Commun. Korean Math. Soc. **34** (2019), No. 4, pp. 1175–1199
https://doi.org/10.4134/CKMS.c180421
pISSN: 1225-1763 / eISSN: 2234-3024

A p-DEFORMED q-INVERSE PAIR AND ASSOCIATED POLYNOMIALS INCLUDING ASKEY SCHEME

Rajesh V. Savalia

Abstract. We construct a general bi-basic inverse series relation which provides extension to several q-polynomials including the Askey-Wilson polynomials and the q-Racah polynomials. We introduce a general class of polynomials suggested by this general inverse pair which would unify certain polynomials such as the q-extended Jacobi polynomials and q-Konhauser polynomials. We then emphasize on applications of the general inverse pair and obtain the generating function relations, summation formulas involving the associated polynomials and derive the p-deformation of some of the q-analogues of Riordan's classes of inverse series relations. We also illustrate the companion matrix corresponding to the general class of polynomials; this is followed by a chart showing the reducibility of the extended p-deformed Askey-Wilson polynomials as well as the extended p-deformed q-Racah polynomials.

1. Introduction

Recently, Díaz and Teruel [5] introduced two parameter deformation of the classical gamma function by means of the q,k-Pochhammer symbol which is denoted and defined by [5, Def. 4, p. 121]

$$[t]_{n,k} = \prod_{j=0}^{n-1}[t+jk]_q, \quad t>0,\ k>0, \tag{1}$$

where $[a]_q = 1-q^a$. Using this, the q,k-generalized gamma function was defined in the form [5, Def. 6, p. 122]:

$$\Gamma_{q,k}(t) = \frac{\left(1-q^k\right)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}}, \quad t>0,\ k>0,$$

where $(1+x)_{q,k}^t = \frac{(1+x)_{q,k}^\infty}{(1+xq^{kt})_{q,k}^\infty}$ and $(x+y)_{q,k}^n = \prod_{j=0}^{n-1}(x+yq^{jk})$.

Received October 7, 2018; Revised December 28, 2018; Accepted January 4, 2019.
2010 *Mathematics Subject Classification.* Primary 33D15, 33D45, 33D65.
Key words and phrases. q,p-gamma function, q,p-Pochhammer symbol, p-deformed q-polynomial, q-inverse series.

Alternatively [5, Lem-2, p. 122],

$$\Gamma_{q,k}(t) = \frac{\left(1-q^k\right)_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}\,(1-q)^{\frac{t}{k}-1}}, \quad t>0,\ k>0. \tag{2}$$

As $q \to 1^-$ from within the interval $(0,1)$, the defining expressions in (1) and (2) yield the k-generalized Pochhammer symbol $(t)_{n,k}$ and the k-deformed classical gamma function $\Gamma_k(t)$ ([5, p. 119] and [4]).

Having motivated by the works of R. Diaz and C. Teruel [5], and R. Diaz and E. Pariguan [4], we provide here the extension to certain classical q-polynomials in the sense of k-deformation and derive their inverse series relation. Further, we obtain the generating function relations of these polynomials; and using the inverse series, we deduce certain summation formulas involving the corresponding polynomials.

2. Notations, formulas and definitions

We replace in the present work, k by p, and write $(q^t;q)_{n,p}$ instead of $[t]_{n,k}$, where $t \in \mathbb{C}$. In the notations of (1) and (2), we have

$$(q^t;q)_{n,p} = (1-q^t)(1-q^{t+p})(1-q^{t+2p})\cdots(1-q^{t+(n-1)p}), \tag{3}$$

$$\Gamma_{q,p}(t) = \frac{(q^p;q)_{\infty,p}(1-q)^{1-t/p}}{(q^t;q)_{\infty,p}}, \quad \Re(t)>0,\ p>0, \tag{4}$$

where

$$(q^a;q)_{n,p} = \begin{cases} 1, & \text{if } n=0,\\ (1-a)(1-aq^p)\cdots(1-aq^{p(n-1)}), & \text{if } n\in\mathbb{Z}_{>0},\\ \left[(1-aq^{-p})(1-aq^{-2p})\cdots(1-aq^{np})\right]^{-1}, & \text{if } n\in\mathbb{Z}_{<0},\\ (a;q)_{\infty,p}/(aq^{np};q)_{\infty,p}, & \text{if } n\in\mathbb{C}, \end{cases}$$

and

$$(q^\alpha;q)_{\infty,p} = \prod_{n=0}^{\infty}(1-q^{\alpha+np}),\ |q|<1.$$

We shall adopt the convention that for a parameter $\alpha \in \mathbb{C}$, q^α will be denoted as α.

In what follows, the following formulas will be used in the work. For $0 < q < 1$,

$$(a;q)_{n+m,p} = (a;q)_{n,p}(aq^{np};q)_{m,p},\ m,n\in\mathbb{N}, \tag{5}$$

$$(aq^{-np};q)_{n,p} = (-1)^n a^n q^{-pn(n+1)/2}\left(\frac{q^p}{a};q\right)_{n,p}, \tag{6}$$

$$(a;q)_{n-k,p} = \left(-\frac{1}{a}\right)^k q^{pk(k+1)/2-nkp}\frac{(a;q)_{n,p}}{(q^{p-np}/a;q)_{k,p}}, \tag{7}$$

$$(a;q)_{-k,p} = \left(-\frac{1}{a}\right)^k q^{pk(k+1)/2} \frac{1}{(q^p/a;q)_{k,p}}. \tag{8}$$

When $p = 1$, these formulas get reduced to those listed in [6, Appendix I, pp. 233–234]. For $\lambda \neq 0$, a general q-binomial coefficient is given as

$$\begin{bmatrix} u \\ v \end{bmatrix}_\lambda = \frac{(q^\lambda;q^\lambda)_u}{(q^\lambda;q^\lambda)_v (q^\lambda;q^\lambda)_{u-v}}.$$

There are two q-exponential functions [6, Eq. (1.3.16), p. 9]:

$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{(q;q)_n}, \quad (x \in \mathbb{R}, |q| < 1), \tag{9}$$

and [6, Eq. (1.3.15), p. 9]

$$e_q(x) = \sum_{n=0}^{\infty} \frac{(0;q)_n}{(q;q)_n} x^n = \frac{1}{(x;q)_\infty}, \quad (|x| < 1, |q| < 1). \tag{10}$$

These two functions are contained in [7, Eq. (3.1), p. 1011]

$$\varepsilon_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \tag{11}$$

where the series converges for $|x| < \frac{1}{|1-q|}$ if $|q| < 1$ and converges for every $x \in \mathbb{C}$ for $|q| > 1$ or $q = 1$. The notation

$$[n]_q! = \prod_{j=1}^{n} \frac{(1-q^j)}{(1-q)^j}$$

for $q \neq 1$ and if $q = 1$, then $[n]_q! = n!$.

In fact [7, Pr. 5.2, p. 1021], $\varepsilon_q(x) = e_q((1-q)x)$ for $|q| < 1$ and $\varepsilon_{1/q}(x) = E_q((1-q)x)$ for $0 < |q| < 1$.

The q-binomial series for $|z| < 1$ and $|q| < 1$ is

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = {}_1\phi_0(a;-;q,z) = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{12}$$

the summation formula [6, Eq. (II.5), p. 236]:

$${}_1\phi_1(a;c;q,c/a) = \frac{(c/a;q)_\infty}{(c;q)_\infty}, \tag{13}$$

and the q-binomial theorem [6, Ex. 1.2(vi), p. 20]

$$\sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} z^k = (z;q)_n. \tag{14}$$

Next, Diaz and Pariguan [4] defined the following generalization of the generalized hypergeometric series.

Definition. For $(a) = (a_1, a_2, \ldots, a_r) \in \mathbb{C}^r$, $(k) = (k_1, k_2, \ldots, k_r) \in (\mathbb{R}^+)^r$, $(b) = (b_1, b_2, \ldots, b_s) \in \mathbb{C}^s \setminus (k\mathbb{Z}^-)^s$ and $l = (l_1, l_2, \ldots, l_s) \in (\mathbb{R}^+)^s$,

$$ {}_rF_s(a,k,b,l)(x) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2}\cdots(a_r)_{n,k_r}}{(b_1)_{n,l_1}(b_2)_{n,l_2}\cdots(b_s)_{n,l_s}n!}x^n. \tag{15}$$

This infinite series converges for all x if $r \le s$, diverges if $r > s+1$, and if $r = s+1$, it converges for $|x| < (l_1 l_2 \cdots l_s)/(k_1 k_2 \cdots k_r)$.

We define its q-analogue in the form of bi-basic series with $k_1 = k_2 = \cdots = k_r = l_1 = l_2 = \cdots = l_s = p \in \mathbb{R}^+$ as follows.

