# CONVEXITY OF INTEGRAL OPERATORS GENERATED BY SOME NEW INEQUALITIES OF HYPER-BESSEL FUNCTIONS 

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#### Abstract

In this article, we deduced some new inequalities related to hyper-Bessel function. By using these inequalities we will find some sufficient conditions under which certain families of integral operators are convex in the open unit disc. Some applications related to these results are also the part of our investigation.


## 1. Introduction

Let $\mathcal{A}$ denote the class of functions $f$ of the form

$$
\begin{equation*}
f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n} \tag{1.1}
\end{equation*}
$$

which are analytic in the open unit disc $\mathcal{U}=\{z:|z|<1\}$ and $\mathcal{S}$ denote the class of all functions which are univalent in $\mathcal{U}$. Let $\mathcal{S}^{*}(\alpha), \mathcal{C}(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of starlike, convex and close-to-convex functions of order $\alpha$ and are defined as:

$$
\begin{aligned}
\mathcal{S}^{*}(\alpha) & =\left\{f: f \in \mathcal{A} \text { and } \operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)>\alpha, \quad z \in \mathcal{U}, \alpha \in[0,1)\right\} \\
\mathcal{C}(\alpha) & =\left\{f: f \in \mathcal{A} \text { and } \operatorname{Re}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>\alpha, \quad z \in \mathcal{U}, \quad \alpha \in[0,1)\right\}
\end{aligned}
$$

and

$$
\mathcal{K}(\alpha)=\left\{f: f \in \mathcal{A} \text { and } \operatorname{Re}\left(\frac{z f^{\prime}(z)}{g(z)}\right)>\alpha, \quad z \in \mathcal{U}, \alpha \in[0,1), g \in \mathcal{S}^{*}\right\}
$$

It is clear that

$$
\mathcal{S}^{*}(0)=\mathcal{S}^{*}, \mathcal{C}(0)=\mathcal{C} \text { and } \mathcal{K}(0)=\mathcal{K} .
$$

[^0]The Bessel function of the first kind $J_{v}$ is defined by

$$
\begin{equation*}
J_{v}(z)=\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!\Gamma(v+n+1)}\left(\frac{z}{2}\right)^{2 n+v} \tag{1.2}
\end{equation*}
$$

where $\Gamma$ stands for Euler gamma function. It is a particular solution of the second order linear homogeneous differential equation

$$
z^{2} w^{\prime \prime}(z)+z w^{\prime}(z)+\left(z^{2}-v^{2}\right) w(z)=0
$$

where $v \in \mathbb{C}$. For some details see $[1,6]$. Bessel functions are indispensable in many branches of mathematics and applied mathematics. Thus, it is important to study their properties in many aspects. Next, we consider the hyper-Bessel function in terms of the hypergeometric functions defined below (for details see [2])

$$
\begin{equation*}
J_{\alpha_{d}}(z)=\frac{\left(\frac{z}{d+1}\right)^{\alpha_{1}+\alpha_{2}+\cdots+\alpha_{d}}}{\prod_{i=1}^{d} \Gamma\left(\alpha_{i}+1\right)}{ }_{0} F_{d}\left(\left(\alpha_{d}^{-}+1\right) ;-\left(\frac{z}{d+1}\right)^{d+1}\right) \tag{1.3}
\end{equation*}
$$

where the notation

$$
\begin{equation*}
{ }_{p} F_{q}\left(\binom{(\beta)_{p}}{(\gamma)_{q}} ; x\right)=\sum_{n=0}^{\infty} \frac{\left(\beta_{1}\right)_{n}\left(\beta_{2}\right)_{n} \cdots\left(\beta_{p}\right)_{n}}{\left(\gamma_{1}\right)_{n}\left(\gamma_{2}\right)_{n} \cdots\left(\gamma_{p}\right)_{n}} \frac{x^{n}}{n!}, \tag{1.4}
\end{equation*}
$$

represents the generalized hypergeometric function, where $(\beta)_{n}$ represents the shifted factorial or pochhammer's symbol and $\alpha_{d}$ represents the array of $d$ parameters $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{d}$. By combining (1.3) and (1.4), we get the following infinite representation of the hyper-Bessel function

$$
\begin{equation*}
J_{\alpha_{d}}(z)=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!\prod_{i=1}^{d} \Gamma\left(\alpha_{i}+n+1\right)}\left(\frac{z}{d+1}\right)^{n(d+1)+\alpha_{1}+\alpha_{2}+\cdots+\alpha_{d}} \tag{1.5}
\end{equation*}
$$

