k^{\text {th }}$-order slant weighted Hankel operator on $L^{2}(\beta)$. Therefore, we have $M_{z}^{\beta} A-A M_{z^{k}}^{\beta} \in$ $\mathcal{K}\left(L^{2}(\beta)\right)$ and $M_{z^{-1}}^{\beta} B=B M_{z^{k}}^{\beta}$. Using this we get that

$$
\begin{aligned}
M_{z}^{\beta} A B-A B M_{z^{k}}^{\beta} & =A M_{z^{k}}^{\beta} B-A\left(B M_{z^{k}}^{\beta}\right)\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =A M_{z^{k}}^{\beta} B-A M_{z^{-1}}^{\beta} B \\
& =A\left(M_{z^{k}}^{\beta}-M_{z^{-1}}^{\beta}\right) B .
\end{aligned}
$$

Hence, $A B \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ if and only if $A\left(M_{z^{k}}^{\beta}-M_{z^{-1}}^{\beta}\right) B \in \mathcal{K}\left(L^{2}(\beta)\right)$.

Theorem 2.18. If an operator $A \in W-\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$ is a $k^{t h}$-order slant Hankel operator, then $A\left(I-\left(M_{z^{k}}^{\beta}\right)^{2}\right) \in \mathcal{K}\left(L^{2}(\beta)\right)$, where $I$ denotes the identity operator on $L^{2}(\beta)$.

Proof. Let $A \in \mathrm{~W}-\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$ be a $k^{t h}$-order slant Hankel operator on $L^{2}(\beta)$. Then the operator $A$ satisfies the relation $M_{z}^{\beta} A-A M_{z^{k}}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)$ and also $M_{z^{-1}}^{\beta} A=A M_{z^{k}}^{\beta}$. Using these relations, we have

$$
M_{z}^{\beta} A-A M_{z^{k}}^{\beta}=A-M_{z^{-1}}^{\beta} A M_{z^{k}}^{\beta}\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right)
$$

$$
\begin{aligned}
& =A-A M_{z^{k}}^{\beta} M_{z^{k}}^{\beta}\left(\bmod \mathcal{K}\left(L^{2}(\beta)\right)\right) \\
& =A\left(I-\left(M_{z^{k}}^{\beta}\right)^{2}\right)
\end{aligned}
$$

Hence, this gives that $A\left(I-\left(M_{z^{k}}^{\beta}\right)^{2}\right) \in \mathcal{K}\left(L^{2}(\beta)\right)$.
We conclude this section by proving that the essential spectrum of a $k^{t h}$-order essentially slant weighted Toeplitz operator contains zero.

Theorem 2.19. Let $A \in W-\operatorname{ESTO}_{k}\left(L^{2}(\beta)\right)$. Then $0 \in \sigma_{e}(A)$, the essential spectrum of $A$.

Proof. Let $A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(L^{2}(\beta)\right)$ and if possible assume that $0 \notin \sigma_{e}(A)$. This means that $A$ is essentially invertible on $L^{2}(\beta)$. Since the operator $A$ satisfies the relation $\left.M_{z}^{\beta} A-A M_{z^{k}}^{\beta} \in \mathcal{K}\left(L^{2}(\beta)\right)\right)$ and therefore it follows that $M_{z}^{\beta}-$ $\left.A M_{z^{k}}^{\beta} A^{-1} \in \mathcal{K}\left(L^{2}(\beta)\right)\right)$. This gives that $M_{z}^{\beta}$ and $M_{z^{k}}^{\beta}$ are essentially similar. But this is not possible as their Fredholm indexes are not same. Hence, we get that $0 \in \sigma_{e}(A)$.

## 3. Compressions of $k^{t h}$-order essentially slant weighted Toeplitz operators

In 2013, Arora and Kathuria [5] obtained a characterization for the compression of a $k^{t h}$-order slant weighted Toeplitz operator to $H^{2}(\beta)$ and they have shown that a bounded linear operator $B$ on $H^{2}(\beta)$ is the compression of a $k^{t h_{-}}$ order slant weighted Toeplitz operator to $H^{2}(\beta)$ if and only if $T_{z}^{\beta} B=B T_{z^{k}}^{\beta}$, where $T_{z}^{\beta}$ and $T_{z^{k}}^{\beta}$ are weighted Toeplitz operators on $H^{2}(\beta)$ induced by $z$ and $z^{k}$ respectively. Motivated by this study, we define the compression of a $k^{\text {th }}$ order essentially slant weighted Toeplitz operator to the weighted space $H^{2}(\beta)$. Let $\mathcal{K}\left(H^{2}(\beta)\right)$ denote the set of all compact operators on the weighted space $H^{2}(\beta)$.
Definition. An operator $B$ on the space $H^{2}(\beta)$ is the compression of a $k^{t h_{-}}$ order essentially slant weighted Toeplitz operator of $L^{2}(\beta)$ to $H^{2}(\beta)$ if $B-$ $T_{z}^{\beta^{*}} B T_{z^{k}}^{\beta} \in \mathcal{K}\left(H^{2}(\beta)\right)$.