Definition. If (a) stands for the array of r parameters $a_1, a_2, \ldots, a_r \in \mathbb{C}^r$, (b) stands for the array of s parameters $b_1, b_2, \ldots, b_s \in C^s \setminus (Z^-)^s$, $p, \alpha \in \mathbb{R}^+$ and $|q| < 1$, then

$$\begin{aligned} &{}_r\phi_s((a);\ (b)\ ; q^p)(x|q,q^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{(a_1;q)_{n,p}(a_2;q)_{n,p}\cdots(a_r;q)_{n,p}}{(b_1;q)_{n,p}(b_2;q)_{n,p}\cdots(b_s;q)_{n,p}(q^\alpha;q^\alpha)_n}\left((-1)^n q^{\alpha\binom{n}{2}}\right)^{1+s-r} x^n. \end{aligned} \tag{16}$$

Note. The case:

$$\lim_{q\to 1^-} {}_r\phi_s((a);\ (b)\ ; q^p)\left((1-q)^{1+s-r}x \middle| q, q^\alpha\right) = {}_rF_s((a),p,(b),p)(x).$$

The series behaves similarly as the series (15). In fact, if

$$ {}_r\phi_s((a);\ (b)\ ; p)(x|q^\alpha) = \sum_{n=0}^{\infty} A_n x^n,$$

then by d'Alembert's ratio test,

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{(1-a_1q^{np})(1-a_2q^{np})\cdots(1-a_rq^{np})q^{\alpha n(s+1-r)}}{(1-b_1q^{np})(1-b_2q^{np})\cdots(1-b_sq^{np})(1-q^{\alpha(n+1)})}x\right|.$$

From this, it follows that the series converges for all x if $r \le s$, and it diverges when $r > s+1$ and $x \ne 0$. If $r = s+1$, then it converges for $|x| < 1$.

In the present work, we propose a general class of q,p-polynomials involving the function (4) and the symbol (3), as follows.

Definition. For $a \in \mathbb{C}$, $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $0 < q < 1$ and $p > 0$,

$$\mathcal{B}^a_{n,m,p}(x|q;l) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}(q^{-nl/m};q^{l/m})_{mk}(q^{a+np};q)_{\frac{kl}{p},p}\ \gamma_k x^k, \tag{17}$$

in which $l = r - m$, $r \in \mathbb{C}\setminus\{m\}$, and the floor function $\lfloor u \rfloor = floor\ u$, represents the greatest integer $\le u$.

This general class extends the q-extended Jacobi polynomials [2, Eq. (3.8)] and hence the q-Brafman polynomials and the little q-Jacobi polynomials [8, Eq. (3.12.1, p. 92)] (also [6, Ex. 1.32, p. 27]). As a limiting case, this general class also extends the q-Konhauser polynomials [1, Eq. (3.1), p. 3] and hence the q-Laguerre polynomials [10]. The main objective of the work is to establish a general inverse series relations (GISR) which would invert the aforesaid polynomials; and furthermore, this GISR would also extend and invert the well known orthogonal polynomials in ${}_4\phi_3$-function forms namely, the Askey-Wilson polynomials [8, Eq. (3.1.1), p. 63] (also [6, Ex. 2.11, p. 51]) and the q-Racah polynomials [8, Eq. (3.2.1), p. 66] (also [6, Ex. 2.10, p. 51]). It is interesting to note that the q-analogues of some of the Riordan's classes of inverse series relations [3] also assume extension by means of this GISR.

The GISR, as a main result, will be stated and proved in Section 3 using Lemma 3.1. Section 4 incorporates several alternative forms of GISR by means of which various particular polynomials will be deduced. In Section 5, we emphasis on applicability of both series of GISR; the one for obtaining generating function relations (GFR) in Subsection 5.1 and the other, that is the inverse series, for deducing the summation formulas in Subsection 5.2. Some of the q-analogues of Riordan's inverse pairs [3] admit deformation which are tabulated in Section 6. In Section 7, the companion matrix [9] for the general class (17) is illustrated. A chart showing the reducibility of the p-deformed Askey-Wilson polynomials and the p-deformed q-Racah polynomials to a number of polynomials is given in the last section that is, in Section 8. This also includes the inter-connections amongst these particular polynomials.

3. Inverse series relations

While proving the main theorem, we shall require the following inverse pair.

Lemma 3.1. *For* $0 < q < 1,\ M \in \mathbb{N} \cup \{0\},\ m \in \mathbb{N},\ \alpha \in \mathbb{C},\ \lambda \in \mathbb{C} \setminus \{0\}$ *and* $p > 0$,

$$(18)\quad g(M) = \sum_{k=0}^{M} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} M \\ k \end{bmatrix}_{q^\lambda} \frac{(1 - q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+(M+mj)\lambda-kp-mjp}; q)_{\infty,p}} f(k)$$

$$\Leftrightarrow$$

$$(19)\quad f(M) = \sum_{k=0}^{M} (-1)^k q^{k\lambda(k-2M+1)/2} \begin{bmatrix} M \\ k \end{bmatrix}_{q^\lambda} (q^{\alpha+k\lambda+mj\lambda+p-(M+mj)p}; q)_{\infty,p}\ g(k).$$

Proof. We first note that the diagonal elements of the coefficient matrix of the first series are

$$(-1)^i q^{i\lambda(i-1)/2} (1 - q^{\alpha+i\lambda+mj\lambda-ip-mjp}) / (q^{\alpha+(i+mj)\lambda-ip-mjp}; q)_{\infty,p}$$

and those of the second series are

$$(-1)^i q^{i\lambda(1-i)/2} (q^{\alpha+i\lambda+mj\lambda+p-(i+mj)p}; q)_{\infty,p}.$$

Since these elements are all non zero; it follows that these matrices have unique inverse. Hence, it suffice to prove that one of these series implies the other. We prefer to show that (18) implies (19). For that we denote the right hand side of (19) by $\Phi(M)$ and substitute for $g(k)$ from (18) to get

$$\begin{aligned}\Phi(M) = &\sum_{k=0}^{M}(-1)^k q^{k\lambda(k-2M+1)/2}\begin{bmatrix}M\\k\end{bmatrix}_{q^\lambda}\left(q^{\alpha+k\lambda+mj\lambda+p-(M+mj)p};q\right)_{\infty,p}\\ &\times\sum_{i=0}^{k}(-1)^i q^{i\lambda(i-1)/2}\begin{bmatrix}k\\i\end{bmatrix}_{q^\lambda}\frac{(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp})}{(q^{\alpha+(k+mj)\lambda-ip-mjp};q)_{\infty,p}}f(i)\\ = f(M)+&\sum_{i=0}^{M-1}\begin{bmatrix}M\\i\end{bmatrix}_{q^\lambda}q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp})f(i)\sum_{k=0}^{M-i}(-1)^k\\ &\times q^{k\lambda(k+2i-2M+1)/2}\begin{bmatrix}M-i\\k\end{bmatrix}_{q^\lambda}\frac{(q^{\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p};q)_{\infty,p}}{(q^{\alpha+(k+i+mj)\lambda-ip-mjp};q)_{\infty,p}}.\end{aligned}$$

Here, the ratio

$$\frac{(q^{\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p};q)_{\infty,p}}{(q^{\alpha+(k+i+mj)\lambda-ip-mjp};q)_{\infty,p}}=\sum_{l=0}^{M-i-1}A_l\ q^{\lambda kl}$$

say, represents a polynomial of degree $M-i-1$ in k, hence we further have

$$\begin{aligned}\Phi(M) = f(M)+&\sum_{i=0}^{M-1}\begin{bmatrix}M\\i\end{bmatrix}_{q^\lambda}q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp})f(i)\\ &\times\sum_{l=0}^{M-i-1}A_l\sum_{k=0}^{M-i}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix}M-i\\k\end{bmatrix}_{q^\lambda}q^{\lambda k(l+i-M+1)}.\end{aligned}\tag{20}$$

The inner most series on the right hand side in (20) may be summed up by means of the q-binomial theorem (14), then we have

$$\begin{aligned}\Phi(M) = f(M)+&\sum_{i=0}^{M-1}\begin{bmatrix}M\\i\end{bmatrix}_{q^\lambda}q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp})f(i)\\ &\times\sum_{l=0}^{M-i-1}A_l\ (q^{\lambda(l+i-M+1)};q^\lambda)_{M-i}\\ = f(M).&\end{aligned}$$

This completes the proof. ☐

Interestingly, this lemma gives rise to the *q-series orthogonality relation.* In fact, the substitution $\begin{bmatrix}0\\M\end{bmatrix}_{q^\lambda}$ for either $f(M)$ or $g(M)$ yields this property. In particular, the following corollary is of our use.