The normalized hyper-Bessel function $J_{\alpha_{d}}(z)$ is defined by

$$
\begin{equation*}
J_{\alpha_{d}}(z)=\frac{\left(\frac{z}{d+1}\right)^{\alpha_{1}+\alpha_{2}+\cdots+\alpha_{d}}}{\prod_{i=1}^{d} \Gamma\left(\alpha_{i}+1\right)} \mathcal{J}_{\alpha_{d}}(z) \tag{1.6}
\end{equation*}
$$

By combining (1.5) and (1.6), we get the following representation of the hyperBessel function

$$
\begin{equation*}
\mathcal{J}_{\alpha_{d}}(z)=1+\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(d+1)^{(n-1)(d+1)} \prod_{i=1}^{d}\left(\alpha_{i}+1\right)_{n-1}} z^{(n-1)(d+1)} \tag{1.7}
\end{equation*}
$$

It is observed that the function $\mathcal{J}_{\alpha_{d}}$ defined in (1.7) does not belong to the class $\mathcal{A}$. Here, we consider the following normalized form of the hyper-Bessel function for our own convenience.

$$
\begin{align*}
\mathcal{H}_{\alpha_{d}}(z) & =z \mathcal{J}_{\alpha_{d}}(z) \\
\text { 8) } & =z+\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(d+1)^{(n-1)(d+1)} \prod_{i=1}^{d}\left(\alpha_{i}+1\right)_{n-1}} z^{(n-1)(d+1)+1} . \tag{1.8}
\end{align*}
$$

Recently, Deniz et al. [3], Din et al. [4] and Din et al. [5] have obtained sufficient conditions for the univalence of certain families of integral operators defined by Bessel, Struve and Dini functions respectively. The families of integral operators are defined below:

$$
\begin{gather*}
F_{\alpha_{1}, \ldots, \alpha_{n}, \zeta}(z)=\left\{\zeta \int_{0}^{z} t^{\zeta-1} \prod_{i=1}^{n}\left(\frac{f_{i}(t)}{t}\right)^{\frac{1}{\alpha_{i}}} d t\right\}^{1 / \zeta},  \tag{1.9}\\
G_{\xi, n}(z)=\left\{(n \xi+1) \int_{0}^{z} \prod_{i=1}^{n}\left(f_{i}(t)\right)^{\xi} d t\right\}^{1 /(n \xi+1)}  \tag{1.10}\\
H_{\delta_{1}, \ldots, \delta_{n}, \mu}(z)=\left\{\mu \int_{0}^{z} t^{\mu-1} \prod_{i=1}^{n}\left(f_{i}^{\prime}(t)\right)^{\delta_{i}} d t\right\}^{1 / \mu} \tag{1.11}
\end{gather*}
$$

and

$$
\begin{equation*}
Q_{\lambda}(z)=\left\{\lambda \int_{0}^{z} t^{\lambda-1}\left(e^{f(t)}\right)^{\lambda} d t\right\}^{1 / \lambda} \tag{1.12}
\end{equation*}
$$

In this paper, we are mainly interested in the convexity of the integral operators involving Dini function $q_{v}$. These integral operators are defined as

$$
\begin{equation*}
F_{v_{1}, \ldots, v_{n}, \alpha_{1}, \ldots, \alpha_{n}}(z)=\int_{0}^{z} \prod_{i=1}^{n}\left(\frac{q_{v_{i}}(t)}{t}\right)^{\alpha_{i}} d t \tag{1.13}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{v_{1}, \ldots, v_{n}, \delta_{1}, \ldots, \delta_{n}}(z)=\int_{0}^{z} \prod_{i=1}^{n}\left(q_{v_{i}}^{\prime}(t)\right)^{\delta_{i}} d t \tag{1.14}
\end{equation*}
$$

Now we prove some functional inequalities which are useful in establishing our main results.

Lemma 1.1. Let $i \in\{1,2,3, \ldots, d\}, \alpha_{i}>-1$ and $\zeta \eta>1$, where $\zeta=(d+1)^{d+1}$, $\eta=\prod_{i=1}^{d}\left(\alpha_{i}+1\right)$. Then $\mathcal{H}_{\alpha_{d}}$ defined in (1.8) satisfy the following inequalities:
(i) $\left|\frac{z \mathcal{H}_{\alpha_{d}}^{\prime}(z)}{\mathcal{H}_{\alpha_{d}}(z)}-1\right| \leq \frac{(d+1)(2 \zeta \eta-1)}{(\zeta \eta-1)(2 \zeta \eta-3)}, \zeta \eta>1$.
(ii) $\left|\frac{z \mathcal{H}_{\alpha_{d}}^{\prime \prime}(z)}{\mathcal{H}_{\alpha_{d}}^{\prime}(z)}\right| \leq \frac{\zeta \eta(2 \zeta \eta-1)(d+1)(2 d+3)}{(\zeta \eta-1)(2 \zeta \eta-3)-(2 \zeta \eta-1)(d+1)}, \zeta \eta>1$.