Since weighted Toeplitz operator $T_{z}^{\beta}$ is essentially unitary in $H^{2}(\beta)$, therefore equivalently we define an operator $B$ on the space $H^{2}(\beta)$ as the compression of a $k^{t h}$-order essentially slant weighted Toeplitz operator to $H^{2}(\beta)$ if $T_{z}^{\beta} B-B T_{z^{k}}^{\beta} \in$ $\mathcal{K}\left(H^{2}(\beta)\right)$. We denote the set of all compression of essentially $k^{\text {th }}$-order slant weighted Toeplitz operators to $H^{2}(\beta)$ by W- $\operatorname{ESTO}_{k}\left(H^{2}(\beta)\right)$. Clearly if $T$ is the compression of a $k^{t h}$-order slant weighted Toeplitz operators to $H^{2}(\beta)$, then $T \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(H^{2}(\beta)\right)$. The class $\mathrm{W}-\mathrm{ESTO}_{k}\left(H^{2}(\beta)\right)$ enjoys the similar properties that holds for the set W-ESTO ${ }_{k}\left(L^{2}(\beta)\right)$.

Proposition 3.1. The set $W$ - $E S T O_{k}\left(H^{2}(\beta)\right)$ satisfies the following properties:
(i) $W$ - $\operatorname{ESTO}_{k}\left(H^{2}(\beta)\right)$ is a norm-closed vector subspace of $\mathcal{B}\left(H^{2}(\beta)\right)$.
(ii) $W$ - $\operatorname{ESTO}_{k}\left(H^{2}(\beta)\right)$ is not an algebra of operators on $H^{2}(\beta)$.
(iii) $W-E S T O_{k}\left(H^{2}(\beta)\right)$ is not a self-adjoint set.
(iv) $\mathcal{K}\left(H^{2}(\beta)\right) \cap W-E S T O_{k}\left(H^{2}(\beta)\right)=\mathcal{K}\left(H^{2}(\beta)\right)$.
(v) Let $A_{1}, A_{2} \in W-E S T O_{k}\left(H^{2}(\beta)\right)$. Then
the operator $A_{1} A_{2} \in W-E S T O_{k}\left(H^{2}(\beta)\right)$ if and only if $A_{1}\left(T_{z^{k}}^{\beta}-T_{z}^{\beta}\right) A_{2} \in$ $\mathcal{K}\left(H^{2}(\beta)\right)$.
(vi) If $A_{1}, A_{2} \in W-\operatorname{ESTO}_{k}\left(H^{2}(\beta)\right)$ such that either $A_{1}$ commutes essentially with $T_{z}^{\beta}$ or $A_{2}$ commutes essentially with $T_{z^{k}}^{\beta}$, then $A_{1} A_{2} \in W$ $\operatorname{ESTO}_{k}\left(H^{2}(\beta)\right)$.
(vii) Let an operator $A \in W-\operatorname{ESTO}_{k}\left(H^{2}(\beta)\right)$ be self adjoint and $T=T_{z}^{\beta}+$ $T_{\bar{z}^{k}}^{\beta}$. Then the operator $T A$ is essentially commutant on $H^{2}(\beta)$.
(ix) Let $A \in W-E S T O_{k}\left(H^{2}(\beta)\right)$. Then $0 \in \sigma_{e}(A)$, the essential spectrum of $A$.
(x) For integers $k_{1}, k_{2} \geq 2$ such that $k_{1} \neq k_{2}$, the following holds $W-\operatorname{ESTO}_{k}\left(H^{2}(\beta)\right) \cap W-E S T O_{k}\left(H^{2}(\beta)\right)=\mathcal{K}\left(H^{2}(\beta)\right)$.
Theorem 3.2. If $T_{\phi}^{\beta}$ is a weighted Toeplitz operator on $H^{2}(\beta)$ induced by symbol $\phi$ in $L^{\infty}(\beta)$ and $A \in W-\operatorname{ESTO}_{k}\left(H^{2}(\beta)\right)$, then $A T_{\phi}^{\beta}$ and $T_{\phi}^{\beta} A$ both are in $W$-ESTO ${ }_{k}\left(H^{2}(\beta)\right)$.