Corollary 3.2. *For* $0\le j\le n, m\in\mathbb{N},\lambda\in\mathbb{C}\backslash\{0\}$ *and* $p>0$,

$$\begin{bmatrix}0\\M\end{bmatrix}_{q^\lambda}=\sum_{k=0}^{M}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}_{q^\lambda}\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}}\tag{21}$$

$$\times (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}.$$

Proof. In (18), the substitution $g(k)=\begin{bmatrix}0\\k\end{bmatrix}_{q^\lambda}$ gives

$$f(k)=(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p},$$

and with these $f(k)$ and $g(k)$, (19) yields the series orthogonality relation. □

We now establish the main GISR as:

Theorem 3.3. *For $0<q<1$, $\lambda\in\mathbb{C}\setminus\{0\}$, $\alpha\in\mathbb{C}$, $n\in\mathbb{N}\cup\{0\}$, $m\in\mathbb{N}$ and $p>0$,*

$$F(n)=\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{mk}q^{mk\lambda(mk-2n+1)/2}\frac{(q^{\alpha+mk\lambda+p-np};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{n-mk}}G(k) \tag{22}$$

$$\Leftrightarrow$$

$$G(n)=\sum_{k=0}^{mn}(-1)^k q^{k\lambda(k-1)/2}\frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{mn-k}(q^{\alpha+mn\lambda-kp};q)_{\infty,p}}F(k) \tag{23}$$

and for $n\neq mr$, $r\in\mathbb{N}$,

$$\sum_{k=0}^{n}(-1)^k q^{k\lambda(k-1)/2}\frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{n-k}(q^{\alpha+n\lambda-kp};q)_{\infty,p}}F(k)=0. \tag{24}$$

Proof. We first show that (22)⇒(23). We denote the right hand side of (23) by $V(n)$ and then substitute for $F(k)$ from (22) to get

$$V(n)=\sum_{k=0}^{mn}(-1)^k q^{k\lambda(k-1)/2}\frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{mn-k}(q^{\alpha+mn\lambda-kp};q)_{\infty,p}}$$
$$\times\sum_{j=0}^{\lfloor k/m\rfloor}(-1)^{mj}q^{mj\lambda(mj-2k+1)/2}\frac{(q^{\alpha+mj\lambda+p-kp};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{k-mj}}G(j).$$

Here making use of the double series relation [14]:

$$\sum_{k=0}^{mn}\sum_{j=0}^{\lfloor k/m\rfloor}A(k,j)=\sum_{j=0}^{n}\sum_{k=0}^{mn-mj}A(k+mj,j),$$

we further get

$$V(n)=\sum_{j=0}^{n}\sum_{k=0}^{mn-mj}(-1)^k q^{(k+mj)\lambda(k+mj-1)/2+mj\lambda(mj-2k-2mj+1)/2} \tag{25}$$
$$\times\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{mn-mj-k}(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}(q^\lambda;q^\lambda)_k}G(j)$$
$$=G(n)+\sum_{j=0}^{n-1}\frac{G(j)}{(q^\lambda;q^\lambda)_{mn-mj}}\sum_{k=0}^{mn-mj}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix}mn-mj\\k\end{bmatrix}_{q^\lambda}$$

$$\times (1 - q^{\alpha+k\lambda+mj\lambda-kp-mjp}) \frac{(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}}.$$

We now show that the inner series in this last expression vanishes. For that we replace $(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}$ by $f(k)$ and denote the inner series by $g(mn-mj)$, then we have

$$g(mn-mj) = \sum_{k=0}^{mn-mj} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_{q^\lambda} \times \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}} f(k). \tag{26}$$

The inverse companion of this series follows at once from Lemma 3.1 in the form:

$$f(mn-mj) = \sum_{k=0}^{mn-mj} (-1)^k q^{k\lambda(k-2mn+2mj+1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_{q^\lambda} \times (q^{\alpha+k\lambda+mj\lambda+p-mnp};q)_{\infty,p}\; g(k). \tag{27}$$

As suggested by Corollary 3.2, we set $g(k) = \begin{bmatrix} 0 \\ k \end{bmatrix}_{q^\lambda}$ in series (27), we then get $f(k) = (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}$ back, and with these $f(k)$ and $g(k)$, the series orthogonality relation occurs from (26) as given below.

$$\begin{bmatrix} 0 \\ mn-mj \end{bmatrix}_{q^\lambda} = \sum_{k=0}^{mn-mj} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_{q^\lambda} \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}} \times (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}.$$

Using this in (25), we get

$$V(n) = G(n) + \sum_{j=0}^{n-1} \frac{G(j)}{(q^\lambda;q^\lambda)_{mn-mj}} \begin{bmatrix} 0 \\ mn-mj \end{bmatrix}_{q^\lambda} = G(n).$$

Thus, (22)⇒(23). We now show that (22)⇒(24). For that let $R(n)$ denote the right hand side of (24) that is,

$$R(n) = \sum_{k=0}^{n} (-1)^k\; q^{k\lambda(k-1)/2}\; \frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{n-k}(q^{\alpha+n\lambda-kp};q)_{\infty,p}}\; F(k). \tag{28}$$

Proceeding as before, that is, substituting for $F(k)$ from (22), we have

$$R(n) = \sum_{k=0}^{n} (-1)^k\; q^{k\lambda(k-1)/2}\; \frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{n-k}(q^{\alpha+n\lambda-kp};q)_{\infty,p}} \times \sum_{j=0}^{\lfloor k/m \rfloor} (-1)^{mj}\; q^{mj\lambda(mj-2k+1)/2}\; \frac{(q^{\alpha+mj\lambda+p-kp};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{k-mj}}\; G(j) \tag{29}$$

$$= \sum_{j=0}^{\lfloor n/m \rfloor} \frac{G(j)}{(q^\lambda;q^\lambda)_{n-mj}} \sum_{k=0}^{n-mj} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} n-mj \\ k \end{bmatrix}_{q^\lambda} \times \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+n\lambda-kp-mjp};q)_{\infty,p}} (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}.$$

We see that the inner series on the right hand side in this last expression differs slightly from the one occurring in (25); that is, instead of $mn-mj$, it is $n-mj$ here. Accordingly, the series orthogonality relation occurs in the form:

$$\sum_{k=0}^{n-mj} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} n-mj \\ k \end{bmatrix}_{q^\lambda} \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+n\lambda-kp-mjp};q)_{\infty,p}} (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p} = \begin{bmatrix} 0 \\ n-mj \end{bmatrix}_{q^\lambda}.$$

This leads us to

$$R(n) = \sum_{j=0}^{\lfloor n/m \rfloor} \frac{G(j)}{(q^\lambda;q^\lambda)_{n-mj}} \begin{bmatrix} 0 \\ n-mj \end{bmatrix}_{q^\lambda}.$$

If $n \neq mr,\ r \in \mathbb{N}$, then the right hand member in (29) vanishes and thus (22)⇒(24); which completes the proof of the first part. For the converse part, assume that (23) and (24) both hold true. In view of (24),

$$R(n) = 0,\ n \neq mr,\ r \in \mathbb{N}, \tag{30}$$

and also,

$$R(mn) = G(n) \tag{31}$$

by comparing (23) with (28). Now, from the inverse pair (26) and (27), taking $j=0$ and $m=1$, we find that

$$R(n) = \sum_{k=0}^{n} (-1)^k\ q^{k\lambda(k-1)/2}\ \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+n\lambda-kp};q)_{\infty,p}(q^\lambda;q^\lambda)_{n-k}}\ F_k$$

$$\Rightarrow \quad F_n = \sum_{k=0}^{n} (-1)^k\ q^{k\lambda(k-2n+1)/2}\ \frac{(q^{\alpha+k\lambda+p-np};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{n-k}}\ R(k).$$

Hence, in view of the relations (30) and (31), we arrive at

$$R(mn) = \sum_{k=0}^{mn} (-1)^k\ q^{k\lambda(k-1)/2}\ \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp};q)_{\infty,p}(q^\lambda;q^\lambda)_{mn-k}}\ F_k$$

$$\Rightarrow \quad F_n = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk}\ q^{mk\lambda(mk-2n+1)/2}\ \frac{(q^{\alpha+mk\lambda+p-np};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{n-mk}}\ R(mk).$$

Thus, the series in (23) with $R(n) = 0, n \neq mr$ for $r \in \mathbb{N}$, implies the series in (22). This proves the converse part and hence the theorem. □

4. Particular cases

In this section, we obtain the alternative forms of Theorem 3.3 by assuming that the condition (24) holds true. Hence for the sake of brevity, we *shall not* mention the condition (24) in each of the following inverse pairs. These alternative forms will be used to deduce the basic analogues of the general class of q, p-polynomials (17) and its particular cases along with their inverse series relations. Besides this, one of such alternative forms will also be used to deduce the *extended p-deformed Askey-Wilson polynomials* and the *extended p-deformed q-Racah polynomials* together with their inverse series.

We begin with Theorem 3.3 and apply the formula

$$(q^{-1}; q^{-1})_n = (-1)^n q^{-n(n+1)/2}(q; q)_n,$$

in it to get

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^n q^{\lambda(2mk-n(n+1)/2)} \frac{(q^{\alpha+mk\lambda+p-np}; q)_{\infty,p}}{(q^{-\lambda}; q^{-\lambda})_{n-mk}} G(k)$$

$$\Leftrightarrow G(n) = \sum_{k=0}^{mn} (-1)^{mn} q^{\lambda(2kmn-mn(mn+1))/2} \frac{(1 - q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp}; q)_{\infty,p}(q^{-\lambda}; q^{-\lambda})_{mn-k}} F(k).$$

Next, using the formula

$$(q^{-n}; q)_k (q; q)_{n-k} = (-1)^k q^{k(k-2n-1)/2}(q; q)_n$$

with q is replaced by $q^{-\lambda}$ and k by mk in this pair, it transforms to

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{n-mk} q^{-\lambda(n(n+1)-mk(mk-1)+2mnk-2mk)/2} \frac{(q^{n\lambda}; q^{-\lambda})_{mk}}{(q^{-\lambda}; q^{-\lambda})_n} \times (q^{\alpha+mk\lambda+p-np}; q)_{\infty,p}\ G(k)$$

$$\Leftrightarrow G(n) = \sum_{k=0}^{mn} (-1)^{mn-k} q^{-\lambda(mn(mn+1)-k(k-1))/2} \frac{(q^{mn\lambda}; q^{-\lambda})_k (1 - q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp}; q)_{\infty,p}(q^{-\lambda}; q^{-\lambda})_{mn}} F(k).$$