Proof. (i) By using the well known triangle inequality and the inequalities

$$
n!\geq n, \quad\left(\alpha_{i}+1\right)_{n} \geq\left(\alpha_{i}+1\right)^{n}, \forall n \in \mathbb{N}
$$

we obtain

$$
\left|\mathcal{H}_{\alpha_{d}}^{\prime}(z)-\frac{\mathcal{H}_{\alpha_{d}}(z)}{z}\right| \leq \frac{(d+1)}{\zeta \eta} \sum_{n \geq 1}\left(\frac{1}{\zeta \eta}\right)^{n-1}
$$

where

$$
\zeta=(d+1)^{d+1} \text { and } \eta=\prod_{i=1}^{d}\left(\alpha_{i}+1\right)
$$

This implies that

$$
\begin{equation*}
\left|\mathcal{H}_{\alpha_{d}}^{\prime}(z)-\frac{\mathcal{H}_{\alpha_{d}}(z)}{z}\right| \leq \frac{d+1}{\zeta \eta-1} . \tag{1.15}
\end{equation*}
$$

Furthermore, if we use the reverse triangle inequality and the inequality

$$
n!\geq 2^{n-1},\left(\alpha_{i}+1\right)_{n} \geq\left(\alpha_{i}+1\right)^{n}, \forall n \in \mathbb{N}
$$

then we get

$$
\begin{aligned}
\left|\frac{\mathcal{H}_{\alpha_{d}}(z)}{z}\right| & =\left|1+\sum_{n \geq 1} \frac{(-1)^{n-1}}{(n-1)!(d+1)^{(n-1)(d+1)} \prod_{i=1}^{d}\left(\alpha_{i}+1\right)_{n-1}} z^{(n-1)(d+1)}\right| \\
& \geq 1-\frac{1}{\zeta \eta} \sum_{n \geq 1}\left(\frac{1}{2 \zeta \eta}\right)^{n-1} \\
& =\frac{2 \zeta \eta-3}{2 \zeta \eta-1} .
\end{aligned}
$$

By combining (1.15) and (1.16), we get

$$
\left|\frac{z \mathcal{H}_{\alpha_{d}}^{\prime}(z)}{\mathcal{H}_{\alpha_{d}}(z)}-1\right| \leq \frac{(d+1)(2 \zeta \eta-1)}{(\zeta \eta-1)(2 \zeta \eta-3)}, \quad \zeta \eta>1
$$

(ii) Consider

$$
\begin{aligned}
\mathcal{H}_{\alpha_{d}}(z) & =\sum_{n \geq 0} \frac{(-1)^{n}}{n!(d+1)^{n(d+1)} \prod_{i=1}^{d}\left(\alpha_{i}+1\right)_{n}} z^{n(d+1)+1}, \\
z \mathcal{H}_{\alpha_{d}}^{\prime \prime}(z) & =\sum_{n \geq 0} \frac{\left\{n^{2}(d+1)^{2}+n(d+1)\right\}(-1)^{n}}{n!(d+1)^{n(d+1)} \prod_{i=1}^{d}\left(\alpha_{i}+1\right)_{n}} z^{n(d+1)},
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{n \geq 1} \frac{(n-1)^{2}(d+1)^{2}}{(n-1)!(d+1)^{(n-1)(d+1)} \prod_{i=1}^{d}\left(\alpha_{i}+1\right)_{n-1}} z^{n(d+1)} \\
& +\sum_{n \geq 1} \frac{(n-1)(d+1)}{(n-1)!(d+1)^{(n-1)(d+1)} \prod_{i=1}^{d}\left(\alpha_{i}+1\right)_{n-1}} z^{n(d+1)} .
\end{aligned}
$$