Proof. Let $A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(H^{2}(\beta)\right)$ and $T_{\phi}^{\beta}$ be a weighted Toeplitz operator on $H^{2}(\beta)$ induced by symbol $\phi$ in $L^{\infty}(\beta)$. By the characterization of weighted Toeplitz operator [4], it follows that the commutator of $T_{\phi}^{\beta}$ and $T_{z}^{\beta}$ is compact. In fact for any positive integer $n$, the commutator of $T_{\phi}^{\beta}$ and $T_{z^{n}}^{\beta}$ is a compact operator on $H^{2}(\beta)$. Then, we have

$$
\begin{aligned}
T_{z}^{\beta}\left(T_{\phi}^{\beta} A\right)-\left(T_{\phi}^{\beta} A\right) T_{z^{k}}^{\beta} & =T_{\phi}^{\beta} T_{z}^{\beta} A-T_{\phi}^{\beta} A T_{z^{k}}^{\beta}\left(\bmod \mathcal{K}\left(H^{2}(\beta)\right)\right) \\
& =T_{\phi}^{\beta}\left(T_{z}^{\beta} A\right)-T_{\phi}^{\beta}\left(A T_{z^{k}}^{\beta}\right)\left(\bmod \mathcal{K}\left(H^{2}(\beta)\right)\right)
\end{aligned}
$$

Therefore, $T_{\phi}^{\beta}\left(T_{z}^{\beta} A-A T_{z^{k}}^{\beta}\right) \in \mathcal{K}\left(H^{2}(\beta)\right)$. Similarly,

$$
\begin{aligned}
T_{z}^{\beta}\left(A T_{\phi}^{\beta}\right)-\left(A T_{\phi}^{\beta}\right) T_{z^{k}} & =T_{z}^{\beta} A T_{\phi}^{\beta}-A T_{z^{k}} T_{\phi}^{\beta}\left(\bmod \mathcal{K}\left(H^{2}(\beta)\right)\right) \\
& =\left(T_{z}^{\beta} A-A T_{z^{k}}\right) T_{\phi}^{\beta}\left(\bmod \mathcal{K}\left(H^{2}(\beta)\right)\right)
\end{aligned}
$$

Therefore, $T_{z}^{\beta}\left(A T_{\phi}^{\beta}\right)-\left(A T_{\phi}^{\beta}\right) T_{z^{k}} \in \mathcal{K}\left(H^{2}(\beta)\right)$.
It may be noted that if $T_{\phi}^{\beta}$ is an invertible weighted Toeplitz operator, then the converse of Theorem 3.2 holds. In the next theorem, we show that a $k^{t h}$-order essentially slant weighted Toeplitz operator can not be a Fredholm operator on the weighted space $H^{2}(\beta)$.
Theorem 3.3. The space $W$-ESTO $O_{k}\left(H^{2}(\beta)\right)$ contains no Fredholm operator.
Proof. Let us suppose that $A \in \mathrm{~W}-\mathrm{ESTO}_{k}\left(H^{2}(\beta)\right)$ be a Fredholm operator of index $n$. Then, $T_{z}^{\beta} A-A T_{z^{k}}^{\beta}=K$ for some compact operator $K$ on $H^{2}(\beta)$.

Since $A$ is a Fredholm operator of index $n$ and $T_{z}^{\beta} A=A T_{z^{k}}^{\beta}+K$, therefore it follows that $T_{z}^{\beta} A$ and $A T_{z^{k}}^{\beta}+K$ are Fredholm operators of index $n-1$ and $n-k$ respectively. This gives that $\operatorname{ind}\left(T_{z}^{\beta} A\right)=n-1=\operatorname{ind}\left(A T_{z^{k}}^{\beta}+K\right)=$ $n-k$, which is absurd. Thus, there is no Fredholm operator in the space $\mathrm{W}-\mathrm{ESTO}_{k}\left(H^{2}(\beta)\right)$.

Remark 3.4. In particular, from the last theorem we can conclude that the set $\mathrm{W}-\mathrm{ESTO}_{k}\left(H^{2}(\beta)\right)$ does not contain weighted shift operators. Indeed, consider the weighted shift $T_{z}^{\beta}$ defined on $H^{2}(\beta)$ by $T_{z}^{\beta}\left(e_{n}\right)=\frac{\beta_{n+1}}{\beta_{n}} e_{n+1}$, which is a Fredholm operator of index -1 . Now consider

$$
\begin{aligned}
\left(T_{z}^{\beta} T_{z}^{\beta}-T_{z}^{\beta} T_{z^{k}}^{\beta}\right)\left(e_{n}(z)\right) & =T_{z}^{\beta}\left(\frac{\beta_{n+1}}{\beta_{n}} e_{n+1}(z)-\frac{\beta_{n+k}}{\beta_{n}} e_{n+k}(z)\right) \\
& =\frac{\beta_{n+2}}{\beta_{n}} e_{n+2}(z)-\frac{\beta_{n+k+1}}{\beta_{n}} e_{n+k+1}(z)
\end{aligned}
$$

Hence, by Theorem 3.3, $T_{z}^{\beta} \notin \mathrm{W}-\mathrm{ESTO}_{k}\left(H^{2}(\beta)\right)$.
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