Here replacing $F(n)$ by $q^{-\lambda n(n+1)/2}F(n)$ and $G(n)$ by

$$(q^{-\lambda(mn(mn+1)/2)}/(q^{\alpha+mn\lambda+p}; q)_{\infty,p})\ G(n),$$

we obtain after little simplification, the pair:

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{-\lambda mnk} \frac{(q^{n\lambda}; q^{-\lambda})_{mk}}{(q^{\alpha+mk\lambda+p}; q)_{-n,p}(q^{-\lambda}; q^{-\lambda})_n} G(k)$$

$$\Leftrightarrow G(n) = \sum_{k=0}^{mn} q^{-k\lambda} \frac{(q^{mn\lambda}; q^{-\lambda})_k (1 - q^{\alpha+k\lambda-kp})(q^{\alpha+mn\lambda+p}; q)_{-k,p}}{(q^{-\lambda}; q^{-\lambda})_{mn}(1 - q^{\alpha+mn\lambda-kp})} F(k).$$

But since

$$\frac{q^{-\lambda mnk}}{(q^{\alpha+mk\lambda+p}; q)_{-n,p}} = \frac{q^{-\lambda mnk}(q^{p-\alpha-mk\lambda-p}; q)_{n,p}}{(-1)^n q^{pn(n+1)/2-(\alpha+mk\lambda+p)n}}$$

$$= (-1)^n q^{-pn(n+1)/2+(\alpha+p)n}(q^{-\alpha-mk\lambda};q)_{n,p}$$

and

$$\begin{aligned} q^{-k\lambda}(q^{\alpha+mn\lambda+p};q)_{-k,p} &= \frac{(-1)^k q^{-k\lambda} q^{pk(k+1)/2-(\alpha+mn\lambda+p)k}}{(q^{p-\alpha-mn\lambda-p};q)_{k,p}} \\ &= \frac{(-1)^k q^{-k\lambda-mnk\lambda} q^{pk(k+1)/2-(\alpha+p)k}}{(q^{-\alpha-mn\lambda};q)_{k,p}}, \end{aligned}$$

consequently, the above pair changes to

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^n q^{-pn(n+1)/2+(\alpha+p)n} \frac{(q^{n\lambda};q^{-\lambda})_{mk}(q^{-\alpha-mk\lambda};q)_{n,p}}{(q^{-\lambda};q^{-\lambda})_n} G(k) \\ \Leftrightarrow G(n) &= \sum_{k=0}^{mn} (-1)^k q^{-k\lambda-mnk\lambda} q^{pk(k+1)/2-(\alpha+p)k} \frac{(q^{mn\lambda};q^{-\lambda})_k (1-q^{\alpha+k\lambda-kp})}{(q^{-\lambda};q^{-\lambda})_{mn}(1-q^{\alpha+mn\lambda-kp})} \\ &\quad \times \frac{F(k)}{(q^{-\alpha-mn\lambda};q)_{k,p}}. \end{aligned}$$

Further, replacing $F(n)$ by $(-1)^n q^{-pn(n+1)/2+(\alpha+p)n} F(n)/(q^{-\lambda};q^{-\lambda})_n$, and noticing that

$$\begin{aligned} &\frac{q^{-k\lambda}(1-q^{\alpha+k\lambda-kp})}{(1-q^{\alpha+mn\lambda-kp})(q^{-\alpha-mn\lambda};q)_{k,p}} \\ &= \frac{q^{-k\lambda} q^{\alpha+k\lambda-kp}(1-q^{-\alpha-k\lambda+kp})}{q^{\alpha+mn\lambda-kp}(1-q^{-\alpha-mn\lambda+kp})(q^{-\alpha-mn\lambda};q)_{k,p}} \\ &= \frac{(1-q^{-\alpha-k\lambda+kp})}{q^{mn\lambda}(q^{-\alpha-mn\lambda};q)_{k+1,p}}, \end{aligned}$$

the above pair assumes the form:

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} (q^{n\lambda};q^{-\lambda})_{mk}(q^{-\alpha-mk\lambda};q)_{n,p}\, G(k) \\ \Leftrightarrow G(n) &= \sum_{k=0}^{mn} \frac{q^{-mnk\lambda}(q^{mn\lambda};q^{-\lambda})_k(1-q^{-\alpha-k\lambda+kp})}{q^{mn\lambda}(q^{-\lambda};q^{-\lambda})_{mn}(q^{-\alpha-mn\lambda};q)_{k+1,p}(q^{-\lambda};q^{-\lambda})_k}\, F(k). \end{aligned}$$

Finally, replacing

$$G(n) \text{ by } G(n)/(q^{mn\lambda}(q^{-\alpha-mn\lambda};q)_{\infty,p}), \quad F(n) \text{ by } F(n)/(q^{a+np};q)_{\infty,p}$$

and substituting $\alpha = -a$, $m\lambda = -l$, where $l = r - m$, in this last pair, we obtain

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}(q^{-nl/m};q^{l/m})_{mk}(q^{a+np};q)_{\frac{kl}{p},p} G(k) \tag{32}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} \frac{q^{nkl}(q^{nl};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} F(k). \tag{33}$$

We first deduce the inverse series of the polynomials (17). In fact, the choice $G(n) = \gamma_n x^n$ in (32) yields the polynomials (17); whereas the same substitution in (33) yields its inverse series:

$$\gamma_n x^n = \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} \mathcal{B}^a_{k,m,p}(x|q;l). \tag{34}$$

Next, regarding $l \in \mathbb{C}$, and putting $a = e$ and

$$\gamma_n = (q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}/((q^{\beta_1};q)_{n,p}\cdots(q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_n)$$

in (17) and (34) provides the basic analogue of the p-deformed extended Jacobi polynomials $\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]$ and its inverse series:

$$\begin{aligned} &\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q] \\ &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)};q^{l/m})_{mk}(q^{e+np};q)_{\frac{kl}{p},p}(q^{\alpha_1};q)_{k,p}\cdots(q^{\alpha_c};q)_{k,p}}{(q^{\beta_1};q)_{k,p}\cdots(q^{\beta_d};q)_{k,p}(q^{l/m};q^{l/m})_k} x^k \end{aligned} \tag{35}$$

$\Leftrightarrow$

$$\begin{aligned} &\frac{(q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}}{(q^{\beta_1};q)_{n,p}\cdots(q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_n} x^n \\ &= \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{e+Lk+kp})}{(q^{l/m};q^{l/m})_{mn}(q^{e+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_k} \mathcal{F}^{(e)}_{k,m,p,l}[(\alpha);(\beta):x|q], \end{aligned} \tag{36}$$

where (α) indicates the array of c parameters $\alpha_1, \alpha_2, \ldots, \alpha_c$ and (β) indicates the array of d parameters $\beta_1, \beta_2, \ldots, \beta_d$. Here the limit $q^e \to 0$ leads us to the bi-basic p-deformed q-Brafman polynomials and its inverse series as follows.

$$\begin{aligned} &B^m_{n,p}[(\alpha);(\beta):xq^l|q] \\ &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)};q^{l/m})_{mk}(q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}}{(q^{\beta_1};q)_{k,p}\cdots(q^{\beta_d};q)_{k,p}(q^{l/m};q^{l/m})_k} \ x^k \\ \Leftrightarrow \quad &\frac{(q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}}{(q^{\beta_1};q)_{n,p}\cdots(q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_n} \ x^n \\ &= \sum_{k=0}^{mn} q^{nkl} \frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} B^m_{k,p}[(\alpha);(\beta):q^l x|q]. \end{aligned}$$

the bi-basic p-deformed q-Brafman polynomials and its inverse series tends to the p-deformed Brafman polynomial [13, Eq. (1.16), p. 228] and its inverse series relation [13, p. 232] as $q \to 1$ with $l = m$. The extended p-deformed little q-Jacobi polynomials (cf. [6, p. 27] with $m = l = p = 1$) and its inverse series may be deduced from (17) and (34) by replacing a by $a+b+p$ and taking $\gamma_n = 1/((aq^p;q)_{n,p}(q^{l/m};q^{l/m})_n)$ which are stated below.

$$\begin{aligned} &p_{n,m,p,l}(x;a,b;q) \\ &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)};q^{l/m})_{mk}(abq^{np+p};q)_{\frac{kl}{p},p}}{(aq^p;q)_{k,p}(q^{l/m};q^{l/m})_k} x^k \\ \Leftrightarrow \quad & \frac{x^n}{(aq^p;q)_{n,p}(q^{l/m};q^{l/m})_n} \\ &= \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-abq^{k(l/m)+kp+p})}{(abq^{kp+p};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} p_{k,m,p,l}(x;a,b;q). \end{aligned}$$

Next, in (17) and (34), making the limit $q^a \to 0$, putting

$$\gamma_n = q^{ln(\alpha+1)-lmn+ln(ln-1)/2}/(p\alpha;q)_{nl,p}(q^l;q^l)_{mn},$$

and replacing l and x by lm and $(xq^n)^l$ respectively, lead us to the inverse pair of the extended p-deformed q-Konhauser polynomial (cf. [1] with $p=1$ and $m=1$) and its inverse series:

$$Z^{(\alpha)}_{n,m,p}(x;l|q) = \frac{(p\alpha;q)_{nl,p}}{(q^l;q^l)_n} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{q^{kl(\alpha+n+1)+kl(kl-1)/2}\,(q^{-nl};q^l)_{mk}}{(p\alpha;q)_{kl,p}(q^l;q^l)_{mk}} x^{kl} \tag{37}$$