By using the well known triangle inequality with inequalities

$$
(n-1)!\geq \frac{(n-1)^{2}}{2},(n-1)!\geq n-1,\left(\alpha_{i}+1\right)_{n} \geq\left(\alpha_{i}+1\right)^{n}, \forall n \in \mathbb{N}
$$

we have

$$
\begin{aligned}
\left|z \mathcal{H}_{\alpha_{d}}^{\prime \prime}(z)\right| & =\left|\begin{array}{c}
\sum_{n \geq 1} \frac{(n-1)^{2}(d+1)^{2}}{(n-1)!(d+1)^{(n-1)(d+1)} \prod_{i=1}^{d}\left(\alpha_{i}+1\right)_{n-1}} z^{n(d+1)} \\
+\sum_{n \geq 1} \frac{(n-1)(d+1)}{(n-1)!(d+1)^{(n-1)(d+1)} \prod_{i=1}^{d}\left(\alpha_{i}+1\right)_{n-1}} z^{n(d+1)}
\end{array}\right| \\
& \leq 2(d+1)^{2} \sum_{n \geq 1}\left(\frac{1}{\zeta \eta}\right)^{n-1}+(d+1) \sum_{n \geq 1}\left(\frac{1}{\zeta \eta}\right)^{n-1},
\end{aligned}
$$

where

$$
\zeta=(d+1)^{d+1} \text { and } \eta=\prod_{i=1}^{d}\left(\alpha_{i}+1\right)
$$

This implies that

$$
\begin{equation*}
\left|z \mathcal{H}_{\alpha_{d}}^{\prime \prime}(z)\right| \leq \frac{\zeta \eta(d+1)(2 d+3)}{(\zeta \eta-1)} \tag{1.17}
\end{equation*}
$$

Furthermore, if we use the reverse triangle inequality and the inequalities

$$
n!\geq n, n!\geq 2^{n-1}, \quad\left(\alpha_{i}+1\right)_{n} \geq\left(\alpha_{i}+1\right)^{n}, \forall n \in \mathbb{N}
$$

then we get

$$
\begin{align*}
\left|\mathcal{H}_{\alpha_{d}}^{\prime}(z)\right| & \geq 1-\sum_{n \geq 1} \frac{n(d+1)+1}{n!\zeta^{n} \eta^{n}} \\
& \geq 1-\frac{d+1}{\zeta \eta} \sum_{n \geq 1}\left(\frac{1}{\zeta \eta}\right)^{n-1}+\frac{1}{\zeta \eta} \sum_{n \geq 1}\left(\frac{1}{2 \zeta \eta}\right)^{n-1} \\
& =\frac{(\zeta \eta-1)(2 \zeta \eta-3)-(2 \zeta \eta-1)(d+1)}{(\zeta \eta-1)(2 \zeta \eta-1)} \tag{1.18}
\end{align*}
$$

By combining (1.17) and (1.18), we get

$$
\left|\frac{z \mathcal{H}_{\alpha_{d}}^{\prime \prime}(z)}{\mathcal{H}_{\alpha_{d}}^{\prime}(z)}\right| \leq \frac{\zeta \eta(2 \zeta \eta-1)(d+1)(2 d+3)}{(\zeta \eta-1)(2 \zeta \eta-3)-(2 \zeta \eta-1)(d+1)}, \quad \zeta \eta>1
$$

## 2. Convexity of integral operators with normalized hyper-Bessel function

The main objective of this paper is to give convexity properties of integral operators involving normalized hyper-Bessel function. The main results are given as follows.

Theorem 2.1. Let $i \in\{1,2,3, \ldots, d\}, \alpha_{i}>-1$ and $\zeta \eta>1$, where $\zeta=$ $(d+1)^{d+1}, \eta=\prod_{i=1}^{d}\left(\alpha_{i}+1\right)$ and $\mathcal{H}_{\alpha_{d}}$ defined in (1.8). Suppose that $\sigma_{1}, \ldots, \sigma_{n}$ be positive real numbers such that these numbers satisfy the following inequality

$$
0 \leq 1-\frac{(d+1)(2 \zeta \eta-1)}{(\zeta \eta-1)(2 \zeta \eta-3)} \sum_{i=1}^{n} \sigma_{i}<1
$$

Then, the function $F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}}$ defined by (1.13), is in the class $\mathcal{C}(\beta)$, where

$$
\beta=1-\frac{(d+1)(2 \zeta \eta-1)}{(\zeta \eta-1)(2 \zeta \eta-3)} \sum_{i=1}^{n} \sigma_{i}
$$

