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ON COMPLETE MONOTONICITY OF LINEAR COMBINATION OF FINITE PSI FUNCTIONS

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Abstract. In the paper, the authors supply complete monotonicity of linear combination of finite psi functions and extend some known results.

1. Preliminaries

It is well-known [1, 12, 14] that the classical gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

or by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$$

and that the logarithmic derivative

$$\psi(z)=[\ln\Gamma(z)]'=\frac{\Gamma'(z)}{\Gamma(z)}$$

is called the psi or digamma function.

From [11, Chapter XIII], [30, Chapter 1], and [31, Chapter IV], we recall that an infinitely differentiable and nonnegative function $f(x)$ is said to be completely monotonic on an interval I if and only if

$$(-1)^{m-1}f^{(m-1)}(x)\ge 0$$

for all $m\in\mathbb{N}$ and $x\in I$. The Bernstein–Widder theorem [31, p. 161, Theorem 12b] states that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0,\infty)$ is that

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\,\mu(t)$$

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for $x \in (0, \infty)$, where μ is non-decreasing and the above integral converges. In other words or simply speaking, a function is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform.

2. Motivations

In [2, Theorem 4.1], it was obtained that, if a_k and b_k for $1 \le k \le m$ satisfy $a_1 \ge a_2 \ge \cdots \ge a_m$ and $b_1 \ge b_2 \ge \cdots \ge b_m > 0$, then

$$\phi_0(x) = \sum_{k=1}^{m} a_k \psi(b_k x)$$

is completely monotonic on $(0, \infty)$ if and only if

$$\sum_{k=1}^{m} a_k = 0 \quad \text{and} \quad \sum_{k=1}^{m} a_k \ln b_k \ge 0.$$

In [8, Lemma 2.1], the function $\phi_0(x)$ was extended as

$$\phi_\delta(x) = \sum_{k=1}^{m} a_k \psi(b_k x + \delta)$$

for $x > 0$ and $\delta \ge 0$ and it was acquired that, if $a_1 \ge a_2 \ge \cdots \ge a_m$ and $b_1 \ge b_2 \ge \cdots \ge b_m \ge 0$ such that $\sum_{k=1}^{m} a_k \ge 0$, then, when $\delta \ge \frac{1}{2}$, the first derivative $\phi_\delta'(x)$ is completely monotonic and, consequently, the function $\phi_\delta(x)$ is increasing and concave, on $(0, \infty)$. These results were applied in [8] to discuss a problem arising in the context of statistical density estimation based on Bernstein polynomials.

In the proof of [29, Theorem 2.2], it was obtained that the function

$$\left(\sum_{i=1}^{m} a_i\right)^2 \psi'\left(1 + x\sum_{i=1}^{m} a_i\right) - \sum_{i=1}^{m} a_i^2 \psi'(1 + a_i x), \quad a_i > 0$$

is completely monotonic and, consequently,

$$\left(\sum_{i=1}^{m} a_i\right) \psi\left(1 + x\sum_{i=1}^{m} a_i\right) - \sum_{i=1}^{m} a_i \psi(1 + a_i x), \quad a_i > 0 \tag{1}$$

is positive, increasing, and concave, with respect to $x \in (0, \infty)$. See also the proof of [13, Theorem 2.1]. In other words, the function defined by (1) is a Bernstein function of $x \in (0, \infty)$. For detailed information on the Bernstein functions, please refer to the monograph [30], the papers [21–23, 25, 26] and closely related references.

In this paper, we supply an alternative proof for the above complete monotonicity of the functions $\phi_0(x)$ and $\phi_\delta(x)$ for $\delta \ge \frac{1}{2}$ on $(0, \infty)$ and obtain slightly extended conclusions.

3. Main results and their proofs

We now state and prove our main results alternatively.

Theorem 1. *If* $\delta \ge \frac{1}{2}$, $(a_i - a_j)(b_i - b_j) \gtreqless 0$ *for all* $1 \le i, j \le m$, *and* $\sum_{k-1}^{m} a_k \gtreqless 0$, *then the first derivative* $\pm\phi'_\delta(x)$ *is completely monotonic and, consequently, the function* $\pm\phi_\delta(x)$ *is increasing and concave, on* $(0,\infty)$.

If $(a_i - a_j)(b_i - b_j) \gtreqless 0$ *for all* $1 \le i, j \le m$, *then the function* $\pm\phi_0(x)$ *is completely monotonic on* $(0,\infty)$ *if and only if* $\sum_{k=1}^{m} a_k = 0$ *and* $\sum_{k=1}^{m} a_k \ln b_k \gtreqless 0$.