$\Leftrightarrow$

$$\frac{q^{ln(\alpha+1)-lmn+ln(ln-1)/2}}{(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}} x^{ln} = \sum_{k=0}^{mn} \frac{(-1)^k q^{kl(kl-1)/2}\, Z^{(\alpha)}_{k,m,p}(x;l|q)}{(\alpha q^p;q)_{kl,p}(q^l;q^l)_{mn-k}}. \tag{38}$$

The series in (37) and (38) provide basic analogues of the extended p-deformed Konhauser polynomial [13, Eq. (1.17), p. 229] and its inverse series [13, Eq. (2.8), p. 232]. The instance $l=1$ is the pair of inverse series relations involving the extended p-deformed q-Laguerre polynomials (cf. [10] with $m=p=1$):

$$\begin{aligned} L^{(\alpha)}_{n,m,p}(x|q) &= \frac{(p\alpha;q)_{n,p}}{(q;q)_n} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(q^{-n};q)_{mk}\; q^{k(\alpha+n+1)+k(k-1)/2}}{(p\alpha;q)_{k,p}(q;q)_{mk}} x^k \\ \Leftrightarrow \frac{q^{n(\alpha+1)-mn+n(n-1)/2}}{(p\alpha;q)_{n,p}(q;q)_{mn}} x^n &= \sum_{k=0}^{mn} \frac{(-1)^k q^{k(k-1)/2}}{(p\alpha;q)_{k,p}(q;q)_{mn-k}} L^{(\alpha)}_{k,m,p}(x|q). \end{aligned}$$

It is noteworthy that the inverse pair (32) and (33) provides the extension to the Askey-Wilson polynomials and q-Racah polynomials to which we call the extended p-deformed Askey-Wilson polynomials and the extended p-deformed q-Racah polynomials; and denote them by

$$p_{n,l,m,p}(\cos\theta;a,b,c,d\,|\,q) \text{ and } R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d\,|\,q),$$

respectively. These polynomials may be deduced from (32) and (33) as follows. First replacing a by $a+b+c+d-p$ and then choosing

$$G(n) = (ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n),$$
$$F(n) = p_{n,l,m,p}(\cos\theta;a,b,c,d|q)a^n/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p})$$

yield the pair:

$$\text{(39)} \quad \frac{p_{n,l,m,p}(\cos\theta;a,b,c,d|q)a^n}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}} = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}\ \frac{(q^{-n(l/m)};q^{l/m})_{mk}}{(q^{l/m};q^{l/m})_k} \times \frac{(abcdq^{np-p};q)_{kl/p,p}\ (ae^{i\theta};q)_{k,p}\ (ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}}$$

$$\Leftrightarrow$$

$$\text{(40)} \quad \frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n} = \sum_{k=0}^{mn} q^{nkl}\ \frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_k} \times \frac{(1-abcdq^{kL+kp-p})\ a^k\ p_{k,l,m,p}(\cos\theta;a,b,c,d|q)}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}}.$$

Likewise, in the inverse pair (32) and (33) if a is replaced by $a+b+p$ and $G(n)$ is chosen as

$$(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}/((aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n),$$

then $F(n) = R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)$ yields the inverse pair:

$$\text{(41)} \quad R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}\ \frac{(q^{-n(l/m)};q^{l/m})_{mk}}{(q^{l/m};q^{l/m})_k} \frac{(abq^{np+p};q)_{\frac{kl}{p},p}(q^{-x};q)_{k,p}(cdq^{x+p};q)_{k,p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}}$$

$$\Leftrightarrow$$

$$\text{(42)} \quad \frac{(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}}{(aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n} = \sum_{k=0}^{mn} q^{nkl} \frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_k} \frac{(1-abq^{kL+kp+p})}{(abq^{kp+p};q)_{ln/p+1,p}(q^{l/m};q^{l/m})_{mn}} \times R_{k,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q),$$

These p-deformed q-polynomials provide p-extension to a number of particular q-polynomials (see [8, pp. 61–62] for complete reducibility chart and [8, Ch. 3]). They include the q-Hahn, dual q-Hahn, continuous q-Hahn, continuous dual

q-Hahn, Meixner-Pollaczek, Meixner, Krawtchouk and Charlier polynomials together with their inverse series relations.

5. Application

We now apply the first series of Theorem 3.3 to derive the generating function relations for the particular q-polynomials; and then apply the second series that is, the inverse series to obtain the summation formulas involving these q-polynomials.

5.1. Generating function relations

The generating function relations for the general class of q-polynomials (17), the extended p-deformed Askey-Wilson polynomials and the extended p-deformed q-Racah polynomials will be derived with the help of the alternative form (32) as follows.

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\ t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor} q^{ln(n-1)/2m}\frac{(q^{-ln/m};q^{l/m})_{mk}}{(q^{l/m};q^{l/m})_n}\ (a;q)_{n+\frac{kl}{p},p}\ q^{klp}G(k)\ t^n.$$

Now, in formula (7), replacing k by mk and taking $p = 1$, it changes to

$$(q^{l/m};q^{l/m})_{n-mk} = (-1)^{mk}q^{lk(mk+1)/2-lk-lnk}\ \frac{(q^{l/m};q^{l/m})_n}{(q^{-ln/m};q^{l/m})_{mk}},$$

from which we have

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\ t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{mk}q^{ln(n-1)/2m+lk(mk-1)/2-lnk}\frac{(a;q)_{n+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_{n-mk}}\ q^{kl}G(k)\ t^n.$$

Here the double sum may be replaced by means of the identity [14, Eq. (5), p. 101] (also [11, Eq. (7), p. 57] for $m = 2$):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk),$$

to get

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\ t^n \tag{43}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}q^{ln(n-1)/2m}\frac{(a;q)_{n+mk+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_n}\ G(k)\ t^{n+mk}.$$

From this, we deduce the GFR of the q-polynomials with the assumption that $|t| < 1$.

(i) **GFR of $\mathcal{B}^{a}_{n,m,p}(x|q;l)$.**
In (43), the choice $G(n) = \gamma_n x^n \Rightarrow F(n) = \mathcal{B}^{a}_{n,m,p}(x|q;l)$ which yields a general generating function relation:

$$(44) \quad \sum_{n=0}^{\infty} q^{ln(n-1)/2m} \frac{(a;q)_{n,p}}{(q^{l/m};q^{l/m})_n} \mathcal{B}^{a}_{n,m,p}(x|q;l)\ t^n$$
$$= \sum_{k=0}^{\infty} (-1)^{mk} (a;q)_{mk+\frac{kl}{p},p}\ {}_1\phi_1\left(aq^{mkp+kl};0;p\right)\left(t\middle|q,q^{l/m}\right)\ \gamma_k\ \left((-t)^m x\right)^k,$$

in which we have used (5). This relation when further specialized appropriately, provides us the GFR of the particular polynomials which are illustrated below.

(ii) **GFR of $\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]$.**
Next, choosing $l \in \mathbb{C}$, $a = e$ and

$$\gamma_n = (\alpha_1;q)_{n,p} \cdots (\alpha_c;q)_{n,p} / ((\beta_1;q)_{n,p} \cdots (\beta_d;q)_{n,p} (q^{l/m};q^{l/m})_n)$$

in (44), we immediately obtain the GFR

$$\sum_{n=0}^{\infty} q^{(l/m)n(n-1)/2} \mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]\ \frac{(e;q)_{n,p}}{(q^{l/m};q^{l/m})_n}\ t^n$$
$$= \sum_{k=0}^{\infty}\ {}_1\phi_1\left(eq^{mkp+kl};0;p\right)\left(t\middle|q,q^{l/m}\right) \frac{(\alpha_1;q)_{k,p} \cdots (\alpha_c;q)_{k,p}\left((-t)^m x\right)^k}{(\beta_1;q)_{k,p} \cdots (\beta_d;q)_{k,p} (q^{l/m};q^{l/m})_n}.$$

The limiting case $e \equiv q^e \to 0$ of this yields

(iii) **GFR of $B^{m}_{n,p}[(\alpha);(\beta):xq^l|q]$.**
We find using (10), that

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m}\ B^{m}_{n,p}[(\alpha);(\beta):xq^l|q]\ \frac{t^n}{(q^{l/m};q^{l/m})_n}$$
$$= \varepsilon_{q^{-l/m}}(t)\ {}_c\phi_d((\alpha);\ (\beta)\ ;p)\left(x(-t)^m q^{-l} x\middle|q,q^{l/m}\right),$$

wherein $c = d+1$ for convergence as well as the validity of the function notation.