Proof. We observe that $\mathcal{H}_{\left(\alpha_{d}\right)_{i}}(z)$ are such that $\mathcal{H}_{\left(\alpha_{d}\right)_{i}}(0)=\mathcal{H}_{\left(\alpha_{d}\right)_{i}}^{\prime}(0)-1=0$. It is also clear that $F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}} \in \mathcal{A}$. That is $F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}}(0)=\left(F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}}\right)^{\prime}(0)-1=$ 0 . On the other hand, it is easy to see that

$$
\begin{equation*}
\left(F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma n}\right)^{\prime}(z)=\prod_{i=1}^{n}\left(\frac{\mathcal{H}_{\left(\alpha_{d}\right)_{i}}(z)}{z}\right)^{\sigma_{i}} \tag{2.1}
\end{equation*}
$$

Differentiating logarithmically, we get

$$
\begin{equation*}
\frac{z\left(F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}}\right)^{\prime \prime}(z)}{\left(F_{\alpha_{1}, \ldots, \sigma_{d}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots}(z)\right.}=\sum_{i=1}^{n} \sigma_{i}\left(\frac{z \mathcal{H}_{\left(\alpha_{d}\right)_{i}}^{\prime}(z)}{\mathcal{H}_{\left(\alpha_{d}\right)_{i}}(z)}-1\right) \tag{2.2}
\end{equation*}
$$

This implies that

$$
\operatorname{Re}\left\{1+\frac{z\left(F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}}\right)^{\prime \prime}(z)}{\left(F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}}\right)^{\prime}(z)}\right\}=\sum_{i=1}^{n} \sigma_{i} \operatorname{Re}\left(\frac{z \mathcal{H}_{\left(\alpha_{d}\right)_{i}}^{\prime}(z)}{\mathcal{H}_{\left(\alpha_{d}\right)_{i}}(z)}\right)+\left(1-\sum_{i=1}^{n} \sigma_{i}\right) .
$$

Now, by using the assertion (i) of Lemma 1.1 for each $M_{i}=(\zeta \eta)_{i}$, where $i=1,2, \ldots, n$, we obtain

$$
\begin{aligned}
\operatorname{Re}\left\{1+\frac{z\left(F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}}\right)^{\prime \prime}(z)}{\left(F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \alpha_{n}}\right)^{\prime}(z)}\right\} & \geq \sum_{i=1}^{n} \sigma_{i}\left(1-\frac{(d+1)\left(2 M_{i}-1\right)}{\left(M_{i}-1\right)\left(2 M_{i}-3\right)}\right)+\left(1-\sum_{i=1}^{n} \sigma_{i}\right) \\
& =1-\sum_{i=1}^{n} \sigma_{i} \frac{(d+1)\left(2 M_{i}-1\right)}{\left(M_{i}-1\right)\left(2 M_{i}-3\right)}
\end{aligned}
$$

Consider the function $\phi:\left(\frac{3}{2}, \infty\right) \rightarrow \mathbb{R}$ defined as

$$
\phi(M)=\frac{(d+1)(2 M-1)}{(M-1)(2 M-3)}
$$

is a decreasing function such that

$$
\frac{(d+1)\left(2 M_{i}-1\right)}{\left(M_{i}-1\right)\left(2 M_{i}-3\right)} \leq \frac{(d+1)(2 M-1)}{(M-1)(2 M-3)}, \forall i=1,2, \ldots, n
$$

Therefore

$$
\operatorname{Re}\left\{1+\frac{z\left(F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}}\right)^{\prime \prime}(z)}{\left(F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}}\right)^{\prime}(z)}\right\}>1-\frac{(d+1)(2 M-1)}{(M-1)(2 M-3)} \sum_{i=1}^{n} \sigma_{i}
$$

Since $0 \leq 1-\frac{(d+1)(2 M-1)}{(M-1)(2 M-3)} \sum_{i=1}^{n} \sigma_{i}<1$, therefore $F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma_{1}, \ldots, \sigma_{n}} \in \mathcal{C}(\beta)$, where

$$
\beta=1-\frac{(d+1)(2 \zeta \eta-1)}{(\zeta \eta-1)(2 \zeta \eta-3)} \sum_{i=1}^{n} \sigma_{i}
$$

which completes the proof.
By setting $\sigma_{1}=\sigma_{2}=\cdots=\sigma_{n}=\sigma$ in Theorem 2.1, we obtain the result given below.