Proof. Using the formula

$$\psi'(z) = \int_0^\infty \frac{t}{1-e^{-t}} e^{-zt} \,\mathrm{d}\, t, \quad \Re(z) > 0$$

in [1, p. 260, 6.4.1] gives

$$\begin{aligned}\psi'(\tau x + \delta) &= \int_0^\infty \frac{t}{1-e^{-t}} e^{-(\tau x+\delta)t} \,\mathrm{d}\, t \\ &= \int_0^\infty \frac{te^{-\delta t}}{1-e^{-t}} e^{-\tau x t} \,\mathrm{d}\, t \\ &= \frac{1}{\tau}\int_0^\infty h\left(\frac{v}{\tau}\right) e^{-vx} \,\mathrm{d}\, v,\end{aligned}$$

where $\tau > 0$ and $h_\delta(t) = \frac{te^{-\delta t}}{1-e^{-t}}$. Hence

$$\phi'_\delta(x) = \sum_{k=1}^{m} a_k b_k \psi'(b_k x + \delta) = \int_0^\infty \left[\sum_{k=1}^{m} a_k h_\delta\left(\frac{v}{b_k}\right)\right] e^{-vx} \,\mathrm{d}\, v.$$

In [9, 32], it was established that the positive function

$$h_\delta(t) = \begin{cases} \dfrac{t}{e^{\delta t} - e^{(\delta-1)t}}, & t \ne 0, \\ 1, & t = 0, \end{cases}$$

is decreasing on $\mathbb{R}$ if $\delta \ge 1$, increasing on $\mathbb{R}$ if $\delta \le 0$, increasing in $(-\infty, 0)$ if $\delta \le \frac{1}{2}$, and decreasing in $(0,\infty)$ if $\delta \ge \frac{1}{2}$. For more information on properties and applications of $h_\delta(t)$, please refer to the papers and review articles [3, 4, 6, 15–19, 24, 28] and closely related references therein. Therefore, by virtue of the Čebyšev inequality in [10, p. 36, Section 2.5, Theorem 1], we acquire

$$\frac{1}{m}\sum_{k=1}^{m} a_k h_\delta\left(\frac{v}{b_k}\right) \gtreqless \left(\frac{1}{m}\sum_{k=1}^{m} a_k\right)\left[\frac{1}{m}\sum_{k=1}^{m} h_\delta\left(\frac{v}{b_k}\right)\right] \gtreqless 0.$$

This means that the first derivative $\pm\phi'_\delta(x)$ is completely monotonic on $(0,\infty)$. Consequently, the function $\pm\phi_\delta(x)$ is increasing and concave on $(0,\infty)$.

When $\delta = 0$, the function $h_0(t)$ is positive and increasing on $[0,\infty)$ and

$$\frac{1}{m}\sum_{k=1}^{m} a_k h_0\left(\frac{v}{b_k}\right) \lesseqqgtr \left(\frac{1}{m}\sum_{k=1}^{m} a_k\right)\left[\frac{1}{m}\sum_{k=1}^{m} h_0\left(\frac{v}{b_k}\right)\right].$$

Accordingly, when $\sum_{k=1}^{m} a_k \lesseqqgtr 0$, the first derivative $\mp\phi_0'(x)$ is completely monotonic and the function $\mp\phi_0(x)$ is increasing on $(0,\infty)$. Furthermore, utilizing

$$\lim_{x\to\infty}[\ln x - \psi(x)] = 0$$

in [5, Theorem 1] and [7, 27] yields that the limit

$$\phi_0(x) = \sum_{k=1}^{m} a_k[\psi(b_k x) - \ln(b_k x)] + \sum_{k=1}^{m} a_k \ln b_k + (\ln x)\sum_{k=1}^{m} a_k \to \sum_{k=1}^{m} a_k \ln b_k$$

as $x \to \infty$ is valid and

$$\mp\phi_0(x) \le \mp\sum_{k=1}^{m} a_k \ln b_k$$

if and only if $\sum_{k=1}^{m} a_k = 0$. When and only when $\sum_{k=1}^{m} a_k \ln b_k \gtreqqless 0$, the function $\pm\phi_0(x)$ is positive and, consequently, completely monotonic, on $(0,\infty)$. The proof of Theorem 1 is complete. □

Remark 1. This paper is a slight revision of the preprint [20].

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