(iv) **GFR of $p_{n,m,p,l}(x;a,b;q)$.**
In (44), we replace a by $a+b+p$ and substitute $\gamma_n = 1/((ap;q)_{n,p}(q^{l/m};q^{l/m})_n)$ to get

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m} \frac{(abq^p;q)_{n,p}}{(q^{l/m};q^{l/m})_n} p_{n,m,p,l}(x;a,b;q)\ t^n$$
$$= \sum_{k=0}^{\infty} \frac{(abq^p;q)_{mk+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_k}\ {}_1\phi_1\left(abq^{p+mkp+kl};0;p\right)(t|q,q^{l/m}).$$

(v) **GFR of $Z^{(\alpha)}_{n,m,p}(x;l|q)$.**
In (44), taking limit $q^a \to 0$, replacing l and x by lm and $(xq^n)^l$, $l \in \mathbb{N}$, and putting $\gamma_n = q^{ln(\alpha+1)-lmn+ln(ln-1)/2}/(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}$, we get

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2} \frac{Z^{(\alpha)}_{n,m,p}(x;l|q)}{(p\alpha;q)_{nl,p}}\, t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk} q^{nl(n-1)/2-klm} \frac{q^{kl(\alpha+n+1)-klm+kl(kl-1)/2}}{(p\alpha;q)_{kl,p}(q^l;q^l)_{mk}(q^l;q^l)_n}\, q^{mkl}x^{kl}\, t^{n+mk}.$$

Here we may put $kl = s$ to get elegant form:

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2} \frac{Z^{(\alpha)}_{n,m,p}(x;l|q)}{(p\alpha;q)_{nl,p}} t^n = \varepsilon_{q^{-l}}(t) \sum_{s=0}^{\infty} \frac{q^{(\alpha+n+1)s+s(s-1)/2-ms}}{(p\alpha;q)_{s,p}(q^l;q^l)_{ms}} x^s(-t)^{ms/l}.$$

(vi) **GFR of $L^{(\alpha)}_{n,m,p}(x|q)$.**
It is the straightforward case $l = 1$ the GFR 5.

The GFR of the polynomials (39) and (41) follow from (43). Here also we assume that $|t| < 1$. Now, if a is replaced by $a+b+c+d-p$ and if

$$G(n) = (ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n),$$

then $F(n) = p_{n,l,m,p}(\cos\theta;a,b,c,d|q)a^n/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p})$ leads us to

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m}(abcdq^{-p};q)_{n,p} \frac{p_{n,l,m,p}(\cos\theta;a,b,c,d|q)a^n}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}\, t^n$$
$$= \sum_{k=0}^{\infty} \frac{(abcdq^{-p};q)_{mk+\frac{kl}{p},p}(ae^{i\theta};q)_{k,p}(ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{l/m};q^{l/m})_k}$$
$$\times {}_1\phi_1\left(abcdq^{mkp+kl-p};0;p\right)(t|q,q^{l/m})(-t)^{mk}.$$

Likewise, replacing a by $a+b+p$ and choosing

$$G(n) = (q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}/((aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n)$$

in (43) implies $F(n) = R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)$. We then find the following GFR.

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m} \frac{(abq^p;q)_{n,p}}{(q^{l/m};q^{l/m})_n} R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)\, t^n$$
$$= \sum_{k=0}^{\infty} \frac{(q^{-x};q)_{k,p}(cdq^{x+p};q)_{k,p}(abq^p;q)_{mk+\frac{kl}{p},p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}(q^{l/m};q^{l/m})_k}$$
$$\times {}_1\phi_1\left(abq^{p+mkp+kl};0;p\right)(t|q,q^{l/m})\,(-t)^{mk}.$$

5.2. Summation formulas

In this section, we illustrate the application of the inverse series of the GISR and in particular the inverse series of the general class (17), to deduce certain summation formulas. The inverse series (23) of Theorem 3.3 provides the sums involving the p-Askey-Wilson polynomials (39) and p-q-Racah polynomials (41); whereas the inverse series (34) takes care of the sums involving the other polynomials.

We begin with the inverse series (34) with the assumption that $\gamma_n \neq 0, \forall n = 0, 1, 2, \ldots$, then we have

$$\frac{1}{\gamma_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) = x^n. \tag{45}$$

In this, multiplying both sides by $(a;q)_n/(q;q)_n$ and taking summation from $n = 0$ to ∞ and then using (12) with $|x| < 1$, we get

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n\gamma_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l)$$
$$= \frac{(ax;q)_\infty}{(x;q)_\infty}.$$

If $x = 0$, then $\mathcal{B}^a_{k,m,p}(0|q;l) = \gamma_0$ simplifies this sum to the form:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n\gamma_0}{(q;q)_n\gamma_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} = 1.$$

Next, multiplying both sides by $1/[n]_q! = 1/(q;q)_n$ and taking summation from $n = 0$ to ∞, in (45) provides

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{\gamma_n(q;q)_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) = \varepsilon_q(x),$$

using (11). Taking summation $n = 0$ to ∞ and assuming $|x| < 1$ in (45) yields

$$\sum_{n=0}^{\infty}\frac{1}{\gamma_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) = \frac{1}{1-x}.$$

By assigning different values to x from $(-1, 1)$, a number of particular summation formulas can be derived. For example, $x = 1/2$ in this formula gives the following one.

$$\sum_{n=0}^{\infty}\frac{1}{\gamma_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}\left(\frac{1}{2}\Big|q;l\right) = 2.$$

The sum of ${}_1\phi_1[*]$ in (13) enables us to obtain one more summation formula by multiplying $\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n}{(c;q)_n(q;q)_n}$ to both sides of (45), replacing x by c/a and then summing-up from $n = 0$ to ∞. We then obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(a;q)_n}{\gamma_n (c;q)_n (q;q)_n} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k (1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} \mathcal{B}^a_{k,m,p}\left(\frac{c}{a}\Big| q;l\right)$$
$$= \frac{(c/a;q)_\infty}{(c;q)_\infty}.$$

The reducibility to all these summation formulas to the particular polynomials may be obtained by making the substitutions as specified in Section 4.

Illustration. Taking $a = e$ and

$$\gamma_n = (q^{\alpha_1};q)_{n,p} \cdots (q^{\alpha_c};q)_{n,p} / ((q^{\beta_1};q)_{n,p} \cdots (q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_{n,p})$$

in (45) yields the summation formula involving the p-deformed extended q-Jacobi polynomials as follows.

$$\frac{(\beta_1;q)_{n,p} \cdots (\beta_d;q)_{n,p}(q^{l/m};q^{l/m})_{n,p}}{(\alpha_1;q)_{n,p} \cdots (\alpha_c;q)_{n,p}} \sum_{k=0}^{mn} q^{kln} \frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_{mn}}$$
$$\times \frac{(1-q^{e+Lk+kp})}{(q^{e+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_k} \mathcal{F}^{(e)}_{k,m,p,l}[(\alpha);(\beta):x|q] = x^n.$$

We now obtain summation formulas involving the extended p-deformed Askey-Wilson polynomials and p-deformed extended q-Racah polynomials. While illustrating the sums, we require the deformed versions of the q-Gauss sum [6, Eq. (1.5.1), p. 10] and corresponding q-Vandermonde's sum [6, Eq. (1.5.2), p. 11] :

$${}_2\phi_1\left(a,b;c;q,\frac{c}{ab}\right) = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty}$$

and

$${}_2\phi_1\left(q^{-n},b;c;q,\frac{cq^n}{b}\right) = \frac{(c/b;q)_n}{(c;q)_n},$$

respectively. For that we notice the relation $(a;q)_{n,p} = (a;q^p)_n$, $p > 0$, thereby transform these sums to the forms:

$$(46) \qquad {}_2\phi_1\left(a,b;c;q^p,\frac{c}{ab}\right) = \frac{(c/a;q)_{\infty,p} (c/b;q)_{\infty,p}}{(c;q)_{\infty,p} (c/ab;q)_{\infty,p}}$$

and

$$(47) \qquad {}_2\phi_1\left(q^{-np},b;c;q^p,\frac{cq^{np}}{b}\right) = \frac{(c/b;q)_{n,p}}{(c;q)_{n,p}}.$$

We rewrite the inverse series (40) by introducing $(q^p;q)_{n,p}$ to get

$$(48) \qquad \frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-ln};q^{l/m})_k (1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$

$$\times \frac{a^k p_{k,l,m,p}(\cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} = \frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(q^p;q)_{n,p}}.$$

We intend to use (46), and for that we multiply both side of (48) by

$$q^{n(b-a-2\cos\theta)}$$

and then take sum from $n = 0$ to ∞, then after little simplification, we find

$$\sum_{n=0}^{\infty} \frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$
$$\times \frac{a^k p_{k,l,m,p}(\cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} q^{n(b-a-2\cos\theta)}$$
$$= \frac{\left(be^{-i\theta};q\right)_{\infty,p}\left(be^{i\theta};q\right)_{\infty,p}}{(ab;q)_{\infty,p}\left(q^{b-a-2\cos\theta};q\right)_{\infty,p}}.$$

In (48), we transfer $(ae^{-i\theta};q)_{n,p}$ to the other side to get

$$\frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$
$$\times \frac{a^k p_{k,l,m,p}(\cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} = \frac{(ae^{i\theta};q)_{n,p}}{(ab;q)_{n,p}(q^p;q)_{n,p}}.$$

In this, multiplying both sides by $(q^{-jp};q)_{n,p}(q^{jp}be^{-i\theta})^n$ and then taking the sum from $n = 0$ to j, then using (47) on the right hand side, we obtain

$$\sum_{n=0}^{j} \frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$
$$\times \frac{a^k p_{k,l,m,p}(\cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}(q^{-jp};q)_{n,p}(q^{jp}be^{-i\theta})^n = \frac{\left(be^{-i\theta};q\right)_{j,p}}{(ab;q)_{j,p}}.$$