Corollary 2.2. Let $i \in\{1,2,3, \ldots, d\}, \alpha_{i}>-1$ and $\zeta \eta>1$, where $\zeta=$ $(d+1)^{d+1}, \eta=\prod_{i=1}^{d}\left(\alpha_{i}+1\right)$ and $\mathcal{H}_{\alpha_{d}}$ defined in (1.8). Suppose that $\sigma$ be positive real number such that satisfy the following inequality

$$
0 \leq 1-\frac{n \sigma(d+1)(2 \zeta \eta-1)}{(\zeta \eta-1)(2 \zeta \eta-3)}<1
$$

Then, the function $F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma}$ defined by

$$
F_{\alpha_{1}, \ldots, \alpha_{d}}^{\sigma}(z)=\int_{0}^{z} \prod_{i=1}^{n}\left(\frac{\mathcal{H}_{\left(\alpha_{d}\right)_{i}}(t)}{t}\right)^{\sigma} d t
$$

is in the class $\mathcal{C}\left(\beta_{1}\right)$, where

$$
\beta=1-\frac{n \sigma(d+1)(2 \zeta \eta-1)}{(\zeta \eta-1)(2 \zeta \eta-3)} .
$$

The next theorem gives convexity properties of the integral operator defined in (1.14). The key tool in the proof is inequality (ii) of Lemma 1.1.

Theorem 2.3. Let $i \in\{1,2,3, \ldots, d\}, \alpha_{i}>-1$ and $\zeta \eta>1$, where $\zeta=$ $(d+1)^{d+1}, \eta=\prod_{i=1}^{d}\left(\alpha_{i}+1\right)$ and $\mathcal{H}_{\alpha_{d}}$ defined in (1.8). Suppose that $\delta_{1}, \ldots, \delta_{n}$ be positive real numbers such that these numbers satisfy the following inequality

$$
0 \leq 1-\frac{\zeta \eta(2 \zeta \eta-1)(d+1)(2 d+3)}{(\zeta \eta-1)(2 \zeta \eta-3)-(2 \zeta \eta-1)(d+1)} \sum_{i=1}^{n} \delta_{i}<1
$$

Then, the function $H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}$ defined by (1.14), is in the class $\mathcal{C}(\gamma)$, where

$$
\gamma=1-\frac{\zeta \eta(2 \zeta \eta-1)(d+1)(2 d+3)}{(\zeta \eta-1)(2 \zeta \eta-3)-(2 \zeta \eta-1)(d+1)} \sum_{i=1}^{n} \delta_{i} .
$$

Proof. It can easily be observed that, the operator defined in (1.14) belongs to class $\mathcal{A}$, that is $H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}(0)=\left(H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}\right)^{\prime}(0)-1=0$. Differentiating (1.14), we have

$$
\left(H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}\right)^{\prime}(z)=\prod_{i=1}^{n}\left(\mathcal{H}_{\left(\alpha_{d}\right)_{i}}^{\prime}(z)\right)^{\delta_{i}}
$$

Differentiating logarithmically, we obtain

$$
\frac{z\left(H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}\right)^{\prime \prime}(z)}{\left(H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}\right)^{\prime}(z)}=\sum_{i=1}^{n} \delta_{i}\left(\frac{z \mathcal{H}_{\left(\alpha_{d}\right)_{i}}^{\prime \prime}(z)}{\mathcal{H}_{\left(\alpha_{d}\right)_{i}}^{\prime}(z)}\right)
$$

This implies that

$$
\operatorname{Re}\left\{1+\frac{z\left(H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}\right)^{\prime \prime}(z)}{\left(H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}\right)^{\prime}(z)}\right\}=1+\sum_{i=1}^{n} \delta_{i} \operatorname{Re}\left(\frac{z \mathcal{H}_{\left(\alpha_{d}\right)_{i}}^{\prime \prime}(z)}{\mathcal{H}_{\left(\alpha_{d}\right)_{i}}^{\prime}(z)}\right)
$$

Now, by using the assertion (ii) of Lemma 1.1 for each $M_{i}=(\zeta \eta)_{i}$, where $i=1,2, \ldots, n$, we obtain

$$
\begin{aligned}
& \operatorname{Re}\left\{1+\frac{z\left(H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}\right)^{\prime \prime}(z)}{\left(H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \alpha_{n}}\right)^{\prime}(z)}\right\} \\
> & 1-\sum_{i=1}^{n} \delta_{i}\left(\frac{M_{i}\left(2 M_{i}-1\right)(d+1)(2 d+3)}{\left(M_{i}-1\right)\left(2 M_{i}-3\right)-\left(2 M_{i}-1\right)(d+1)}\right) .
\end{aligned}
$$