We proceed in a similar manner to derive summation formulas from the inverse series (42). We rewrite it by introducing the factor $(q^p;q)_{n,p}$ and transfer the factors $(cdq^{x+p};q)_{n,p}$, $(bdq^p;q)_{n,p}$, $(cq^p;q)_{n,p}$ and $(q^{l/m};q^{l/m})_n$ to the other side to get

$$\frac{(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n}{(cdq^{x+p};q)_{n,p}(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abq^{kL+kp+p})}{(abq^{kp+p};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}}$$
$$\times \frac{R_{k,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)}{(q^{l/m};q^{l/m})_k} = \frac{(q^{-x};q)_{n,p}}{(aq^p;q)_{n,p}(q^p;q)_{n,p}}.$$

Now multiplying both sides by $(q^{-jp};q)_{n,p}(axq^{jp+p})^n$ and then taking the summation from $n = o$ to j, we obtain

$$\sum_{n=0}^{j} \frac{(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n}{(cdq^{x+p};q)_{n,p}(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-abq^{kL+kp+p})}{(abq^{kp+p};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}}$$

$$\times \frac{(q^{-jp};q)_{n,p}(axq^{jp+p})^n}{(q^{l/m};q^{l/m})_k} R_{k,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q) = \frac{(aq^{x+p};q)_{j,p}}{(aq^p;q)_{j,p}}.$$

6. Extension of Riordan's q-inverse pairs

Apart from yielding the extended p-deformed q-polynomials, Theorem 3.3 and its alternative forms also provide an effective tool for carrying out the extension of certain inverse series relations belonging to q-Riordan's classification [3, Tables 2, 5, 7, pp. 17–20] in the sense of p-deformation (also see [12] for *ordinary* forms). For instance, replacing $G(n)$ and $F(n)$ by

$$q^{-\lambda mn(mn+1)/2}(q^p;q)_{\alpha/p+\lambda mn-mn,p}G(n)/(q^p;q)_{\infty,p} \text{ and } q^{-\lambda n(n+1)/2}(-1)^n F(n)$$

in (32) and (33), yield the
* Inverse pair - 1

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{-\lambda(mk(mk-1))/2)} \frac{(q^p;q)_{\frac{\alpha+\lambda mk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+mk\lambda-np}{p},p}(q^{-\lambda};q^{-\lambda})_{n-mk}} G(k)$$
$$\Leftrightarrow G(n) = \sum_{k=0}^{mn} (-1)^{mn+k} q^{-\lambda k(k-2mn+1)/2} \frac{(1-q^{\alpha+k\lambda-kp})(q^p;q)_{\frac{\alpha+mn\lambda-kp-p}{p},p}}{(q^p;q)_{\frac{\alpha+\lambda mn-mnp}{p},p}(q^{-\lambda};q^{-\lambda})_{mn-k}} F(k).$$

Next, replacing α by $\alpha+p, F(n)$ by $F(n)/(1-q^{\alpha+\lambda n-np+p})$ and $G(n)$ by $G(n)/(1-q^{\alpha+\lambda mn-mnp+p})$ in inverse pair - 1, yields
* Inverse pair - 2

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{-\lambda mk(mk-1)/2} \frac{(1-q^{\alpha+\lambda n-np+p})(q^p;q)_{\frac{\alpha+\lambda mk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+mk\lambda-np+p}{p},p}(q^{-\lambda};q^{-\lambda})_{n-mk}} G(k)$$
$$\Leftrightarrow G(n) = \sum_{k=0}^{mn} (-1)^{mn+k} q^{-\lambda k(k-2mn+1)/2} \frac{(q^p;q)_{\frac{\alpha+mn\lambda-kp}{p},p}}{(q^p;q)_{\frac{\alpha+\lambda mn-mnp}{p},p}(q^{-\lambda};q^{-\lambda})_{mn-k}} F(k).$$

Here inverting the base q, and then replacing $G(n)$ by

$$G(n)/(q^{-\alpha-\lambda mn+mn-1};q^{-1})_{\infty,p},$$

this pair transforms to the
* Inverse pair - 3

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{\lambda mk(mk-1)/2} \frac{(q^p;q)_{\frac{-\alpha+np-\lambda mk-p}{p},p}}{(q^p;q)_{\frac{-\alpha-mk\lambda+mkp-p}{p},p}(q^{\lambda};q^{\lambda})_{n-mk}} G(k)$$
$$\Leftrightarrow G(n) = \sum_{k=0}^{mn} (-1)^{mn+k} q^{\lambda k(k-2mn+1)/2} \frac{(1-q^{-\alpha-\lambda k+kp})(q^p;q)_{\frac{-\alpha-\lambda mn+mnp-p}{p},p}}{(q^p;q)_{\frac{-\alpha-\lambda mn+kp}{p},p}(q^{\lambda};q^{\lambda})_{mn-k}} F(k).$$

Finally, replacing $F(n)$ by $F(n)/(1-q^{-\alpha-\lambda np+np})$ in this last pair, we get
∗ Inverse pair - 4

$$F(n)=\sum_{k=0}^{\lfloor n/m\rfloor} q^{\lambda mk(mk-1))/2}\frac{(1-q^{-\alpha-\lambda n+np})(q^p;q)_{\frac{-\alpha+np-\lambda mk-p}{p},p}}{(q^p;q)_{\frac{-\alpha-mk\lambda+mkp}{p},p}(q^\lambda;q^\lambda)_{n-mk}}G(k)$$

$$\Leftrightarrow G(n)=\sum_{k=0}^{mn}(-1)^{mn+k}q^{\lambda k(k-2mn+1)/2}\frac{(q^p;q)_{\frac{-\alpha-mn\lambda+mnp}{p},p}}{(q^p;q)_{\frac{-\alpha-\lambda mn+kp}{p},p}(q^\lambda;q^\lambda)_{mn-k}}F(k).$$

Inverse pairs - 1 to 4 lead us to the p-deformed versions of certain inverse pairs of q-Riordan classes as shown in the following table.

TABLE 1. p-deformed extension of certain q-Riordan inverse pairs

$$F(n)=\sum_{k=0}^{\lfloor n/m\rfloor} q^{\beta mk(mk-1)/2}A_{n,k}G(k);$$
$$G(n)=\sum_{k=0}^{mn}(-1)^{mn+k}q^{\beta k(k-2mn+1)/2}B_{n,k}F(k)$$

Inverse pair -	β	α	λ	$A_{n,k}$	$B_{n,k}$	p-deformed extension of q-class (inverse pair no.) as in Table 2, 5 and 7 [3]
1	$-l$	α	l	$\frac{(q^p;q)_{\frac{\alpha+lmk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+lmk-np}{p},p}} \times\frac{1}{(q^{-l};q^{-l})_{n-mk}}$	$\frac{(1-q^{\alpha+lk-kp})}{(q^p;q)_{\frac{\alpha+lmn-mnp}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+lmn-kp-p}{p},p}}{(q^{-l};q^{-l})_{mn-k}}$	q-Gold Class(1) Table 2
2	$-l$	α	l	$\frac{(1-q^{\alpha+ln-np+p})}{(q^p;q)_{\frac{\alpha+lmk-np+p}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+lmk-mkp}{p},p}}{(q^{-l};q^{-l})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+lmn-kp}{p},p}}{(q^p;q)_{\frac{\alpha+lmn-mnp}{p},p}} \times\frac{1}{(q^{-l};q^{-l})_{mn-k}}$	q-Gold Class(2) Table 2
3	$p-2$	$-\alpha-p$	$p-2$	$\frac{(q^p;q)_{\frac{\alpha+np+2mk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+2mk}{p},p}} \times\frac{1}{(q^{(p-2)};q^{(p-2)})_{n-mk}}$	$\frac{(1-q^{\alpha+2k+p})}{(q^p;q)_{\frac{\alpha+2mn-mnp+kp+p}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+2mn}{p},p}}{(q^{(p-2)};q^{(p-2)})_{mn-k}}$	q-Simpler Legendre Class(1) Table 5
4	$p-2$	$-\alpha$	$p-2$	$\frac{(1-q^{\alpha+2n})}{(q^p;q)_{\frac{\alpha+2mk}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+np-mkp+2mk-p}{p},p}}{(q^{(p-2)};q^{(p-2)})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+2mn}{p},p}}{(q^p;q)_{\frac{\alpha+2mn-mnp+kp}{p},p}} \times\frac{1}{(q^{(p-2)};q^{(p-2)})_{mn-k}}$	q-Simpler Legendre Class(2) Table 5
4	$p-c$	$-\alpha$	$p-c$	$\frac{(1-q^{\alpha+cn})}{(q^p;q)_{\frac{\alpha+cmk}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+np+cmk-mkp-p}{p},p}}{(q^{(p-c)};q^{(p-c)})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+cmn}{p},p}}{(q^p;q)_{\frac{\alpha+cmn-mnp+kp}{p},p}} \times\frac{1}{(q^{(p-c)};q^{(p-c)})_{mn-k}}$	q-Legendre-Chebyshev Class(1) Table 7
1	$-p-c$	α	$p+c$	$\frac{(q^p;q)_{\frac{\alpha+cmk}{p},p}}{(q^p;q)_{\frac{\alpha+cmk+mkp-np}{p},p}} \times\frac{1}{(q^{-(c+p)};q^{-(c+p)})_{n-mk}}$	$\frac{(1-q^{\alpha+ck})}{(q^p;q)_{\frac{\alpha+cmn}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+cmn+mnp-kp-p}{p},p}}{(q^{-(c+p)};q^{-(c+p)})_{mn-k}}$	q-Legendre -Chebyshev Class(3) Table 7
3	$p-c$	$-\alpha-p$	$p-c$	$\frac{(q^p;q)_{\frac{\alpha+np+cmk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+cmk}{p},p}}$	$\frac{(1-q^{\alpha+ck+p})}{(q^p;q)_{\frac{\alpha+cmn-mnp+kp+p}{p},p}}$	q-Legendre-Chebyshev