Consider the function $\varphi:\left(\frac{(2 d+7)+\sqrt{4 d^{2}+20 d+25}}{4}, \infty\right) \rightarrow \mathbb{R}$ defined as

$$
\varphi(M)=\frac{M(2 M-1)(d+1)(2 d+3)}{(M-1)(2 M-3)-(2 M-1)(d+1)}
$$

is a decreasing function such that

$$
\begin{aligned}
& \frac{M_{i}\left(2 M_{i}-1\right)(d+1)(2 d+3)}{\left(M_{i}-1\right)\left(2 M_{i}-3\right)-\left(2 M_{i}-1\right)(d+1)} \\
\leq & \frac{M(2 M-1)(d+1)(2 d+3)}{(M-1)(2 M-3)-(2 M-1)(d+1)}, \forall i=1,2, \ldots, n .
\end{aligned}
$$

It follows that

$$
\operatorname{Re}\left\{1+\frac{z\left(H_{\alpha_{1}}^{\delta_{1}, \ldots, \delta_{n}}\right)^{\prime}, \alpha_{n}}{\left(H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}}\right)^{\prime}(z)}\right\}>1-\frac{M(2 M-1)(d+1)(2 d+3)}{(M-1)(2 M-3)-(2 M-1)(d+1)} \sum_{i=1}^{n} \delta_{i}
$$

Since

$$
0 \leq 1-\frac{M(2 M-1)(d+1)(2 d+3)}{(M-1)(2 M-3)-(2 M-1)(d+1)} \sum_{i=1}^{n} \delta_{i}<1
$$

therefore $H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta_{1}, \ldots, \delta_{n}} \in \mathcal{C}(\gamma)$, where

$$
\gamma=1-\frac{M(2 M-1)(d+1)(2 d+3)}{(M-1)(2 M-3)-(2 M-1)(d+1)} \sum_{i=1}^{n} \delta_{i},
$$

which completes the proof.
By setting $\delta_{1}=\delta_{2}=\delta_{n}=\delta$ in Theorem 2.3, we obtain the result given below.

Corollary 2.4. Let $i \in\{1,2,3, \ldots, d\}, \alpha_{i}>-1$ and $\zeta \eta>1$, where $\zeta=$ $(d+1)^{d+1}, \eta=\prod_{i=1}^{d}\left(\alpha_{i}+1\right)$ and $\mathcal{H}_{\alpha_{d}}$ defined in (1.8). Suppose that $\delta_{1}, \ldots, \delta_{n}$ be positive real numbers such that these numbers satisfy the following inequality

$$
0 \leq 1-\frac{n \delta \zeta \eta(2 \zeta \eta-1)(d+1)(2 d+3)}{(\zeta \eta-1)(2 \zeta \eta-3)-(2 \zeta \eta-1)(d+1)}<1
$$

Then, the function $H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta}$ defined as

$$
H_{\alpha_{1}, \ldots, \alpha_{n}}^{\delta}(z)=\int_{0}^{z} \prod_{i=1}^{n}\left(\mathcal{H}_{\left(\alpha_{d}\right)_{i}}^{\prime}(t)\right)^{\delta} d t
$$

is in the class $\mathcal{C}\left(\gamma_{1}\right)$, where

$$
\gamma_{1}=1-\frac{n \delta \zeta \eta(2 \zeta \eta-1)(d+1)(2 d+3)}{(\zeta \eta-1)(2 \zeta \eta-3)-(2 \zeta \eta-1)(d+1)}
$$

## 3. Applications

When we put $d=1$ and $\alpha_{1}=v$ in (1.8), we get the classical Bessel function defined in (1.2). The normalized classical Bessel function can also be obtained by putting $d=1$ and $\alpha_{1}=v$ in (1.8). The normalized classical Bessel function is defined as:

$$
\begin{equation*}
\psi_{v}(z)=\sum_{n \geq 1} \frac{(-1)^{n}}{4^{n} n!(v+1)_{n}} z^{2 n+1} \tag{3.1}
\end{equation*}
$$

By choosing $v=\frac{1}{2}$ and $v=\frac{3}{2}$ in (3.1), we get the following forms of the normalized hyper-Bessel function

$$
\begin{aligned}
\psi_{\frac{1}{2}}(z) & =\sin z \\
\psi_{\frac{3}{2}} & (z)
\end{aligned}=3\left(\frac{\sin z}{z^{2}}-\frac{\cos z}{z}\right) .
$$