Table 1. – *Continue*

Inverse pair -	β	α	λ	$A_{n,k}$	$B_{n,k}$	p-deformed extension of q-class (inverse pair no.) as in Table 2, 5 and 7[3]
				$\times \frac{1}{(q^{(p-c)};q^{(p-c)})_{n-mk}}$	$\times \frac{(q^p;q)_{\frac{\alpha+cmn}{p},p}}{(q^{(p-c)};q^{(p-c)})_{mn-k}}$	Class(5) Table 7
2	$-p-c$	α	$p+c$	$\frac{(1-q^{\alpha+cn+p})}{(q^p;q)_{\frac{\alpha+cmk+mkp-np+p}{p},p}}$	$\frac{(q^p;q)_{\frac{\alpha+cmn+mnp-kp}{p},p}}{(q^p;q)_{\frac{\alpha+cmn}{p},p}}$	q-Legendre-Chebyshev
				$\times \frac{(q^p;q)_{\frac{\alpha+cmk}{p},p}}{(q^{(c+p)};q^{(c+p)})_{n-mk}}$	$\times \frac{1}{(q^{(c+p)};q^{(c+p)})_{mn-k}}$	Class(7) Table 7

7. Companion matrix

The companion matrix of a polynomial is defined as follows [9, p. 39].

Definition. If a polynomial $f(x) = \delta_0 + \delta_1 x + \delta_2 x^2 + \cdots + \delta_{j-1} x^{j-1} + x^j \in K[X], K$ is a field, then to f there is associated the $j \times j$ matrix $C(f(x))$, called the companion matrix of $f(x)$, is denoted and defined by

$$C(f(x)) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\delta_0 & -\delta_1 & -\delta_2 & \dots & -\delta_{j-1} \end{bmatrix}.$$

Note. The eigenvalues of the companion matrix $C(f(x))$ are precisely the zeros of $f(x)$, counting the multiplicity. The characteristic polynomial of $C(f(x))$ is therefore, $f(x)$.

We have the following [9, Prop. 1.5.14, p. 39].

Proposition. *If $f(x) \in K[X]$ is non constant and $A = C(f(x))$ is the companion matrix of $f(x)$, then $f(A) = O$, where O is a zero matrix.*

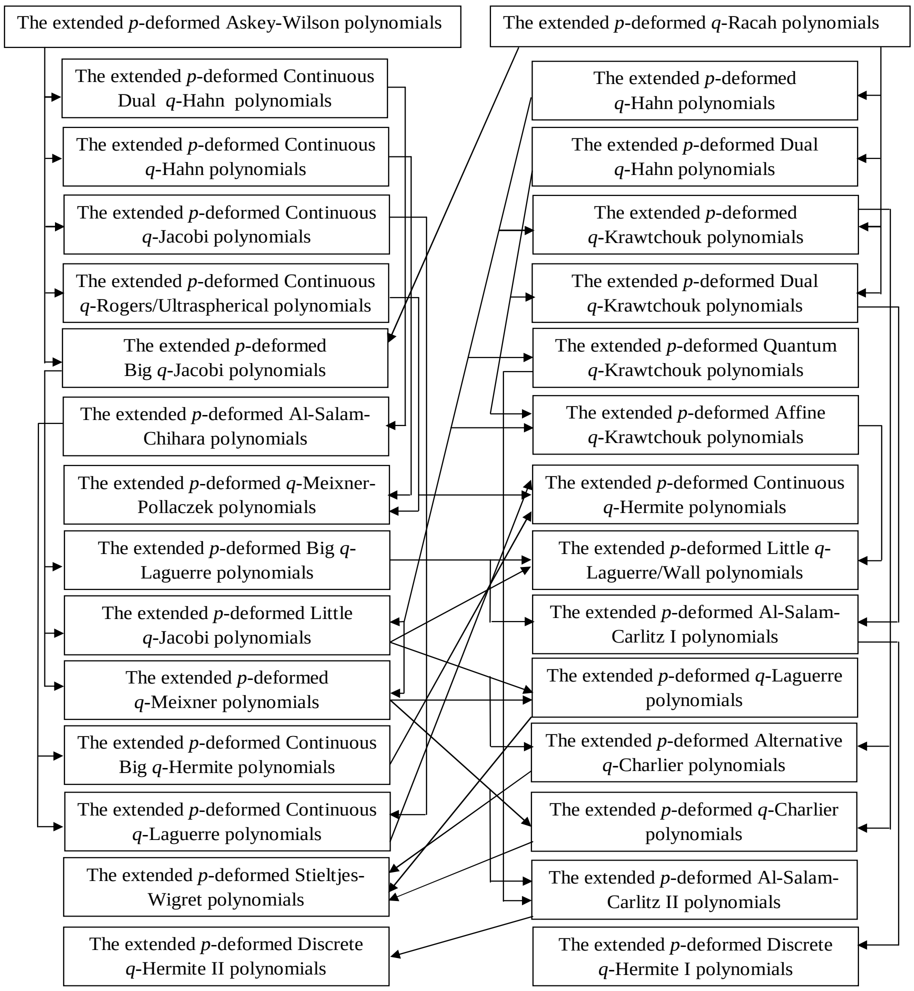
Taking $\lfloor n/m \rfloor = N$ in (17) and converting it to the monic form $\widetilde{\mathcal{B}}^a_{n,m,p}(x|q;l)$, we get

$$\widetilde{\mathcal{B}}^a_{n,m,p}(x|q;l) = \sum_{k=0}^{N} \delta_k \; x^k,$$

where

$$\delta_k = \frac{q^{kl}(q^{-n(l/m)};q^{l/m})_{mk}(q^{a+np};q)_{\frac{kl}{p},p}\gamma_k}{q^{lN}(q^{-(l/m)n};q^{(l/m)})_n(q^{a+np};q)_{\frac{Nl}{p},p}\gamma_N}.$$

Thus, $\widetilde{B}^a_{n,m,p}(x|q;l)$ is of the form as stated in Definition 7. The eigenvalues of this matrix will be then precisely the zeros of the polynomial $\widetilde{\mathcal{B}}^a_{n,m,p}(x|q;l)$.



8. p-deformed polynomials' reducibility chart

Scheme of p-deformed extended basic hypergeometric orthogonal polynomials

Acknowledgement. Author sincerely thanks the referee for the valuable suggestions; and also thanks his guide B. I. Dave for his help during the preparation of the manuscript.

References

[1] W. A. Al-Salam and A. Verma, *q-Konhauser polynomials*, Pacific J. Math. **108** (1983), no. 1, 1–7. http://projecteuclid.org/euclid.pjm/1102720467

[2] B. I. Dave, *A general q-inversion formula and extensions of certain polynomial systems*, J. Indian Math. Soc. (N.S.) **65** (1998), no. 1-4, 119–126.
[3] ______, *A general q-inverse series relation*, Bol. Soc. Mat. Mex. (3) **24** (2018), no. 2, 279–299. https://doi.org/10.1007/s40590-017-0172-8
[4] R. Díaz and E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulg. Mat. **15** (2007), no. 2, 179–192.
[5] R. Díaz and C. Teruel, *q, k-generalized gamma and beta functions*, J. Nonlinear Math. Phys. **12** (2005), no. 1, 118–134. https://doi.org/10.2991/jnmp.2005.12.1.10
[6] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, **35**, Cambridge University Press, Cambridge, 1990.
[7] R. Goldman, P. Simeonov, and Y. Simsek, *Generating functions for the q-Bernstein bases*, SIAM J. Discrete Math. **28** (2014), no. 3, 1009–1025. https://doi.org/10.1137/130921623
[8] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report 98–17, TU Delft University of Technology, The Netherlands, 1998.
[9] M. Mignotte and D. Ştefănescu, *Polynomials*, Springer Series in Discrete Mathematics and Theoretical Computer Science, Springer-Verlag Singapore, Singapore, 1999.
[10] D. S. Moak, *The q-analogue of the Laguerre polynomials*, J. Math. Anal. Appl. **81** (1981), no. 1, 20–47. https://doi.org/10.1016/0022-247X(81)90048-2
[11] E. D. Rainville, *Special Functions*, The Macmillan Co., New York, 1960.
[12] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, Inc., New York, 1968.
[13] R. V. Savalia and B. I. Dave, *p-deformation of a general class of polynomials and its properties*, J. Indian Math. Soc. (N.S.) **85** (2018), no. 1-2, 226–240. https://doi.org/10.18311/jims/2018/17945
[14] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1984.

Rajesh V. Savalia
Research Scholar at Department of Mathematics
Faculty of Science
The Maharaja Sayajirao University of Baroda
Vadodara-390 002, India
and
Department of Mathematical Sciences
P. D. Patel Institute of Applied Sciences
Faculty of Applied Sciences
Charotar University of Science and Technology
Changa-388 421, Dist: Anand, India
Email address: rajeshsavalia.maths@charusat.ac.in