In particular, the results of the above mentioned theorems are given below.
Corollary 3.1. Let $v_{1}, v_{2}, \ldots, v_{n}>-\frac{5}{8}$ and $\psi_{v}(z): \mathcal{U} \rightarrow \mathbb{C}$ be defined as

$$
\psi_{v}(z)=\sum_{n \geq 1} \frac{(-1)^{n}}{4^{n} n!(v+1)_{n}} z^{2 n+1}
$$

Let $v=\min \left\{v_{1}, \ldots, v_{n}\right\}$. Suppose that $\sigma_{1}, \ldots, \sigma_{n}$ be positive real numbers such that these numbers satisfy the following inequality

$$
0 \leq 1-\frac{2(8 v+7)}{(4 v+3)(8 v+5)} \sum_{i=1}^{n} \sigma_{i}<1
$$

Then, the function $F_{v}^{\sigma}$ defined by

$$
F_{v}^{\sigma}(z)=\int_{0}^{z}\left(\frac{\psi_{v}(t)}{t}\right)^{\sigma} d t
$$

is in the class $\mathcal{C}\left(\beta_{2}\right)$, where

$$
\beta_{2}=1-\frac{2(8 v+7)}{(4 v+3)(8 v+5)} \sum_{i=1}^{n} \sigma_{i} .
$$

In particular,
(i) if $\sigma \leq \frac{45}{22}$, then the function $F_{\frac{1}{2}}^{\sigma}: \mathcal{U} \rightarrow \mathbb{C}$ defined by

$$
F_{\frac{1}{2}}^{\sigma}(z)=\int_{0}^{z}(\sin t)^{\sigma} d t
$$

is in the class $\mathcal{C}\left(\beta_{3}\right)$, where $\beta_{3}=1-\frac{22 \alpha}{45}$;
(ii) if $\alpha \leq \frac{153}{38}$, then the function $F_{\frac{3}{2}}^{\sigma}: \mathcal{U} \rightarrow \mathbb{C}$ defined by

$$
F_{\frac{3}{2}}^{\sigma}(z)=\int_{0}^{z} 3\left(\frac{\sin t}{t^{2}}-\frac{\cos t}{t}\right)^{\sigma} d t
$$

is in the class $\mathcal{C}\left(\beta_{4}\right)$, where $\beta_{4}=1-\frac{38 \alpha}{153}$.
Corollary 3.2. Let $v_{1}, v_{2}, \ldots, v_{n}>\frac{-2+\sqrt{3}}{4}$ and $\psi_{v}(z): \mathcal{U} \rightarrow \mathbb{C}$ be defined as

$$
\psi_{v}(z)=\sum_{n \geq 1} \frac{(-1)^{n}}{4^{n} n!(v+1)_{n}} z^{2 n+1}
$$

Let $v=\min \left\{v_{1}, \ldots, v_{n}\right\}$. Suppose that $\delta_{1}, \ldots, \delta_{n}$ be positive real numbers such that these numbers satisfy the following inequality

$$
0 \leq 1-\frac{10(4 v+4)(8 v+7)}{(4 v+3)(4 v+5)-2(8 v+7)} \sum_{i=1}^{n} \sigma_{i}<1
$$

Then, the function $H_{v}^{\delta}$ defined as

$$
H_{v}^{\delta}(z)=\int_{0}^{z}\left(\psi_{v}^{\prime}(t)\right)^{\delta} d t
$$

is in the class $\mathcal{C}\left(\gamma_{2}\right)$, where

$$
\gamma_{2}=1-\frac{10(4 v+4)(8 v+7)}{(4 v+3)(4 v+5)-2(8 v+7)} \sum_{i=1}^{n} \sigma_{i} .
$$

In particular,
(i) if $\delta \leq \frac{13}{660}$, then the function $H_{\frac{1}{2}}^{\delta}: \mathcal{U} \rightarrow \mathbb{C}$ defined by

$$
H_{\frac{3}{2}, \alpha}(z)=\int_{0}^{z}(\cos t)^{\delta} d t
$$

is in the class $\mathcal{C}\left(\gamma_{3}\right)$, where $\gamma_{3}=1-\frac{13}{660} \delta$;
(ii) if $\delta \leq \frac{61}{1900}$, then the function $H_{\frac{3}{2}}^{\delta}: \mathcal{U} \rightarrow \mathbb{C}$ defined by

$$
H_{\frac{3}{2}, \alpha}(z)=\int_{0}^{z}\left(\psi_{\frac{3}{2}}^{\prime}(t)\right)^{\delta} d t
$$

is in the class $\mathcal{C}\left(\gamma_{4}\right)$, where $\gamma_{4}=1-\frac{1900}{61} \delta$.

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